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A RELATIVE EXTENDING MODULE AND TORSION PRECOVERS

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Abstract. We first characterize τ-complemented modules with relative (pre)covers. We also introduce an extending module relative to τ-pure submodules on a hereditary torsion theory τ and give its relationship with τ-complemented modules.
Keywords: τ-pure submodule, extending module, torsion theory, (pre)covers, τ-cocompact module.
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1. **Introduction**

In this note all rings are associative with identity and all modules are unitary right modules. For a ring R, let $\tau := (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on the right module category Mod-R. Modules in $\mathcal{T}$ will be called τ-*torsion* and modules in $\mathcal{F}$ will be called τ-*torsionfree*. Given an R-module M, $\tau(M)$ will denote the τ-torsion submodule of M. Then $\tau(M)$ is necessarily the unique largest τ-torsion submodule of M and $\tau(M/\tau(M)) = 0$. For the torsion theory $\tau := (\mathcal{T}, \mathcal{F})$, $\mathcal{T} \cap \mathcal{F} = 0$ and the torsion class $\mathcal{T}$ is closed under homomorphic images, direct sums and extensions; and the torsionfree class $\mathcal{F}$ is closed under submodules, direct products and extensions; extension means by short exact sequence. If a torsion class $\mathcal{T}$ is closed under submodules, the torsion theory τ on Mod-R is called hereditary [2].

Let R be any ring and let τ be a torsion theory on Mod-R. For an R-module M, a submodule N of M is called τ-*dense* (respectively, τ-*pure*) in M if M/N is τ-torsion (respectively, τ-torsionfree). A submodule N of a module M is called τ-*essential* (in M) if N is an essential and τ-dense submodule of M (see [2]). A submodule N of a module M is called τ-*closed* in M if L is a submodule of M such that N is essential and τ-dense in L, one has $N = L$ (i.e., type 1

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τ-closed in the sense of [6]) . Let N be a submodule of M, a submodule K of M containing N is called a τ-*closure* of N in M if N is τ-dense in K and K is τ-pure in M [5, Definition 2.1].

Clearly $\tau(M)$ and M itself are τ-pure submodules of M. The class of τ-pure submodules of M is closed under intersections and $\tau(M)$ coincides with the intersection of all τ-pure submodules of M. An R-module M is τ-*injective* if and only if $Ext^1_R(T, M) = 0$ for all τ-torsion R-module T (see [2] or [4]). Equivalently, M is τ-injective if and only if M is τ-pure submodule of $E(M)$ (see [4, Theorem 2.1.1 (ii)]). For unexplained terminology see also [1], [2], [8], [11] and [13].

In this work, we proceed with the study of relative extending modules. In Section 2, we introduce a τ-torsion precover and some of its applications. In Section 3, we define τ-PES extending module and give some fundamental properties of them. This extending module is different from some other relative extending modules (see [7]). In Section 4, we give some characterization of τ-PES extending modules with τ-cocompact modules.

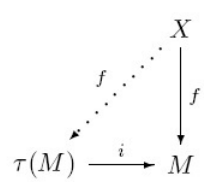
2. **Monic τ-torsion precovers**

Let R be a ring and $\mathcal{X}$ be any class of R-modules that it contains zero module and it is closed up to isomorphisms. For an R-module M, $\varphi : X \to M$ with $X \in \mathcal{X}$ is called an $\mathcal{X}$-*precover* of M if there exists a homomorphism $f : X' \to X$ with $X' \in \mathcal{X}$ for every homomorphism $\varphi' : X' \to M$ such that $\varphi f = \varphi'$. Here a module X is also called a $\mathcal{X}$-*precover* of M. If moreover, an $\mathcal{X}$-precover $\varphi \in \mathrm{Hom}(X, M)$ of M is called an $\mathcal{X}$-*cover* if every endomorphism $g : X \to X$ with $\varphi g = \varphi$ is an automorphism. We mean τ-*torsion precover* if class $\mathcal{X}$ replaced with torsion class $\mathcal{T}$ of a torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ (see [10] or [15]).

The following torsion theoretic result is adapted from Garcia and Torrecillas' result in categorical terms (see [9, Proposition 4]).

Proposition 2.1. *Let $\tau := (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on Mod-R and let M be an R-module. Then $\tau(M)$ is a τ-torsion precover of M.*

Proof. Notice that $\tau(M)$ is the unique largest τ-torsion submodule of M. Now consider the following diagram



with any τ-torsion module X and arbitrary morphism $f : X \to M$. Since $f(X) \in \mathcal{T}$, $f(X)$ is a submodule of $\tau(M)$. This finishes the proof. □

The existence of τ-pure submodules can be related to the existence of monic τ-torsion (pre)covers in the following way. The next result which is an analogue of Crivei's theorem has also a similar proof, hence its proof is not given (see [5, Lemma 2.2 and Theorem 2.4]).

Theorem 2.2. *Let $\tau := (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on Mod-R, let M be a module and let N be a submodule of M. Then the following assertions are equivalent:*

(1) *M has a τ-pure submodule K containing N such that N is τ-dense in K (i.e., N has a τ-closure in M).*
(2) *M/N has a monic τ-torsion precover.*

Recall from [14] that a module M is called τ-*complemented* if every submodule of M is τ-dense in a direct summand of M. One of the basic characterization of τ-complemented modules is as follows:

Lemma 2.3. *(* [14, Proposition 1.6]*) A module M is τ-complemented if and only if every τ-pure submodule of M is a direct summand of M.*

Remark 2.4. Recall from [11] that the τ-closure of a module M in the injective hull $E(M)$ of M is called a τ-*injective hull* of M and is denoted by $E_\tau(M)$. $E_\tau(M)$ is unique up to isomorphism. Clearly M is a τ-dense essential submodule of $E_\tau(M)$ and $E_\tau(M)/M = \tau(E(M)/M)$. Notice that by Proposition 2.1, $E_\tau(M)/M$ is a τ-torsion precover of $E(M)/M$.

Proposition 2.5. *Let M be a τ-complemented module such that for every submodule N of M, M/N has a monic τ-torsion precover. Then M is an extending module.*

Proof. Let N be a closed submodule of M. By hypothesis, Theorem 2.2 implies that M has a τ-pure submodule K containing N such that N is τ-dense in K. Since M is τ-complemented, by Lemma 2.3, K is a direct summand of M. By [4, Proposition 1.7.6] K is a τ-complemented module. Notice that, since N is a closed submodule of M, N is a closed submodule of K, so it is complemented in K. Therefore there exists a submodule X of K such that N is maximal with respect to the property $N \cap X = 0$. In this case $N \oplus X$ is an essential submodule of K. Since also K/N is τ-torsion, $K/(N \oplus X)$ is τ-torsion as an homomorphic image of K/N. Thus $N \oplus X$ is a τ-essential submodule of K. By using [4, Lemma 2.2.5], we have

$$E_\tau(K) = E_\tau(N \oplus X) = E_\tau(N) \oplus E_\tau(X).$$

Now

$$X \cong X/(N \cap X) \cong (N + X)/N \leq K/N.$$

By a hereditary torsion theory τ, X is a τ-torsion module. Thus X and $E_\tau(X)$ are submodules of $\tau(M)$. Since N is essential in $E_\tau(N)$, N is essential in $K \cap E_\tau(N)$, so we have $N = K \cap E_\tau(N)$.

On the other hand, since K is τ-pure in M we have $\tau(M)$ is a submodule of K. Now by modularity law, $K = K \cap E_\tau(K) = K \cap (E_\tau(X) \oplus E_\tau(N)) = E_\tau(X) \oplus (K \cap E_\tau(N)) = E_\tau(X) \oplus N$. Thus, N is a direct summand of M. □

3. τ-PES-extending modules

Recall that a nonzero submodule of a module M is *essential* in M if it has nonzero intersection with all nonzero submodules of M. A submodule K of a module M is *closed* if K has no proper essential extension in M i.e., if whenever there exists a submodule L of M containing K such that K is essential in L, one has $K = L$.

We say that a submodule N of a module M is *τ-PES-submodule* if N has nonzero intersection with every nonzero τ-pure submodule L of M, denoted by $N \leq_{\tau-PES} M$. Since $\tau(M)$ coincides with the intersection of all τ-pure submodules of M, clearly it is a τ-PES-submodule of M. In addition, a submodule K of M is called *τ-PES-closed in M* provided K has no proper τ-PES extension in M i.e., if whenever K is a τ-PES-submodule in a submodule L of M, one has $K = L$. It is also easy to see that $\tau(M)$ is not τ-PES-closed in M. Note that every module is a τ-PES-submodule of itself, a τ-PES-submodule of a module cannot be zero similarly, to essential submodules.

Definition 3.1. An R-module M is called *τ-PES-extending* if every τ-PES-closed submodule is a direct summand of M.

Proposition 3.2. *Let R be a ring. We have the following statements for a hereditary torsion theory τ on Mod-R.*

(1) *Every essential submodule of a module M is a τ-PES-submodule of M.*
(2) *Every τ-PES-closed submodule is a closed submodule.*
(3) *Every extending module is τ-PES-extending.*
(4) $N \leq_{\tau-PES} K \leq_{\tau-PES} M$ *implies* $N \leq_{\tau-PES} M$.
(5) *Every τ-dense, closed submodule of a τ-torsionfree module M is τ-PES-closed in M.*

Proof. (1) Let N be an essential submodule of M. Let $0 \neq L \leq M$ and let $M/L \in \mathcal{F}$. By hypothesis $0 \neq L \cap N$. Therefore N is a τ-PES-submodule of M. (2) and (3) are clear by (1).

(4) Let L be a nonzero submodule of M with $M/L \in \mathcal{F}$. Since K is a τ-PES-submodule of M, $K \cap L \neq 0$. Thus $K/(K \cap L) \cong (K + L)/L \leq M/L$ and $M/L \in \mathcal{F}$ implies $K/(K \cap L) \in \mathcal{F}$. Since also N is a τ-PES-submodule of K, $N \cap (K \cap L) \neq 0$. Thus we have $0 \neq (N \cap K) \cap L \subseteq (N \cap L)$. Therefore $N \cap L \neq 0$.

(5) By the fact that every τ-dense submodule of a τ-torsionfree module M is essential in M. □

Proposition 3.3. *If N is a τ-PES-closed submodule of an R- module M, then there is a submodule K of M such that N is maximal in M with respect to the property $N \cap K = 0$. In particular, $N \oplus K$ is a τ-PES submodule of M.*

Proof. If N is a τ-PES-closed submodule of M then N is closed in M by Proposition 3.2 (2). Thus N is a complement submodule of M. So there is a submodule K of M such that N is a maximal submodule of M and $N \cap K = 0$. By [8, p.6], $N \oplus K$ is essential in M. Thus by Proposition 3.2(1), $N \oplus K$ is a τ-PES-submodule of M. □

The following corollary is an immediate consequence of Proposition 2.5 and Proposition 3.2.

Corollary 3.4. *Let M be a τ-complemented module such that for every submodule N of M, M/N has a monic τ-torsion precover. Then M is τ-PES-extending.*

Example 3.5. (1) *Every semisimple module is τ-PES-extending because every submodule is a direct summand. Also, injective modules and uniform modules provide examples of modules that are τ-PES-extending by Proposition 3.2.*

(2) *Recall that a non zero module M is called* τ-cocritical *if M is τ-torsionfree and every non zero submodule of M is τ-dense in M. Since every proper submodule of a τ-cocritical module is essential by [4, Proposition 1.4.2 (iii)], so it is uniform; thus, it is extending and, consequently, it is τ-PES-extending.*

The converse of Proposition 3.2 (3) is not true in general. See the following example.

Example 3.6. *([6, Example 2.4]) Consider the ring* $R = \begin{bmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{bmatrix}$, *where F is a field. If τ_I is the torsion theory on Mod-R corresponding to the idempotent ideal* $I = \begin{bmatrix} F & F & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, *that is*
$\mathcal{T}_I := \{N \in Mod\text{-}R : NI = 0\}$, *then the followings hold for the right R-module $M := R_R$.*

(1) *M is a τ_I-PES-extending module, but it is not extending.*
(2) *M has direct summands that are not a τ_I-PES-closed submodule.*
(3) *The τ_I-torsion submodule $\tau_I(M)$ of M is neither a direct summand nor τ_I-PES-closed in M.*

Proof. (1) Since there is no τ_I-PES-closed submodule of M at all, we say that M is a τ_I-PES-extending module. We know that it is not extending by [12].

(2) Consider $M = I_1 \oplus I$ for $I_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & F \end{bmatrix}$ and $I = \begin{bmatrix} F & F & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

It is easy to see that the τ_I-torsion submodule of M is $\tau_I(M) = \begin{bmatrix} 0 & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{bmatrix}$.

Neither I nor I_1 is a τ_I-PES-closed submodule of M since I is a τ_I-PES-submodule of M and I_1 is τ_I-PES-submodule in $\tau_I(M)$.

(3) It is also easy to see that $\tau_I(M)$ is not a direct summand of M. Since $\tau_I(M)$ is τ_I-PES-submodule in M, it is not τ_I-PES-closed in M either. □

Lemma 3.7. *For every submodule N of a module M, there exists a τ-PES-closed submodule K of M such that N is a τ-PES-submodule of K.*

Proof. It is easy to see by Zorn's Lemma as follows.
Let $\mathcal{S} = \{H \leq M : N \leq_{\tau-PES} H\}$. Since N is a τ-PES-submodule in N itself, N belongs to $\mathcal{S}$, thus $\mathcal{S}$ is non empty. Let $\mathcal{C}$ be a chain in $\mathcal{S}$ and let $C = \bigcup_{C_i \in \mathcal{C}} C_i$. Claim: N is τ-PES-submodule in C. We know that $N \cap L_i \neq 0$ for every $C_i/L_i \in \mathcal{F}$. Let L be a non-zero submodule of C such that C/L is τ-torsion-free. On the other hand, $0 \neq L \cap C_i \leq C_i$ and $C_i/(L \cap C_i) \cong (C_i + L)/L \leq C/L \in \mathcal{F}$ implies $C_i/(L \cap C_i)$ is τ-torsion-free. Thus $N \cap (L \cap C_i) \neq 0$ for every C_i in $\mathcal{C}$. Assume that $N \cap L = 0$. Then $0 = (N \cap L) \cap C_i \neq 0$ which is a contradiction. Therefore $N \cap L \neq 0$. By Zorn's Lemma there exists a maximal submodule $K$of M. □

Definition 3.8. The τ-PES-closed submodule K of M, as it is constructed in Lemma 3.7, is called a *τ-PES-closure* of N in M.

As a result of Lemma 3.7, every submodule of a module M has a τ-PES-closure in M.

Proposition 3.9. *A module M is τ-PES-extending if and only if every submodule of M is τ-PES-submodule in a direct summand of M.*

Proof. Let K be any τ-PES-closed submodule of M. By hypothesis, there exists a direct summand L of M such that K is a τ-PES submodule of L. Since K is a τ-PES-closed submodule of M, we have $K = L$. Therefore M is τ-PES-extending. The converse is clear by Lemma 3.7. □

4. τ-cocompact modules

Recall from [3, Definition 2.3] that a module M is τ-*compact* if every nonzero submodule of M is τ-dense in M.

Corollary 4.1. *If a module M is τ-compact and τ-PES-extending then it is τ-complemented.*

Proof. Let N be any submodule of M. Since M is τ-PES-extending, by Proposition 3.9 there exists a direct summand D of M such that N is a τ-PES submodule of D. Because M is τ-compact, then N is τ-dense in M and consequently N is τ-dense in D. Thus, N is τ-dense in a direct summand D of M. □

As a dual of τ-compact module, we give the following definition.

Definition 4.2. We say that a module M is called τ-*cocompact* if every nonzero submodule of M is τ-pure in M.

It is easy to check that if $B \leq B'$ and B is τ-pure in M, then B is τ-pure in B'. Thus, if M is a τ-cocompact module, then every submodule of M is a τ-cocompact module.

Now we have the following easy observation.

Corollary 4.3. *If a module M is τ-cocompact, then every quotient module of M has a no non-zero τ-torsion precover.*

Proof. Let N be an arbitrary submodule of M. Since M is a τ-cocompact, then we have N is τ-pure in M, hence $\tau(M/N) = 0$. Then apply Proposition 2.1. □

Proposition 4.4. *If a module M is τ-cocompact then every τ-PES-submodule of M is essential in M.*

Proof. Let N be a τ-PES submodule of M. Then N has nonzero intersection with every nonzero τ-pure submodule L of M. Since M is τ-cocompact, every submodule of M is τ-pure, hence N has nonzero intersection with every submodule L of M. This completes the proof. □

Corollary 4.5. *Let M be a τ-cocompact module. Then M is extending if and only if M is τ-PES-extending.*

Proof. If M is extending, it is τ-PES-extending by Proposition 3.2. Conversely, let K be a closed submodule of M. Claim: K is τ-PES-closed in M. Suppose not. Then there exists a submodule L of M such that K is a τ-PES-submodule in L. Since M is a τ-cocompact, every τ-PES-submodule of M is essential in M by Proposition 4.4. Therefore K is essential in L. This is a contradiction. Thus K is τ-PES-closed in M. By hypothesis K is a direct summand of M. Hence, M is an extending module. □

Let U and M both be R-modules. Recall that U is M-*injective* if, for every submodule N of M, every homomorphism $\varphi : N \to U$ can be extended to a homomorphism $\psi : M \to U$ such that $\psi(x) = \varphi(x)$, for all x element of N. A class of R-modules $\{M_i : i \in I\}$, where I is an index set, is called *relatively injective* if M_i is M_j-injective for every pair of distinct $i, j \in I$ [12].

As a consequence of definition of τ-cocompact modules and by using Corollary 4.5 we have the following fact:

Theorem 4.6. *Let M be a τ-cocompact module and let $M = M_1 \oplus M_2 \oplus ... \oplus M_n$ finite direct sum of relatively injective modules M_i for $i = 1, ..., n$. Then M is τ-PES-extending if and only if all M_i are τ-PES-extending.*

Proposition 4.7. *If a module M is τ-cocompact and τ-complemented then it is semisimple.*

Proof. Let N be an arbitrary submodule of M. Since M is τ-cocompact, N is τ-pure in M. Also, since M is τ-complemented module by Lemma 2.3 N is a direct summand of M. □

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