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TRIANGULARIZABILITY OF ALGEBRAS OVER DIVISION RINGS

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Abstract. Let $\mathcal{V}$ be a finite-dimensional right vector space over a division ring D and let $\mathcal{C}$ be a collection of linear transformations on $\mathcal{V}$. In case of vector spaces over fields some authors have derived conditions on $\mathcal{C}$ which imply its triangularizability. Here, we will generalize some of these results to the case of vector spaces over division rings. We let $\mathcal{C}$ be a left artinian ring of linear transformations and prove a block triangularization theorem for $\mathcal{C}$. The theorem is then used to extend two well-known results in the theory of triangularization.

1. Introduction

Simultaneous triangularization of a collection $\mathcal{C}$ of operators on a finite-dimensional vector space over a field has been studied extensively by several mathematicians. One approach in this area is to derive various conditions on $\mathcal{C}$ implying its triangularizability. The well-known theorems of Kaplansky, Levitzki, Kolchin, Guralnick, and Radjavi are of this type. A survey of such results can be found in [5]. In case of finite-dimensional vector spaces over a division ring D the problem is much harder and, in most cases, extra conditions are required to obtain

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the corresponding results [7, 9, 10, 11]. For instance, in [10] $\mathcal{C}$ is further assumed to be an algebra of triangularizable operators with inner eigenvalues in the center of D and in [7] triangularization over any extension of D is allowed. In contrast to the algebraic methods of [7], the methods employed in [9, 10, 11] have a linear algebraic nature.

In the present article our aim is to generalize some of the triangularizability results mentioned in [5] to the case of vector spaces over division rings. In spite of the validity of few theorems such as Levitzki's theorem in the noncommutative case, in most cases, we need to impose extra conditions on $\mathcal{C}$ to prove those reducibility or triangularizability results which were ordinarily true in the commutative case. Various obstacles appear when a field F is replaced by a division ring D. For example, if $\mathcal{A}$ is an F-algebra in $M_n(F)$, then $\mathcal{A}$ is reducible if and only if one finds a nonzero element $x \in F^n$ such that $\mathcal{A}x \neq F^n$. That is, in case of fields, irreducibility and transitivity are equivalent. But, when F is replaced by D, the set $\mathcal{A}x$ is not necessarily a right vector space. To be more precise, in order to establish the reducibility of $\mathcal{A}$ one should find a proper $(\mathcal{A}, D)$-submodule of D^n. In [4], the distinctions of the notions of reducibility and transitivity for algebras are illustrated.

We start off with some definitions and notations. Throughout this article, F is a field, D is a general division ring and $\mathcal{V}$ is an n-dimensional right vector space over D. We denote the set of all linear transformations on $\mathcal{V}$ by $\mathcal{L}(\mathcal{V})$. The linear transformations act from the left side, while scalars are multiplied from the right side of a vector. If we choose a basis β for $\mathcal{V}$, then the map which assigns an element of $\mathcal{L}(\mathcal{V})$ to its matrix representation with respect to β in $M_n(D)$ is a ring isomorphism. If A and B are matrix representations of a linear transformation with respect to the bases β and β' respectively, then as in the case of fields, there exists an invertible matrix S such that $SAS^{-1} = B$.

Let $\mathcal{C}$ be a nonempty collection in $\mathcal{L}(\mathcal{V})$. A subspace $\mathcal{M}$ of $\mathcal{V}$ is said to be *invariant* under $\mathcal{C}$ if for all $T \in \mathcal{C}$, $T\mathcal{M} \subseteq \mathcal{M}$. In this case, $\mathcal{M}$ is called a $\mathcal{C}$-*invariant* subspace. The collection $\mathcal{C}$ is said to be *reducible* if $\mathcal{C} = \{0\}$ or there exists a $\mathcal{C}$-invariant subspace of $\mathcal{V}$ different from $\{0\}$ and $\mathcal{V}$. We say that $\mathcal{C}$ is *irreducible* if it is not reducible. The collection $\mathcal{C}$ is called *simultaneously triangularizable* or simply *triangularizable*, if there exists a maximal chain of subspaces of $\mathcal{V}$ each of which is invariant under $\mathcal{C}$. Therefore, a linear transformation is triangularizable if and only if its matrix representation with respect to some basis is an upper triangular matrix. Similarly, a matrix A as a linear transformation on

the right vector space D^n is triangularizable if and only if there exists an invertible matrix $S \in M_n(D)$ such that SAS^{-1} is an upper triangular matrix.

2. **Basic concepts**

In this section we present a theorem which is crucial in establishing our main results. This theorem relates noncommutative ring theory to operator theory over division rings and is a direct consequence of the Density Theorem [3, *p.* 181]. Here, for the sake of completeness, we give a simple proof. In the following, a ring $\mathcal{R}$ is called a prime ring if for any two ideals I and J of $\mathcal{R}$, $IJ = 0$ implies that either $I = 0$ or $J = 0$. Now, motivated by [6], we have the following lemma.

Lemma 2.1. *Let D be a division ring, $\mathcal{V}$ be a finite-dimensional right vector space over D, and $\mathcal{R}$ be an irreducible ring of linear transformations on $\mathcal{V}$. Then, $\mathcal{R}$ is a prime ring.*

Proof. Let I and J be two ideals of $\mathcal{R}$ such that $IJ = 0$. It is not hard to see that $I\mathcal{V}$ and $J\mathcal{V}$ are $\mathcal{R}$-invariant subspaces of $\mathcal{V}$. If $J \neq 0$, then $J\mathcal{V} \neq 0$. Thus, the irreducibility of $\mathcal{R}$ implies that $J\mathcal{V} = \mathcal{V}$. Now, $0 = (IJ)\mathcal{V} = I(J\mathcal{V}) = I\mathcal{V}$. Hence, $I = 0$. $\square$

To prove our theorem it is essential to remind some Wedderburn-Artin theory [2]. If $\mathcal{R}$ is a left artinian ring, then the Jacobson radical of $\mathcal{R}$, denoted by $\mathrm{J}(\mathcal{R})$, is a nilpotent ideal; i.e., $\mathrm{J}(\mathcal{R})^n = 0$ for some $n \in \mathbb{N}$. On the other hand, Wedderburn-Artin Theorem asserts that a nonzero ring $\mathcal{R}$ is semisimple left artinian if and only if

$$\mathcal{R} \simeq M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_k}(D_k)$$

for some natural numbers $n_1, n_2, \cdots, n_k$ and division rings $D_1, D_2, \cdots D_k$. Moreover, $\mathcal{R}$ is simple left artinian if and only if $\mathcal{R} \simeq M_n(D)$ for some natural number n and some division ring D.

Theorem 2.2. *Let D be a division ring containing F as a subfield of its center, $\mathcal{V}$ be a finite-dimensional right vector space over D and $\mathcal{R}$ be an irreducible ring (F-algebra) of linear transformations on $\mathcal{V}$. If $\mathcal{R}$ is left artinian, then $\mathcal{R}$ is simple left artinian and consequently there exist a natural number m and a division ring (F-algebra) E such that $\mathcal{R} \simeq M_m(E)$. Furthermore, m and E are unique.*

Proof. By Lemma 2.1, $\mathcal{R}$ is a prime ring. On the other hand, since $\mathcal{R}$ is left artinian, $\mathrm{J}(\mathcal{R})$ is a nilpotent ideal [2, *p.* 430]. Therefore, $\mathrm{J}(\mathcal{R}) = 0$ and hence $\mathcal{R}$ is a semisimple left artinian ring. Now, by Wedderburn-Artin Theorem we have the ring (F-algebra) isomorphism

$$\mathcal{R} \simeq M_{n_1}(D_1) \times M_{n_2}(D_2) \times \cdots \times M_{n_k}(D_k),$$

for some unique natural numbers $n_1, n_2, \cdots, n_k$ and unique division rings (F-algebras) $D_1, D_2, \cdots, D_k$. Since $\mathcal{R}$ is a prime ring, $k = 1$. □

Remark 2.3. If $\mathcal{R}$ is a nonzero irreducible subring of $M_n(D)$ which is also left artinian, then $\mathcal{R}$ contains the identity matrix I. We present a proof similar to the one given in [11]. From Theorem 2.2, $\mathcal{R}$ is a ring with identity. Let I be the identity of $\mathcal{L}(\mathcal{V})$ and I' be the identity of $\mathcal{R}$. Since $(I - I')\mathcal{R} = 0$, $I \neq I'$ implies that $\mathrm{Ker}(I - I')$ is a nontrivial $\mathcal{R}$-invariant subspace, contradicting the irreducibility of $\mathcal{R}$.

3. Main results

In case of fields, a number of triangularizability results can be derived from the Block Triangularization Theorem [8]. In order to state a corresponding theorem in the noncommutative case, we need to impose a chain condition on the ring to be put into block triangular form. So, we consider left artinian rings of matrices. Motivated by [5], this enables us to prove a version of the Block Triangularization Theorem.

Theorem 3.1. (Block Triangularization Theorem) *Let $n \in \mathbb{N}$, D be a division ring, and $\mathcal{R}$ be a left artinian subring of $M_n(D)$. Then, there exists an invertible matrix $S \in M_n(D)$ such that for any matrix $A \in \mathcal{R}$, SAS^{-1} has the block upper triangular form,*

$$SAS^{-1} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1k} \\ 0 & A_{22} & A_{23} & \cdots & A_{2k} \\ 0 & 0 & A_{33} & \cdots & A_{3k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{kk} \end{bmatrix}, \tag{3.1}$$

where the set $\{1, 2, \cdots, k\}$ is the disjoint union of subsets $J_1, J_2, \cdots, J_l$ with the following properties:

(i) *For any* i, $1 \leq i \leq k$, *the collection* $\mathcal{R}_i = \{A_{ii} : A \in \mathcal{R}\}$ *is either zero and* $n_i = 1$ *or an irreducible subring of* $M_{n_i}(D)$, *where* n_i *is the size of the* A_{ii}. *Moreover, each nonzero* $\mathcal{R}_i$ *is a simple left artinian ring.*

(ii) *If* i *and* j *are both in the same* J_s, *then* $\mathcal{R}_i \simeq \mathcal{R}_j$.

(iii) *If* i *and* j *are in different subsets* J_s *and* J_t, *then the set of pairs* (A_{ii}, A_{jj}) *that arise as* A *ranges over* $\mathcal{R}$ *is* $\mathcal{R}_i \times \mathcal{R}_j$.

(iv) *If* $1 \leq t \leq l$ *is such that* $\mathcal{R}_i$ *is nonzero for all* $i \in J_t$, *then there exists* $A \in \mathcal{R}$ *such that* $A_{ii} = I_{n_i}$ *for all* $i \in J_t$, *and* $A_{jj} = 0$ *for all* $j \notin J_t$.

Proof. Let

$$\{0\} = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_k = D^n \tag{3.2}$$

be a maximal chain of distinct $\mathcal{R}$-invariant subspaces of D^n. Take a basis β_1 of $\mathcal{V}_1$ and extend it to a basis β_2 of $\mathcal{V}_2$ and so on to get bases $\beta_1 \subset \beta_2 \subset \cdots \subset \beta_k$ for $\mathcal{V}_1$, $\mathcal{V}_2$, $\cdots$, $\mathcal{V}_k$, respectively. Put $\beta = \beta_k$. Obviously, with respect to the basis β, each element of $\mathcal{R}$ has the upper triangular form (3.1). Also, it is evident that for $1 \leq i \leq k$,

$$\mathcal{R}_i = \{A_{ii} : A \in \mathcal{R}\}$$

is isomorphic to the ring of matrices induced by $\mathcal{R}$ on $\mathcal{V}_i/\mathcal{V}_{i-1}$ with respect to the basis induced by $\beta_i \backslash \beta_{i-1}$. For $1 \leq i \leq k$, define the surjection,

$$\varphi_i : \mathcal{R} \longrightarrow \mathcal{R}_i, \qquad \varphi_i(A) = A_{ii}.$$

Now, since $\mathcal{R}$ is left artinian, then $\mathcal{R}_i$ is also left artinian. On the other hand, the maximality of chain (3.2) implies that either $\mathcal{R}_i = \{0\}$ with corresponding n_i equal to 1, or $\mathcal{R}_i$ is an irreducible ring. Therefore, by Theorem 2.2, $\mathcal{R}_i$ is either zero or simple left artinian ($1 \leq i \leq k$). This proves (i).

In order to prove (ii) and (iii), we define a relation on the set $\{1, 2, \cdots, k\}$. We say i is linked to j if either $\mathcal{R}_i$ and $\mathcal{R}_j$ are both zero or $\mathcal{R}_i$ and $\mathcal{R}_j$ are both nonzero and for any $A \in \mathcal{R}$, $A_{jj} = 0$ whenever $A_{ii} = 0$. Now, if i is linked to j, then the surjection

$$\varphi_{ij} : \mathcal{R}_i \longrightarrow \mathcal{R}_j, \qquad \varphi_{ij}(A_{ii}) = A_{jj},$$

is well-defined and is an isomorphism, since $\mathcal{R}_i$ is a simple ring. This shows that the defined relation is an equivalence relation. Now, it is clear that this relation gives the desired partition $J_1, J_2, \cdots, J_l$. This proves part (ii).

For part (iii), we say i and j are independent if they are not linked. Let i and j be independent. Without loss of generality, assume that $\mathcal{R}_j$ is nonzero. Since i and j are not linked, then there exists $B \in \mathcal{R}$ such that $B_{ii} = 0$ and $B_{jj} \neq 0$. Consider the set,

$$I = \{A_{jj} : A_{ii} = 0, A \in \mathcal{R}\}.$$

Obviously, I is a nonzero ideal of the simple ring $\mathcal{R}_j$. Thus,

$$I = \{A_{jj} : A_{ii} = 0, A \in \mathcal{R}\} = \mathcal{R}_j. \tag{3.3}$$

If $\mathcal{R}_i = \{0\}$, then the assertion of part (iii) holds. If not, we would similarly have

$$\mathcal{R}_i = \{A_{ii} : A_{jj} = 0, A \in \mathcal{R}\}. \tag{3.4}$$

Now, the fact that $\mathcal{R}$ is closed under addition establishes part (iii).

To establish the last part, let t, $1 \leq t \leq l$, be such that $\mathcal{R}_i \neq 0$, for all $i \in J_t$. We have observed in the previous section that for all $i \in J_t$, $\mathcal{R}_i$ contains I_{n_i}, the identity matrix of $M_{n_i}(D)$. According to parts (ii) and (iii), for any $u \neq t$ there exists $B^u \in \mathcal{R}$ such that $B^u_{ii} = I_{n_i}$ for all $i \in J_t$ and $B^u_{jj} = 0$ for all $j \in J_u$. Now, define $X^t \in \mathcal{R}$ by

$$X^t = \prod_{u \neq t} B^u.$$

Clearly X^t has the desired property. □

Remark 3.2. Yahaghi [10] has shown that if F is a subfield of the center of D and $\mathcal{A}$ is an F-algebra of triangularizable matrices with inner eigenvalues in F, then we would have a block triangularization theorem stronger than the one given here.

Our next result is a generalization of the following theorem [5]: Let F be a field and $\mathcal{A}$ be an F-algebra of triangularizable matrices in $M_n(F)$. Then, the following assertions are equivalent.

(i) $\mathcal{A}$ is triangularizable.

(ii) $A + B$ is nilpotent whenever A and B are nilpotent matrices in $\mathcal{A}$.

(iii) For A, $B \in \mathcal{A}$, AB is nilpotent whenever A or B is nilpotent.

First, we need to prove the following lemma.

Lemma 3.3. *Let $n \in \mathbb{N}$, D be a division ring, and $\mathcal{R}$ be a left artinian subring of $M_n(D)$. Assume that for any two nilpotent matrices A, $B \in$*

$\mathcal{R}$, $A + B$ is nilpotent. If we put $\mathcal{R}$ into block triangular form as in Theorem 3.1, then each $\mathcal{R}_i$ is either zero or a division ring.

Proof. Using the notations of Theorem 3.1, we assume that $\mathcal{R}_i \neq 0$ for some i, $1 \leq i \leq n$. Let A_i and B_i be two nilpotent matrices in $\mathcal{R}_i$. In view of parts (ii) and (iv) of Theorem 3.1, there exist nilpotent matrices A, $B \in \mathcal{R}$ such that $A_{ii} = A_i$ and $B_{ii} = B_i$. Since $A + B$ is nilpotent, then $A_i + B_i$ is also nilpotent. Thus, $\mathcal{R}_i$ has the property that the sum of its nilpotent elements is again nilpotent, but part (i) of Theorem 3.1 together with Theorem 2.2 imply that $\mathcal{R}_i \simeq M_{k_i}(D_i)$ for some division ring D_i and some natural number k_i. If $k_i > 1$, then one can find two nilpotent matrices in $M_{k_i}(D_i)$ the sum of which does not happen to be nilpotent and thus, $\mathcal{R}_i$ is a division ring.

Remark 3.4. A similar proof shows that the following property leads us to the same conclusion: For A, $B \in \mathcal{R}$, AB is nilpotent if A or B is nilpotent.

Theorem 3.5. *Let D be a division ring, $n \in \mathbb{N}$, and $\mathcal{R}$ be a left artinian subring of $M_n(D)$. If $\mathcal{R}$ contains a nilpotent matrix N such that $N^{n-1} \neq 0$, then the following assertions are equivalent.*

(i) *$\mathcal{R}$ is triangularizable*

(ii) *$A + B$ is nilpotent whenever A and B are nilpotent matrices in $\mathcal{R}$.*

(iii) *For A, $B \in \mathcal{R}$, AB is nilpotent whenever A or B is nilpotent.*

Proof. We use the notations of Theorem 3.1. Clearly (i) implies (ii) and (iii). Assume that any of (ii) or (iii) holds. By Lemma 3.2 and its following remark $\mathcal{R}_i$ is either zero or a division ring. Let i, $1 \leq i \leq n$, be arbitrary and consider the block triangular form of N. Since N is nilpotent, then N_{ii} is also nilpotent. Hence, the fact that $\mathcal{R}_i$ is either zero or a division ring implies that N_{ii} is equal to zero. Now, if $n_j > 1$ for some j, $1 \leq j \leq n$, then we would have $N^{n-1} = 0$, which is a contradiction. Thus, for all j, $1 \leq j \leq n$, $n_j = 1$ which means that $\mathcal{R}$ is triangularizable. □

A direct consequence of Guralnick's theorem [1] is that if $\mathcal{A}$ is a subalgebra of $M_n(F)$ such that any pair of matrices in $\mathcal{A}$ are simultaneously triangularizable, then $\mathcal{A}$ is triangularizable. In the next theorem, we extend this result to subalgebras of $M_n(D)$. Recall that a field F is called *perfect* if its finite field extensions are all simple extensions.

Theorem 3.6. *Let D be a division ring and $\mathcal{A}$ be a finite-dimensional F-algebra in $M_n(D)$, where F is a perfect field contained in the center of D. Then, $\mathcal{A}$ is triangularizable if and only if for any pair of matrices $A, B \in \mathcal{A}$, the set $\{A, B\}$ is triangularizable.*

Proof. The "only if" part is trivial. For the "if part" again we use Theorem 3.1. We assume that $\mathcal{A}$ has the block triangular form (3.1). It is easy to see that each $\mathcal{A}_i$ is a finite-dimensional F-algebra. On the other hand, the hypothesis of the theorem implies that $A + B$ is nilpotent whenever A and B are nilpotent matrices in $\mathcal{A}$. Therefore, by Lemma 3.2, each $\mathcal{A}_i$ is a finite-dimensional division F-algebra, and since $\mathcal{A}_i$ contains the identity matrix I_{n_i}, then we can regard F as all elements of the form aI_{n_i}, $a \in F$. By contradiction, let $n_j > 1$ for some j, $1 \leq j \leq n$. For $\mathcal{A}_j$, a field, since F is a perfect field, then $\mathcal{A}_j = F[T]$ for some $T \in \mathcal{A}_j$. On the other hand, it is well-known that triangularizability is inherited by quotients. Hence, T and consequently $\mathcal{A}_j$ are triangularizable, which is a contradiction. If $\mathcal{A}_j$ is a noncommutative division ring, then it can be generated by two elements as an algebra over its center [3, *p.* 246]. Since any pair of matrices in $\mathcal{A}_j$ are simultaneously triangularizable, then $\mathcal{A}_j$ is also triangularizable, which again is a contradiction. □

If we denote the division ring of quaternions by $\mathbb{H}$, then the following result is a direct consequence of Theorem 3.6. Recall that the center of $\mathbb{H}$ is the field $\mathbb{R}$ of real numbers.

Corollary 3.7. *Let $n \in \mathbb{N}$ and $\mathcal{A}$ be an $\mathbb{R}$-algebra in $M_n(\mathbb{H})$. Then, $\mathcal{A}$ is triangularizable if and only if for any A and B in $\mathcal{A}$ the pair $\{A, B\}$ is triangularizable.*

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