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ON GENERALIZED REDUCED REPRESENTATIONS OF RESTRICTED LIE SUPERALGEBRAS IN PRIME CHARACTERISTIC

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Abstract. Let $\mathbb{F}$ be an algebraically closed field of prime characteristic $p > 2$ and $(\mathfrak{g}, [p])$ a finite-dimensional restricted Lie superalgebra over $\mathbb{F}$. It is shown that any finite-dimensional indecomposable $\mathfrak{g}$-module is a module for a finite-dimensional quotient of the universal enveloping superalgebra of $\mathfrak{g}$. These quotient superalgebras are called the generalized reduced enveloping superalgebras, which generalize the notion of reduced enveloping superalgebras. Properties and representations of these generalized reduced enveloping superalgebras are studied. Moreover, each such superalgebra can be identified as a reduced enveloping superalgebra of the associated restricted Lie superalgebra.
Keywords: Restricted Lie superalgebra, generalized reduced representation, indecomposable module, p-character, block.
MSC(2010): Primary: 17B10; Secondary: 17B35, 17B50.

1. Introduction

The finite-dimensional simple Lie superalgebras over the field of complex numbers were classified by Kac in the 1970s (cf. [8]). Although until now, the classification of finite-dimensional simple (restricted) Lie superalgebras over a field of prime characteristic has not yet been completed, there has been increasing interest in modular representation theory of restricted Lie superalgebras in recent years. W. Wang and L. Zhao [12, 13] initiated and developed systematically the modular representations of Lie superalgebras over an algebraically closed field of characteristic $p > 2$. In [12], the super version of the celebrated Kac-Weisfeiler Property was shown to be held for the basic classical Lie superalgebras, which by definition admit an even non-degenerate supersymmetric bilinear form and whose even subalgebras are reductive. There also has been

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increasing interest [2–5, 9, 11] in modular representation theory of algebraic supergroups in connection with other areas in recent years. Indeed, the modular representation theory of supergroups and Lie superalgebras has found remarkable applications to classical mathematics (see [11] for references about some historical remarks).

Motivated by the work [1] of C. P. Bendel on generalized reduced enveloping algebras for restricted Lie algebras, we further consider the case of restricted Lie superalgebras in this paper. Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie superalgebra over an algebraically closed field $\mathbb{F}$ of characteristic $p > 2$. It is obvious that for each $x \in \mathfrak{g}_{\bar{0}}$, the element $x^p - x^{[p]}$ is even and central in the universal enveloping superalgebra $U(\mathfrak{g})$. Let Z denote the central subalgebra of $U(\mathfrak{g})$ generated by all the elements $x^p - x^{[p]}$ with $x \in \mathfrak{g}_{\bar{0}}$, which is the so-called p-center. Since each irreducible $\mathfrak{g}$-module is finite-dimensional (cf. [12, 14]), the Lie superalgebra version of Schur's Lemma [8, Subsection 1.1.6] implies that the p-center Z acts by scalars on any irreducible $\mathfrak{g}$-module M. Then there exists a unique $\chi \in \mathfrak{g}_{\bar{0}}^*$ such that $x^p \cdot v - x^{[p]} \cdot v = \chi(x)^p v, \forall\, x \in \mathfrak{g}_{\bar{0}}, v \in M$. Therefore, M is a module for the finite-dimensional superalgebra $U_\chi(\mathfrak{g}) = U(\mathfrak{g})/(x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{g}_{\bar{0}})$, where $(x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{g}_{\bar{0}})$ denotes the ideal of $U(\mathfrak{g})$ generated by all the elements $x^p - x^{[p]} - \chi(x)^p$ with $x \in \mathfrak{g}_{\bar{0}}$. The superalgebra $U_\chi(\mathfrak{g})$ is called the χ-reduced enveloping superalgebra. More generally, a $\mathfrak{g}$-module M is said to have a p-character χ provided that $x^p \cdot v - x^{[p]} \cdot v = \chi(x)^p v, \forall\, x \in \mathfrak{g}_{\bar{0}}, v \in M$, or equivalently, it is a $U_\chi(\mathfrak{g})$-module.

While each simple $\mathfrak{g}$-module is a $U_\chi(\mathfrak{g})$-module for a unique $\chi \in \mathfrak{g}_{\bar{0}}^*$, this is not necessary true for an arbitrary indecomposable $\mathfrak{g}$-module. Indeed, for any indecomposable $\mathfrak{g}$-module M, there exists a unique $\chi \in \mathfrak{g}_{\bar{0}}^*$ and a least positive integer r such that $x^{p^r} v - (x^{[p]})^{p^{r-1}} v = \chi(x)^{p^r} v$ for all $x \in \mathfrak{g}_{\bar{0}}, v \in M$ (see Theorem 3.4), i.e., it is a module for a finite-dimensional quotient superalgebra $U_{\chi^r}(\mathfrak{g}) = U(\mathfrak{g})/\big((x^p - x^{[p]} - \chi(x)^p)^{p^{r-1}} \mid x \in \mathfrak{g}_{\bar{0}}\big)$, where $\big((x^p - x^{[p]} - \chi(x)^p)^{p^{r-1}} \mid x \in \mathfrak{g}_{\bar{0}}\big)$ denotes the ideal of $U(\mathfrak{g})$ generated by all the elements $(x^p - x^{[p]} - \chi(x)^p)^{p^{r-1}}$ with $x \in \mathfrak{g}_{\bar{0}}$. Each superalgebra $U_{\chi^r}(\mathfrak{g})$ is called a generalized χ-reduced enveloping superalgebra. In particular, when $r = 1$, it coincides with the usual χ-reduced enveloping superalgebra. In some sense, the family $\{U_{\chi^r}(\mathfrak{g}) \mid \chi \in \mathfrak{g}_{\bar{0}}, r \in \mathbb{N}\}$ encompasses the representation theory of all finite-dimensional indecomposable $\mathfrak{g}$-modules (see Remark 3.5).

This paper is structured as follows. In Section 2, we recall some basic notations and properties for restricted Lie superalgebras. Section 3 is devoted to studying indecomposable representations of restricted Lie superalgebras. Our main results imply that any finite-dimensional indecomposable module for a finite-dimensional restricted Lie superalgebra $\mathfrak{g}$ over an algebraically closed field is a module for a finite-dimensional quotient of the universal enveloping superalgebra. These quotient superalgebras form a two-parameter family which generalize the notion of reduced enveloping superalgebras. They are called the generalized reduced enveloping superalgebras. In the final section, representation theory of the generalized χ-reduced enveloping superalgebra $U_{\chi^r}(\mathfrak{g})$ is studied. It is shown that $U_{\chi^r}(\mathfrak{g})$ is a Frobenius superalgebra for any $r \in \mathbb{N}$. Then the projective objects coincide with the injective objects in the $U_{\chi^r}(\mathfrak{g})$-module category. Moreover, with some super trace condition on $\mathfrak{g}$, the generalized reduced enveloping superalgebras are further proved to be symmetric superalgebras. For any positive integer r, the collection of simple $U_{\chi^r}(\mathfrak{g})$-modules is precisely the set of simple $U_{\chi}(\mathfrak{g})$-modules regarded as $U_{\chi^r}(\mathfrak{g})$-modules (see Theorem 4.5). Moreover, the block structure in $U_{\chi^r}(\mathfrak{g})$ coincides with the one in $U_{\chi}(\mathfrak{g})$. Finally, we show that each generalized χ-reduced enveloping superalgebra can be identified as the reduced enveloping superalgebra of the associated restricted Lie superalgebra (see Theorem 4.13). Hence the representation theory of this more general family of superalgebras $\{U_{\chi^r}(\mathfrak{g})\}$ is reduced in some sense to the representation theory of reduced enveloping superalgebras.

2. Preliminaries on restricted Lie superalgebras

Throughout this paper, $\mathbb{F}$ is assumed to be an algebraically closed field of prime characteristic $p > 2$. All modules (vector spaces) are over $\mathbb{F}$ and finite-dimensional.

The following notion of restricted Lie superalgebras is a generalization of the one for restricted Lie algebras (see [7]).

Definition 2.1. *(cf. [10]) A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is called a restricted Lie superalgebra if there is a so-called p-mapping $[p]$ on $\mathfrak{g}_{\bar{0}}$ satisfying the following conditions:*

(i) $(adx)^p y = ad(x^{[p]})y, \forall\, x \in \mathfrak{g}_{\bar{0}}$ *and* $y \in \mathfrak{g}$.

(ii) $(kx)^{[p]} = k^p x^{[p]}, \forall\, k \in \mathbb{F}, x \in \mathfrak{g}_{\bar{0}}$.

(iii) $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum\limits_{i=1}^{p-1} s_i(x,y), \forall\, x, y \in \mathfrak{g}_{\bar{0}}$,

where $is_i(x,y)$ is the coefficient of λ^{i-1} in $ad(\lambda x + y)^{p-1}(x)$ and λ is an indeterminant.

Remark 2.2. *The condition (iii) in Definition 2.1 is equivalent to the following condition.*

(iii′) *We have the following relation in the universal enveloping superalgebra $U(\mathfrak{g})$:*

$$(x+y)^p - x^p - y^p = (x+y)^{[p]} - x^{[p]} - y^{[p]}, \forall\, x, y \in \mathfrak{g}_{\bar{0}}.$$

Remark 2.3. *In short, a restricted Lie superalgebra is a Lie superalgebra whose even subalgebra is a restricted Lie algebra and the odd part is a restricted module over the even subalgebra by the adjoint action.*

Example 2.4. *Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be any associative $\mathbb{F}$-superalgebra. Then A admits the structure of a Lie superalgebra by defining the bracket operation as $[x, y] = xy - (-1)^{\bar{x}\bar{y}}yx$ for any homogeneous elements $x, y \in A$ with $\bar{x}, \bar{y}$ denoting the parity of x and y respectively. Furthermore, this becomes a restricted Lie superalgebra with the p-mapping given by $x^{[p]} = x^p$ for any $x \in A_{\bar{0}}$, i.e., the p-mapping is just taken as the pth power in the superalgebra A. As a special case, let*

$$\mathfrak{g} = \mathfrak{sl}(2|1) = \left\{ \left(\begin{array}{cc|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \end{array} \right) \middle| a_{ij} \in \mathbb{F}, a_{11} + a_{22} - a_{33} = 0 \right\}$$

with

$$\mathfrak{g}_{\bar{0}} = span_{\mathbb{F}}\{e_{11} + e_{33}, e_{22} + e_{33}, e_{12}, e_{21}\}$$

and

$$\mathfrak{g}_{\bar{1}} = span_{\mathbb{F}}\{e_{13}, e_{23}, e_{31}, e_{32}\},$$

where the e_{ij} denotes the 3×3 matrix with 1 in the (i, j)-position and 0 in the other positions for $1 \le i, j \le 3$. Then $\mathfrak{g}$ is an associative $\mathbb{F}$-superalgebra in the natural way. Consequently, $\mathfrak{g}$ admits a structure of a Lie superalgebra with $[x, y] = xy - (-1)^{\bar{x}\bar{y}}yx$ for any $x, y \in \mathfrak{g}_{\bar{0}} \cup \mathfrak{g}_{\bar{1}}$. Moreover, $\mathfrak{g}$ is a restricted Lie superalgebra with the p-mapping defined as follows:

$$(e_{11} + e_{33})^{[p]} = e_{11} + e_{33}, \quad (e_{22} + e_{33})^{[p]} = e_{22} + e_{33}, \quad e_{12}^{[p]} = e_{21}^{[p]} = 0.$$

This restricted Lie superalgebra $\mathfrak{g}$ is a simple Lie superalgebra of classical type $A(1, 0)$.

Example 2.5. *The Lie superalgebra of an algebraic supergroup is a restricted Lie superalgebra (see [11]).*

We will make use of the following generalization of the equivalent condition (iii)′ for the definition of a restricted Lie superalgebra.

Lemma 2.6. *Let $(\mathfrak{g}, [p])$ be a restricted Lie superalgebra over $\mathbb{F}$. Let $x = \sum_{i=1}^{n} k_i x_i \in \mathfrak{g}_{\bar{0}}$ with $k_i \in \mathbb{F}$ and $x_i \in \mathfrak{g}_{\bar{0}}$. Then*

$$x^p - x^{[p]} = \sum_{i=1}^{n} k_i^p (x_i^p - x_i^{[p]})$$

holds in the universal enveloping superalgebra $U(\mathfrak{g})$.

Proof. We use induction on the number n of summands in the expression of x to show the conclusion.

The conclusion obviously holds for $n = 1$. Assume that it holds for $n < m$, where $m \ge 2$ is a positive integer. Next we claim that the statement is also valid for $n = m$. Indeed, write x as

$$x = \sum_{i=1}^{m} k_i x_i = \Big(\sum_{i=1}^{m-1} k_i x_i\Big) + k_m x_m.$$

Then

$$\begin{aligned} x^p - x^{[p]} =& \Big(\big(\sum_{i=1}^{m-1} k_i x_i\big) + k_m x_m \Big)^p - \Big(\big(\sum_{i=1}^{m-1} k_i x_i\big) + k_m x_m \Big)^{[p]} \\ =& \big(\sum_{i=1}^{m-1} k_i x_i\big)^p + k_m^p x_m^p - \big(\sum_{i=1}^{m-1} k_i x_i\big)^{[p]} - k_m^p x_m^{[p]} \\ =& \big(\sum_{i=1}^{m-1} k_i x_i\big)^p - \big(\sum_{i=1}^{m-1} k_i x_i\big)^{[p]} + k_m^p (x_m^p - x_m^{[p]}) \\ =& \sum_{i=1}^{m} k_i^p (x_i^p - x_i^{[p]}), \end{aligned}$$

as desired. □

Let $(\mathfrak{g}, [p])$ be a restricted Lie superalgebra and $\chi \in \mathfrak{g}_{\bar{0}}^*$. A $\mathfrak{g}$-module M is said to be χ-reduced if $x^p \cdot v - x^{[p]} \cdot v = \chi(x)^p v$ for all $x \in \mathfrak{g}_{\bar{0}}, v \in M$. In particular, it is called a restricted module for $\chi = 0$. As in the case of restricted Lie algebras, one can define the so-called χ-reduced enveloping superalgebra $U_\chi(\mathfrak{g})$ to be the quotient of $U(\mathfrak{g})$ by the ideal generated by $\{x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{g}_{\bar{0}}\}$, where $U(\mathfrak{g})$ denotes the universal enveloping superalgebra of $\mathfrak{g}$, i.e., $U_\chi(\mathfrak{g}) = U(\mathfrak{g})/(x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{g}_{\bar{0}})$. If $\chi = 0$, the superalgebra $U_0(\mathfrak{g})$ is called the restricted enveloping superalgebra and denoted by $u(\mathfrak{g})$ for brevity. All the χ-reduced (resp. restricted) $\mathfrak{g}$-modules constitute a full subcategory of the $\mathfrak{g}$-module category, which coincides with the $U_\chi(\mathfrak{g})$ (resp. $u(\mathfrak{g})$)-module category. Each simple $\mathfrak{g}$-module is a $U_\chi(\mathfrak{g})$-module for a unique $\chi \in \mathfrak{g}_{\bar{0}}^*$ (cf. [12, 14]).

3. Indecomposable representations of restricted Lie superalgebras

In this section, we show that every finite-dimensional indecomposable module for a finite-dimensional restricted Lie superalgebra is a module for a certain finite-dimensional superalgebra, which is a quotient of the universal enveloping superalgebra.

Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie superalgebra over $\mathbb{F}$. By the condition (i) in the Definition 2.1, $x^p - x^{[p]}$ is central in $U(\mathfrak{g})$ for any $x \in \mathfrak{g}_{\bar{0}}$. Fix a basis $\{x_1, \cdots, x_m, y_1, \cdots, y_n\}$ of $\mathfrak{g}$ with $x_i \in \mathfrak{g}_{\bar{0}}$ and $y_j \in \mathfrak{g}_{\bar{1}}$ for $1 \leq i \leq m, 1 \leq j \leq n$. Set $z_i = x_i^p - x_i^{[p]} \in U(\mathfrak{g})$ for $1 \leq i \leq m$. Let Z denote the even central subalgebra of $U(\mathfrak{g})$ generated by $x^p - x^{[p]}$ for all $x \in \mathfrak{g}_{\bar{0}}$. Then by Lemma 2.6 and the PBW Theorem, Z is a polynomial algebra $\mathbb{F}[z_1, \cdots, z_m]$. Moreover, $U(\mathfrak{g})$ is free over Z of rank $p^m 2^n$. More generally, for each positive integer r, define Z_r to be the even central subalgebra of $U(\mathfrak{g})$ generated by $(x^p - x^{[p]})^{p^{r-1}}$

for all $x \in \mathfrak{g}_{\bar{0}}$. Then Z_r is a polynomial algebra $\mathbb{F}[z_1^{p^{r-1}}, \cdots, z_m^{p^{r-1}}]$ and $U(\mathfrak{g})$ is free over Z_r of rank $p^{rm}2^n$. In particular, $Z_1 = Z$. We have the following easy observation.

Lemma 3.1. *Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie superalgebra and r be a positive integer. Let $S^*(\mathfrak{g}_{\bar{0}}^{(r)})$ be the symmetric algebra on the vector space $\mathfrak{g}_{\bar{0}}^{(r)}$, where $\mathfrak{g}_{\bar{0}}^{(r)}$ is the r-twist of $\mathfrak{g}_{\bar{0}}$ with the underlying space $\mathfrak{g}_{\bar{0}}$ and $k \in \mathbb{F}$ acting by $k^{p^{-r}}$. Then the natural map $S^*(\mathfrak{g}_{\bar{0}}^{(r)}) \longrightarrow Z_r$ defined on generators by $x \longmapsto (x^p - x^{[p]})^{p^{r-1}}$ for all $x \in \mathfrak{g}_{\bar{0}}$ is an isomorphism of $\mathbb{F}$-algebras.*

The following lemma is an easy generalization of Lemma 2.6, the proof of which is completely similar.

Lemma 3.2. *Let $(\mathfrak{g}, [p])$ be a restricted Lie superalgebra over $\mathbb{F}$ and $r \in \mathbb{N}$. Let $x = \sum_{i=1}^{n} k_i x_i \in \mathfrak{g}_{\bar{0}}$ with $k_i \in \mathbb{F}$ and $x_i \in \mathfrak{g}_{\bar{0}}$. Then*

$$(x^p - x^{[p]})^{p^{r-1}} = \sum_{i=1}^{n} k_i^{p^r}(x_i^p - x_i^{[p]})^{p^{r-1}}$$

holds in the universal enveloping superalgebra $U(\mathfrak{g})$.

In the sequel, we always assume that $(\mathfrak{g}, [p])$ is a finite-dimensional restricted Lie superalgebra and $\chi \in \mathfrak{g}_{\bar{0}}^*$ is a p-character of $\mathfrak{g}$ and r is a positive integer. A $\mathfrak{g}$-module M is called a generalized χ-reduced $\mathfrak{g}$-module if $x^{p^r}v - (x^{[p]})^{p^{r-1}}v = \chi(x)^{p^r}v$ for all $x \in \mathfrak{g}_{\bar{0}}, v \in M$. Define $U_{\chi^r}(\mathfrak{g})$ to be the quotient superalgebra $U(\mathfrak{g})/I_{\chi^r}(\mathfrak{g})$, where $I_{\chi^r}(\mathfrak{g})$ is the ideal in $U(\mathfrak{g})$ generated by the set $\{(x^p - x^{[p]} - \chi(x)^p)^{p^{r-1}} \mid x \in \mathfrak{g}_{\bar{0}}\}$. The superalgebra $U_{\chi^r}(\mathfrak{g})$ is called a generalized χ-reduced enveloping superalgebra. In particular, if $\chi = 0$, it is called the generalized restricted enveloping superalgebra, and denoted by $u_r(\mathfrak{g})$ for brevity. A generalized χ-reduced $\mathfrak{g}$-module is a $U_{\chi^r}(\mathfrak{g})$-module for some $r \in \mathbb{N}$, and the vice versa also holds. We have the following PBW Theorem for the superalgebra $U_{\chi^r}(\mathfrak{g})$.

Lemma 3.3. *Let $(\mathfrak{g}, [p])$ be a restricted Lie superalgebra with an $\mathbb{F}$-basis $\{x_1, \cdots, x_m, y_1, \cdots, y_n\}$ in which $x_i \in \mathfrak{g}_{\bar{0}}$ and $y_j \in \mathfrak{g}_{\bar{1}}$ for $1 \leq i \leq m, 1 \leq j \leq n$. Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ and $r \in \mathbb{N}$. Then the generalized χ-reduced enveloping superalgebra $U_{\chi^r}(\mathfrak{g})$ has an $\mathbb{F}$-basis $\{x_1^{s_1} \cdots x_m^{s_m} y_1^{t_1} \cdots y_n^{t_n} \mid 0 \leq s_i \leq p^r - 1, t_j = 0 \, or \, 1\}$, where we abusively denote the coset representative in $U_{\chi^r}(\mathfrak{g})$ of any element $x \in \mathfrak{g} \hookrightarrow U(\mathfrak{g})$ simply by x.*

Now we are in a position to present one of the main results, which asserts that any finite-dimensional indecomposable $\mathfrak{g}$-module is a $U_{\chi^r}(\mathfrak{g})$-module for a unique $\chi \in \mathfrak{g}_{\bar{0}}^*$ and a least positive integer r.

Theorem 3.4. *Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie superalgebra over an algebraically closed field $\mathbb{F}$ of prime characteristic $p > 2$. Then any finite-dimensional $\mathfrak{g}$-module M can be decomposed as a direct sum of indecomposable modules $M = \oplus M_i$, where each M_i is a $U_{\chi_i^{r_i}}(\mathfrak{g})$-module for a unique $\chi_i \in \mathfrak{g}_{\bar{0}}^*$ and a least positive integer r_i.*

Proof. Take a basis $\{x_1, \cdots, x_m, y_1, \cdots, y_n\}$ of $\mathfrak{g}$ with $x_i \in \mathfrak{g}_{\bar{0}}$ and $y_j \in \mathfrak{g}_{\bar{1}}$ for $1 \leq i \leq m, 1 \leq j \leq n$. Set $z_i = x_i^p - x_i^{[p]}$ for $1 \leq i \leq m$. Then all z_i $(1 \leq i \leq m)$ are even central elements in $U(\mathfrak{g})$.

Let M be any finite-dimensional $\mathfrak{g}$-module. Since $z_1 \cdot M \subseteq M$, we can decompose M as a direct sum of generalized eigenspaces for the element z_1 regarded as a transformation on M, i.e.,

$$M = \bigoplus_{i=1}^{s} M_{\lambda_i},$$

where

$$M_{\lambda_i} = \{v \in M \mid (z_1 - \lambda_i)^{t_i} v = 0 \text{ for some } t_i \in \mathbb{N}\}.$$

Note that all $z_i\,(1 \leq i \leq m)$ are even central elements in $U(\mathfrak{g})$, each M_{λ_i} is invariant under the action of z_2, i.e., $z_2 \cdot M_{\lambda_i} \subseteq M_{\lambda_i}$ for $1 \leq i \leq s$. Hence, each such generalized eigenspace M_{λ_i} can be further decomposed into a direct sum of generalized eigenspaces for the element z_2. We can continue in this manner and then M can be decomposed as a direct sum of generalized eigenspaces $M = \oplus M_{\Lambda_i}$ over some collection of m-tuples in $\mathbb{F}^m$, where for $\Lambda_i = (\lambda_{i_1}, \cdots, \lambda_{i_m}) \in \mathbb{F}^m$,

$$M_{\Lambda_i} = \{v \in M \mid (z_j - \lambda_{i_j})^{t_{i_j}} v = 0 \,\text{for}\, 1 \leq j \leq m, t_{i_j} \in \mathbb{N}\}.$$

Moreover, it is easy to check that all M_{Λ_i}'s are $\mathfrak{g}$-modules.

By Lemma 3.2, we can identify each M_{Λ_i} as the following generalized eigenspace

$$M_{\chi_i} = \{v \in M \mid (x^p - x^{[p]} - \chi_i(x)^p)^{p^{r_i - 1}} v = 0 \text{ for all } x \in \mathfrak{g}_{\bar{0}}\}$$

for a unique $\chi_i \in \mathfrak{g}_{\bar{0}}^*$ and the smallest positive integer r_i such that $\chi_i(x_j) = \sqrt[p]{\lambda_{i_j}}$. Then M_{χ_i} is a $U_{\chi_i^{r_i}}(\mathfrak{g})$-module. Therefore, M can be decomposed as a direct sum of $\mathfrak{g}$-modules

$$M = \oplus M_{\chi_i} \tag{3.1}$$

in which the sum runs over a collection of p-characters in $\mathfrak{g}_{\bar{0}}^*$.

In particular, if M is an indecomposable $\mathfrak{g}$-module, then there is only one summand in (3.1), i.e., there exists a unique p-character $\chi \in \mathfrak{g}_{\bar{0}}^*$ and a least positive integer $r \in \mathbb{N}$ such that M is a $U_{\chi^r}(\mathfrak{g})$-module. □

Remark 3.5. *By Theorem 3.4, studying finite-dimensional representation theory of a restricted Lie superalgebra $\mathfrak{g}$ can be reduced to studying the representation theory of the family of finite-dimensional generalized reduced enveloping* *superalgebras* $\{U_{\chi^r}(\mathfrak{g}) \mid \chi \in \mathfrak{g}_{\bar{0}}^*, r \in \mathbb{N}\}$. *As such, in the following, we proceed to study such superalgebras and their representation theory.*

4. Representation theory of the generalized reduced enveloping superalgebras

As before, we always assume that $(\mathfrak{g}, [p])$ is a finite-dimensional restricted Lie superalgebra over $\mathbb{F}$, and $\chi \in \mathfrak{g}_{\bar{0}}^*, r \in \mathbb{N}$. Note that a $U_{\chi^r}(\mathfrak{g})$-module M is simply a $\mathfrak{g}$-module for which $x^{p^r}v - (x^{[p]})^{p^{r-1}}v = \chi(x)^{p^r}v$ holds for any $x \in \mathfrak{g}_{\bar{0}}$ and $v \in M$. Then it is easy to check that the dual module M^* is a $U_{(-\chi)^r}(\mathfrak{g})$-module. Moreover, if M' is a $U_{\chi'^r}(\mathfrak{g})$-module, then $M \otimes M'$ is a $U_{(\chi+\chi')^r}(\mathfrak{g})$-module. For positive integers $s < r$, a $U_{\chi^s}(\mathfrak{g})$-module is necessarily a $U_{\chi^r}(\mathfrak{g})$-module. However, the converse is not true.

Recall that the universal enveloping superalgebra $U(\mathfrak{g})$ admits the structure of a cocommutative Hopf superalgebra. Specifically, the comultiplication Δ, the antipode σ and the counit ϵ are defined for any homogeneous element $x \in \mathfrak{g}$ by

$$\Delta(x) = 1 \otimes x + (-1)^{\bar{x}} x \otimes 1, \sigma(x) = -x, \epsilon(x) = 0$$

and extended multiplicatively to all elements in $U(\mathfrak{g})$. It is easy to check that the ideal $I_{0^r}(\mathfrak{g}) = \big((x^p - x^{[p]})^{p^{r-1}} \mid x \in \mathfrak{g}_{\bar{0}}\big)$ defining the generalized restricted enveloping superalgebra $u_r(\mathfrak{g})$ is a Hopf ideal. However, if $0 \neq \chi \in \mathfrak{g}_{\bar{0}}^*$, the ideal $I_{\chi^r}(\mathfrak{g})$ is not a Hopf ideal. We then obtain the following lemma.

Lemma 4.1. *For each positive integer r, the generalized restricted enveloping superalgebra $u_r(\mathfrak{g})$ is a finite-dimensional cocommutative Hopf superalgebra upon restriction of the usual Hopf superalgebra structure on $U(\mathfrak{g})$.*

Although the general generalized χ-reduced enveloping superalgebra $U_{\chi^r}(\mathfrak{g})$ need not be a Hopf superalgebra, it is a Frobenius superalgebra in the following sense.

Definition 4.2. *A finite-dimensional associative $\mathbb{F}$-superalgebra A is said to be a Frobenius superalgebra if it admits a non-degenerate associative bilinear form. Moreover, it is further called a symmetric superalgebra if the bilinear form is supersymmetric.*

Theorem 4.3. *Let r be a positive integer. Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie superalgebra and $\chi \in \mathfrak{g}_{\bar{0}}^*$. Then the generalized χ-reduced enveloping superalgebra $U_{\chi^r}(\mathfrak{g})$ is a Frobenius superalgebra. Moreover, if $str(\mathrm{ad}\, x) = 0$ for all $x \in \mathfrak{g}$, then $U_{\chi^r}(\mathfrak{g})$ is a symmetric superalgebra, where the supertrace $str(X)$ of an endomorphism X on a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is defined as $str(X) = tr(X|_{V_{\bar{0}}}) - tr(X|_{V_{\bar{1}}})$. The supertrace condition holds if $\mathfrak{g}$ is a basic classical Lie superalgebra or $\mathfrak{g}$ is p-nilpotent (i.e., some iterate of the p-mapping on $\mathfrak{g}$ is zero).*

Proof. Take a basis $\{x_1, \cdots, x_m, y_1, \cdots, y_n\}$ of $\mathfrak{g}$ with $x_i \in \mathfrak{g}_{\bar{0}}, y_j \in \mathfrak{g}_{\bar{1}}$ for $1 \leq i \leq m, 1 \leq j \leq n$. Define the Z_r-linear map from $U(\mathfrak{g})$ to Z_r as follows

$$\begin{aligned} \Phi_r : \quad & U(\mathfrak{g}) \longrightarrow Z_r \\ x_1^{s_1} \cdots x_m^{s_m} y_1^{t_1} \cdots y_n^{t_n} & \longmapsto \delta_{s_1, p^r-1} \cdots \delta_{s_m, p^r-1} \delta_{t_1, 1} \cdots \delta_{t_n, 1}, \end{aligned}$$

where $\delta_{i,j}$ is defined as

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Then Φ_r induces the following $\mathbb{F}$-linear map on the quotient superalgebra

$$\bar{\Phi}_r : \quad U(\mathfrak{g})/I_{\chi^r}(\mathfrak{g}) = U_{\chi^r}(\mathfrak{g}) \longrightarrow Z_r/\mathcal{I}_r \cong \mathbb{F},$$

where $\mathcal{I}_r$ is an ideal of Z_r generated by $(x_i^p - x_i^{[p]} - \chi(x_i)^p)^{p^{r-1}}$ for $1 \leq i \leq m$. Define a bilinear form $B(.,.)$ on $U_{\chi^r}(\mathfrak{g})$ as follows

$$\begin{aligned} B(.,.) : \quad U_{\chi^r}(\mathfrak{g}) \times U_{\chi^r}(\mathfrak{g}) & \longrightarrow \mathbb{F} \\ (u, v) & \longmapsto \bar{\Phi}_r(uv). \end{aligned}$$

Then

$$B(uw, v) = \bar{\Phi}_r((uw)v) = \bar{\Phi}_r(u(wv)) = B(u, wv), \ \forall\, u, w, v \in U_{\chi^r}(\mathfrak{g}).$$

Moreover, the bilinear form $B(.,.)$ is non-degenerate, since for any

$$x_1^{s_1} \cdots x_m^{s_m} y_1^{t_1} \cdots y_n^{t_n}, x_1^{k_1} \cdots x_m^{k_m} y_1^{l_1} \cdots y_n^{l_n} \in U_{\chi^r}(\mathfrak{g}),$$

we have

$$B(x_1^{s_1} \cdots x_m^{s_m} y_1^{t_1} \cdots y_n^{t_n}, x_1^{k_1} \cdots x_m^{k_m} y_1^{l_1} \cdots y_n^{l_n}) = \begin{cases} \pm 1, & \text{if } s_i + k_i = p^r - 1, t_j + l_j = 1 \\ & \text{for } 1 \leq i \leq m, 1 \leq j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Now suppose that $\mathrm{str}(\mathrm{ad}\, x) = 0$ for all $x \in \mathfrak{g}$. We then claim that the bilinear form $B(.,.)$ defined above is symmetric. Indeed, since $B(.,.)$ is non-degenerate, for each homogeneous element $u \in U_{\chi^r}(\mathfrak{g})$, there exists a unique homogeneous $u^* \in U_{\chi^r}(\mathfrak{g})$ satisfying that $\bar{u} = \overline{u*}$ and $B(u, w) = (-1)^{\bar{u}\bar{w}} B(w, u^*)$ for all $w \in U_{\chi^r}(\mathfrak{g})$. Hence, for any homogeneous elements $u_1, u_2, w \in U_{\chi^r}(\mathfrak{g})$, we have

$$\begin{aligned} B(u_1 u_2, w) &= B(u_1, u_2 w) \\ &= (-1)^{\bar{u_1}(\bar{u_2} + \bar{w})} B(u_2 w, u_1^*) \\ &= (-1)^{\bar{u_1}(\bar{u_2} + \bar{w})} B(u_2, w u_1^*) \\ &= (-1)^{\bar{u_1}(\bar{u_2} + \bar{w})} (-1)^{\bar{u_2}(\bar{w} + \bar{u_1})} B(w u_1^*, u_2^*) \\ &= (-1)^{\bar{w}(\bar{u_1} + \bar{u_2})} B(w, u_1^* u_2^*). \end{aligned}$$

On the other hand, $B(u_1 u_2, w) = (-1)^{\bar{w}(\bar{u_1} + \bar{u_2})} B(w, (u_1 u_2)^*)$. Hence, $B(w, u_1^* u_2^*) = B(w, (u_1 u_2)^*)$ for all $w \in U_{\chi^r}(\mathfrak{g})$. Then the non-degeneracy of the bilinear form $B(.,.)$ implies that $(u_1u_2)^* = u_1^*u_2^*$ for all $u_1, u_2 \in U_{\chi^r}(\mathfrak{g})$, i.e.,

$$\varphi: \quad U_{\chi^r}(\mathfrak{g}) \longrightarrow U_{\chi^r}(\mathfrak{g})$$
$$u \longmapsto u^*$$

is an superalgebra endomorphism.

A straightforward computation implies that for any $x \in \mathfrak{g}$, we have

$$B(x, x_1^{s_1} \cdots x_m^{s_m} y_1^{t_1} \cdots y_n^{t_n}) = (-1)^{\bar{x}(\sum_{j=1}^n t_j)} B(x_1^{s_1} \cdots x_m^{s_m} y_1^{t_1} \cdots y_n^{t_n}, x) - \mathrm{str}(\mathrm{ad}\, x).$$

Therefore, if $\mathrm{str}(\mathrm{ad}\, x) = 0$ for all $x \in \mathfrak{g}$, then

$$B(x, x_1^{s_1} \cdots x_m^{s_m} y_1^{t_1} \cdots y_n^{t_n}) = (-1)^{\bar{x}(\sum_{j=1}^n t_j)} B(x_1^{s_1} \cdots x_m^{s_m} y_1^{t_1} \cdots y_n^{t_n}, x)$$

holds for any $x \in \mathfrak{g}$ and $x_1^{s_1} \cdots x_m^{s_m} y_1^{t_1} \cdots y_n^{t_n} \in U_{\chi^r}(\mathfrak{g})$. Thus $x^* = x$ for any $x \in \mathfrak{g}$. Since $\mathfrak{g}$ generates $U_{\chi^r}(\mathfrak{g})$, we conclude that $u^* = u$ for all $u \in U_{\chi^r}(\mathfrak{g})$. This shows that $B(.,.)$ is symmetric, i.e., $U_{\chi^r}(\mathfrak{g})$ is a symmetric superalgebra.

The proof is completed. □

As a direct consequence, we have

Corollary 4.4. *Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie superalgebra. Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ and $r \in \mathbb{N}$. Then*

(1) *A $U_{\chi^r}(\mathfrak{g})$-module is projective if and only if it is injective.*
(2) *If str$(\mathrm{ad}\, x) = 0$ for all $x \in \mathfrak{g}$, then the Cartan matrix of $U_{\chi^r}(\mathfrak{g})$ is symmetric.*

A fundamental question in representation theory is the identification of the simple modules. Any simple $U_\chi(\mathfrak{g})$-module obviously remains simple when considered as a $U_{\chi^r}(\mathfrak{g})$-module for any $r > 1$. Indeed, it is simple if simply considered as a $\mathfrak{g}$-module. The following result implies that these are precisely all the simple $U_{\chi^r}(\mathfrak{g})$-modules.

Theorem 4.5. *Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie superalgebra over an algebraically closed field $\mathbb{F}$ of prime characteristic $p > 2$. Let $\chi \in \mathfrak{g}_{\bar{0}}^*$ and $r \in \mathbb{N}$. Then the collection of simple $U_{\chi^r}(\mathfrak{g})$-modules is precisely the set of simple $U_\chi(\mathfrak{g})$-modules regarded as $U_{\chi^r}(\mathfrak{g})$-modules.*

Proof. Let S be any simple $U_{\chi^r}(\mathfrak{g})$-module. If S is a $U_\chi(\mathfrak{g})$-module, then it is simple as a $U_\chi(\mathfrak{g})$-module, since $U_\chi(\mathfrak{g})$-submodule of S would be a $U_{\chi^r}(\mathfrak{g})$-submodule. Therefore, we need to show that S is in fact a $U_\chi(\mathfrak{g})$-module.

Since S is a simple $U_{\chi^r}(\mathfrak{g})$-module, S is just a simple $\mathfrak{g}$-module satisfying that

$$x^{p^r} v - (x^{[p]})^{p^{r-1}} v = \chi(x)^{p^r} v, \ \forall\, x \in \mathfrak{g}_{\bar{0}}, v \in M. \tag{4.1}$$

By [12, 14], there exists a unique p-character $\chi' \in \mathfrak{g}_{\bar{0}}^*$ such that S is a $U_{\chi'}(\mathfrak{g})$-module, i.e.,

$$x^p v - x^{[p]} v = \chi'(x)^p v, \ \forall\, x \in \mathfrak{g}_{\bar{0}}, v \in M.$$

Hence we have

$$x^{p^r}v-(x^{[p]})^{p^{r-1}}v=\chi'(x)^{p^r}v,\ \forall\, x\in\mathfrak{g}_{\bar 0}, v\in M. \tag{4.2}$$

By (4.1) and (4.2), we then have

$$\chi(x)^{p^r}=\chi'(x)^{p^r},\ \forall\, x\in\mathfrak{g}_{\bar 0}.$$

Since p^rth roots are unique in a field of characteristic p, this implies that $\chi(x)=\chi'(x)$ for any $x\in\mathfrak{g}_{\bar 0}$, i.e., $\chi=\chi'$, as required.

□

The following result implies that the block structure in $U_{\chi^r}(\mathfrak{g})$ is independent of the positive integer r. Indeed, for any $r\in\mathbb{N}$, the block structure in $U_{\chi^r}(\mathfrak{g})$ coincides with the one in $U_\chi(\mathfrak{g})$.

Theorem 4.6. *Let $(\mathfrak{g},[p])$ be a finite-dimensional restricted Lie superalgebra over $\mathbb{F}$. Let $\chi\in\mathfrak{g}_{\bar 0}^*$ and $r\in\mathbb{N}$. Then any pair of simple $U_{\chi^r}(\mathfrak{g})$-modules S and T lie in the same block over $U_{\chi^r}(\mathfrak{g})$ if and only if they lie in the same block over $U_\chi(\mathfrak{g})$.*

Proof. If S and T lie in the same block over $U_\chi(\mathfrak{g})$, then there exists a chain of simple $U_\chi(\mathfrak{g})$-modules: $S=S_1,S_2,\cdots,S_k=T$ such that for any $1\le i\le k-1$,

$$\operatorname{Ext}^1_{U_\chi(\mathfrak{g})}(S_i,S_{i+1})\neq 0 \text{ or } \operatorname{Ext}^1_{U_\chi(\mathfrak{g})}(S_{i+1},S_i)\neq 0.$$

As any $U_\chi(\mathfrak{g})$-module is a $U_{\chi^r}(\mathfrak{g})$-module, then for any $1\le i\le k-1$,

$$\operatorname{Ext}^1_{U_{\chi^r}(\mathfrak{g})}(S_i,S_{i+1})\neq 0 \text{ or } \operatorname{Ext}^1_{U_{\chi^r}(\mathfrak{g})}(S_{i+1},S_i)\neq 0.$$

Hence S and T lie in the same block over $U_{\chi^r}(\mathfrak{g})$.

If S and T lie in the different block over $U_\chi(\mathfrak{g})$, we claim that they lie in different block over $U_{\chi^r}(\mathfrak{g})$. Suppose they lie in the same block over $U_{\chi^r}(\mathfrak{g})$, then there exists s chain of simple $U_{\chi^r}(\mathfrak{g})$-modules (in fact simple $U_\chi(\mathfrak{g})$-modules by Theorem 4.5): $S=S_1,S_2,\cdots,S_k=T$ such that

$$\operatorname{Ext}^1_{U_{\chi^r}(\mathfrak{g})}(S_i,S_{i+1})\neq 0 \text{ or } \operatorname{Ext}^1_{U_{\chi^r}(\mathfrak{g})}(S_{i+1},S_i)\neq 0 \text{ for any } 1\le i\le k-1.$$

Since S and T lie in different block over $U_\chi(\mathfrak{g})$, there exists some $j\le k-1$ such that

$$\operatorname{Ext}^\bullet_{U_\chi(\mathfrak{g})}(S_j,S_{j+1})=\operatorname{Ext}^\bullet_{U_\chi(\mathfrak{g})}(S_{j+1},S_j)=0.$$

Similar to [6, Proposition 5.3], for any $U_\chi(\mathfrak{g})$-modules M and N, we have the following convergent spectral sequence:

$$E_2^{s,t}(M,N)=\operatorname{Ext}^s_{U_\chi(\mathfrak{g})}(M,N)\otimes\bigwedge^t\mathfrak{g}_{\bar 0}^{*(1)}\Rightarrow\operatorname{Ext}^{s+t}_{U(\mathfrak{g})}(M,N).$$

Take $M=S_j$ and $N=S_{j+1}$, or $M=S_{j+1}$ and $N=S_j$, we obtain that

$$\operatorname{Ext}^\bullet_{U(\mathfrak{g})}(S_j,S_{j+1})=\operatorname{Ext}^\bullet_{U(\mathfrak{g})}(S_{j+1},S_j)=0.$$

In particular,

$$\operatorname{Ext}^1_{U(\mathfrak{g})}(S_j,S_{j+1})=\operatorname{Ext}^1_{U(\mathfrak{g})}(S_{j+1},S_j)=0,$$

i.e., neither nontrivial extension of S_j by S_{j+1} over $U(\mathfrak{g})$ nor nontrivial extension of S_{j+1} by S_j over $U(\mathfrak{g})$ exists. While any nontrivial $U_{\chi^r}(\mathfrak{g})$-extension is necessarily a nontrivial extension over $U(\mathfrak{g})$. Hence

$$\mathrm{Ext}\,^1_{U_{\chi^r}(\mathfrak{g})}(S_j, S_{j+1}) = \mathrm{Ext}\,^1_{U_{\chi^r}(\mathfrak{g})}(S_{j+1}, S_j) = 0.$$

It is a contradiction. Therefore, S and T lie in different blocks over $U_{\chi^r}(\mathfrak{g})$.

The proof is completed. □

Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie superalgebra, $\chi \in \mathfrak{g}_{\bar{0}}^*$ and $r \in \mathbb{N}$. In the following, we will realize the generalized χ-reduced enveloping superalgebra $U_{\chi^r}(\mathfrak{g})$ as a restricted enveloping superalgebra of the associated restricted Lie superalgebra $\mathfrak{g}_r$ defined as below.

Definition 4.7. *Let r be a positive integer. Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie superalgebra over $\mathbb{F}$ with an $\mathbb{F}$-basis $\{x_1, \cdots, x_m, y_1, \cdots, y_n\}$ where $x_i \in \mathfrak{g}_{\bar{0}}, y_j \in \mathfrak{g}_{\bar{1}}$ for $1 \le i \le m, 1 \le j \le n$. Consequently, the following set*

$$\{x_1^{s_1} \cdots x_m^{s_m} y_1^{t_1} \cdots y_n^{t_n} \mid 0 \le s_i \le p^r - 1, t_j = 0 \; or \, 1\}$$

is a basis for the generalized restricted enveloping superalgebra $u_r(\mathfrak{g})$. Define $\mathfrak{g}_r$ to be the $\mathbb{F}$-subsuperspace of $u_r(\mathfrak{g})$ with the following $\mathbb{F}$-basis

$$\{x_i, y_j, x_i^p, x_i^{p^2}, \cdots, x_i^{p^{r-1}} \mid 1 \le i \le m, 1 \le j \le n\}$$

such that the even part $(\mathfrak{g}_r)_{\bar{0}} = span_{\mathbb{F}}\{x_i, x_i^p, \cdots, x_i^{p^{r-1}} \mid 1 \le i \le m\}$ and the odd part $(\mathfrak{g}_r)_{\bar{1}} = span_{\mathbb{F}}\{y_j \mid 1 \le j \le n\}$.

Remark 4.8. *In the case that $r = 1$, the subsuperspace $\mathfrak{g}_1$ is just the original restricted Lie superalgebra $\mathfrak{g}$.*

Remark 4.9. *As before, set $z_i = x_i^p - x_i^{[p]}$ for $1 \le i \le m$. It is a routine to check that the subset $\{x_i, y_j, z_i, z_i^p, \cdots, z_i^{p^{r-2}} \mid 1 \le i \le m, 1 \le j \le n\}$ of $u_r(\mathfrak{g})$ is also an $\mathbb{F}$-basis for the vector superspace $\mathfrak{g}_r$.*

Proposition 4.10. *Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie superalgebra over $\mathbb{F}$ and $r \in \mathbb{N}$. Then the $\mathbb{F}$-subsuperspace $\mathfrak{g}_r = (\mathfrak{g}_r)_{\bar{0}} \oplus (\mathfrak{g}_r)_{\bar{1}}$ of $u_r(\mathfrak{g})$ is preserved under the natural bracket operation and the p-mapping on $u_r(\mathfrak{g})$. Hence $\mathfrak{g}_r$ is a restricted Lie superalgebra. Moreover, the Lie superalgebra $\mathfrak{g}$ embeds as a Lie subsuperalgebra in $\mathfrak{g}_r$, but not as a restricted Lie subsuperalgebra.*

Proof. Take any $w \in \mathfrak{g}_r$. By Remark 4.9, w can be expressed as $w = w_1 + w_2$ with

$$w_1 = \sum_{i=1}^{m} a_i x_i + \sum_{j=1}^{n} b_j y_j, \;\; w_2 = \sum_{i=1,\, j=0}^{m,\, r-2} c_{ij} z_i^{p^j},$$

i.e., w_1 is the unique part of w which lies in $\mathfrak{g}$. Similarly, we may write $v = v_1 + v_2 \in \mathfrak{g}_r$. Then

$$[w, v] = [w_1 + w_2, v_1 + v_2] = [w_1, v_1] \in \mathfrak{g} \subseteq \mathfrak{g}_r.$$

Therefore, $\mathfrak{g}_r$ is closed under the bracket operation. Moreover, $[\mathfrak{g}_r, \mathfrak{g}_r] \subseteq \mathfrak{g}$.

Take any $u \in (\mathfrak{g}_r)_{\bar{0}}$. Then by Remark 4.9, u can be written as $u = \sum_{i=1}^m a_i x_i + \sum_{i=1,\, j=0}^{m,\, r-2} c_{ij} z_i^{p^j}$. Since the restricted structure on $u_r(\mathfrak{g})$ is just taken as the pth power. We have

$$u^p = (\sum_{i=1}^m a_i x_i)^p + (\sum_{i=1,\, j=0}^{m,\, r-2} c_{ij} z_i^{p^j})^p = \sum_{i=1}^m a_i^p z_i + (\sum_{i=1}^m a_i x_i)^{[p]} + \sum_{i=1,\, j=0}^{m,\, r-2} c_{ij}^p z_i^{p^{j+1}} \in \mathfrak{g}_r.$$

Thus $\mathfrak{g}_r$ is closed under the natural p-mapping on $u_r(\mathfrak{g})$.

Finally, it is obvious that the embedding $\mathfrak{g} \hookrightarrow U(\mathfrak{g}) \twoheadrightarrow u_r(\mathfrak{g})$ of $\mathfrak{g}$ into $\mathfrak{g}_r$ as an $\mathbb{F}$-subsuperspace preserves the Lie bracket structure, but not the restricted p-mapping structure. □

Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie superalgebra and $r > 1$ be an integer. Let $\mathfrak{g}_r$ be defined as above with a basis $\{x_i, y_j, z_i^p, \cdots, z_i^{p^{r-2}} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. Set $z_{i,0} = x_i$ and $z_{i,j} = z_i^{p^{j-1}}$ for $1 \leq i \leq m, 1 \leq j \leq r-1$. Then the set $\{z_{i,j}, y_k \mid 1 \leq i \leq m, 0 \leq j \leq r-1, 1 \leq k \leq n\}$ is a basis for $\mathfrak{g}_r$. To avoid possible confusion, we denote the p-mapping on $\mathfrak{g}_r$ by $[[p]]$. If $x_i^{[p]} = \sum_{k=1}^m a_k x_k$ holds in $\mathfrak{g}$ for $1 \leq i \leq m$, we denote by $z_{i,0}^{[p]}$ the element $\sum_{k=1}^m a_k z_{k,0} \in \mathfrak{g}_r$. Then $z_{i,0}^{[[p]]} = z_{i,0}^{[p]} + z_{i,1}$ for $1 \leq i \leq m$, while $z_{i,j}^{[[p]]} = z_{i,j+1}$ for $1 \leq i \leq m, 1 \leq j \leq r-2$ and $z_{i,r-1}^{[[p]]} = 0$ for $1 \leq i \leq m$. We define an associated character χ_r on $\mathfrak{g}_r$ as follows.

Definition 4.11. *Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie superalgebra, $\chi \in \mathfrak{g}_{\bar{0}}^*$ and $r > 1$ be an integer. Define $\chi_r \in (\mathfrak{g}_r)_{\bar{0}}^*$ as follows.*

(1) $\chi_r(z_{i,j}) = 0$ *for any* $1 \leq i \leq m$ *and* $0 \leq j \leq r-2$.

(2) $\chi_r(z_{i,r-1}) = \chi(x_i)^{p^{r-1}}$ *for any* $1 \leq i \leq m$.

The natural embedding of Lie superalgebras $\mathfrak{g} \hookrightarrow \mathfrak{g}_r$ sending $x_i \in \mathfrak{g}$ to $z_{i,0} \in \mathfrak{g}_r$ induces a homomorphism of superalgebras $\phi_r : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}_r)$. Let ψ_r denote the composite homomorphism $\psi_r : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}_r) \twoheadrightarrow U_{\chi_r}(\mathfrak{g}_r)$.

Lemma 4.12. *Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie superalgebra, $\chi \in \mathfrak{g}_{\bar{0}}^*$ and $r > 1$ be an integer. Let ψ_r be defined as above. Then we have $\psi_r(z_i^{p^j}) = z_{i,j+1}$ for $1 \leq i \leq m, 0 \leq j \leq r-2$, where $z_{i,j+1}$ abusively denotes the image of the element $z_{i,j+1} \in \mathfrak{g}_r$ under the composite $\mathfrak{g}_r \hookrightarrow U(\mathfrak{g}_r) \twoheadrightarrow U_{\chi_r}(\mathfrak{g}_r)$.*

Proof. Since $\psi_r : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}_r) \twoheadrightarrow U_{\chi_r}(\mathfrak{g}_r)$ is a superalgebra homomorphism, we have the following computation for $1 \leq i \leq m$,

$$\begin{aligned} \psi_r(z_i) &= \psi_r(x_i^p - x_i^{[p]}) \\ &= \psi_r(x_i)^p - \psi_r(x_i^{[p]}) \\ &= z_{i,0}^p - z_{i,0}^{[p]}. \end{aligned}$$

By the definition of χ_r, in the superspace $U_{\chi_r}(\mathfrak{g}_r)$, $z_{i,0}^p = z_{i,0}^{[[p]]}$ holds. On the other hand, $z_{i,0}^{[[p]]} = z_{i,0}^{[p]} + z_{i,1}$ holds in $\mathfrak{g}_r$. Hence

$$\psi_r(z_i) = z_{i,0}^p - z_{i,0}^{[p]} = z_{i,0}^{[[p]]} - z_{i,0}^{[p]} = z_{i,1},$$

i.e., the statement holds for $1 \le i \le m, j = 0$.

Next assume that $1 \le j \le r-2$. Then

$$\psi_r(z_i^{p^j}) = \psi_r(z_i)^{p^j} = z_{i,1}^{p^j}.$$

By the definition of χ_r, the identity $z_{i,l}^p - z_{i,l}^{[[p]]} = 0$ holds in $U_{\chi_r}(\mathfrak{g}_r)$ for $1 \le i \le m, 1 \le l \le r-2$. Note that $z_{i,l}^{[[p]]} = z_{i,l+1}$ for $1 \le l \le r-2$, then $z_{i,l}^p = z_{i,l+1}$ for $1 \le i \le m, 1 \le l \le r-2$. So $z_{i,1}^{p^j} = z_{i,j+1}$ holds in $U_{\chi_r}(\mathfrak{g}_r)$ for $1 \le i \le m, 1 \le j \le r-2$. Therefore,

$$\psi_r(z_i^{p^j}) = z_{i,1}^{p^j} = z_{i,j+1},\ 1 \le i \le m,\ 0 \le j \le r-2.$$

The proof is completed. □

Theorem 4.13. *Let $(\mathfrak{g}, [p])$ be a finite-dimensional restricted Lie superalgebra, $\chi \in \mathfrak{g}_{\bar{0}}^*$ and $r > 1$ be an integer. Let $\mathfrak{g}_r$ be the associated restricted Lie superalgebra and $\chi_r \in (\mathfrak{g}_r)_{\bar{0}}^*$ defined as before. Then there is an isomorphism of superalgebras $\bar{\psi}_r : U_{\chi^r}(\mathfrak{g}) \xrightarrow{\sim} U_{\chi_r}(\mathfrak{g}_r)$.*

Proof. Recall the homomorphism $\psi_r : U(\mathfrak{g}) \longrightarrow U_{\chi_r}(\mathfrak{g}_r)$ defined above. We claim that it induces a homomorphism $\bar{\psi}_r : U_{\chi^r}(\mathfrak{g}) \longrightarrow U_{\chi_r}(\mathfrak{g}_r)$. For that we need to show that $\psi_r(I_{\chi^r}(\mathfrak{g})) = 0$. Since ψ_r is a superalgebra homomorphism, it suffices to show that ψ_r is zero on the set of generators $\{(x_i^p - x_i^{[p]} - \chi(x_i)^p)^{p^{r-1}} \mid 1 \le i \le m\}$ for the ideal $I_{\chi^r}(\mathfrak{g})$. In fact, for any $1 \le i \le m$, we have

$$\begin{aligned}
\psi_r\big((x_i^p - x_i^{[p]} - \chi(x_i)^p)^{p^{r-1}}\big) &= \psi_r\big(z_i^{p^{r-1}} - \chi(x_i)^{p^r}\big) \\
&= \psi_r\big((z_i^{p^{r-2}})^p\big) - \chi(x_i)^{p^r} \\
&= \big(\psi_r(z_i^{p^{r-2}})\big)^p - \chi(x_i)^{p^r} \\
&= z_{i,r-1}^p - \chi(x_i)^{p^r} \\
&= z_{i,r-1}^p - z_{i,r-1}^{[[p]]} - \chi(z_{i,r-1})^p \\
&= 0.
\end{aligned}$$

Hence $\psi_r : U(\mathfrak{g}) \longrightarrow U_{\chi_r}(\mathfrak{g}_r)$ induces the corresponding superalgebra homomorphism $\bar{\psi}_r : U_{\chi^r}(\mathfrak{g}) \longrightarrow U_{\chi_r}(\mathfrak{g}_r)$. Note that $\bar{\psi}_r(x_i) = z_{i,0}, \bar{\psi}_r(z_i^{p^{j-1}}) = z_{i,j}$ and $\bar{\psi}_r(y_k) = y_k$ for $1 \le i \le m, 1 \le j \le r-1, 1 \le k \le n$. Therefore, $\bar{\psi}_r$ is surjective. Since

$$\dim U_{\chi^r}(\mathfrak{g}) = \dim U_{\chi_r}(\mathfrak{g}_r) = p^{rm}2^n,$$

the homomorphism $\bar{\psi}_r$ is indeed an isomorphism.

□

Remark 4.14. *By Theorem 4.13, the study of representation theory of the generalized χ-reduced enveloping superalgebra $U_{\chi^r}(\mathfrak{g})$ can be reduced to the study of the representation theory of the χ_r-reduced enveloping superalgebra $U_{\chi_r}(\mathfrak{g}_r)$ for the associated restricted Lie superalgebra $\mathfrak{g}_r$.*

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