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HYPERSURFACES OF A SASAKIAN SPACE FORM WITH RECURRENT SHAPE OPERATOR

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Abstract. Let (M^{2n}, g) be a real hypersurface with recurrent shape operator and tangent to the structure vector field ξ of the Sasakian space form $\widetilde{M}(c)$. We show that if the shape operator A of M is recurrent then it is parallel. Moreover, we show that M is locally a product of two constant $\phi-$sectional curvature spaces.
Keywords: Recurrent hypersurfaces, Sasakian manifold.
MSC(2010): Primary: 53C25; Secondary: 53C40.

1. Introduction

The notion of recurrent tensor field of type (r, s) on a differentiable manifold M with a linear connection was introduced in [9] and [14]. A non-zero tensor field K of type (r, s) on M is said to be recurrent if there exists a 1-form ω such that

$$\nabla K = \omega \otimes K.$$

We denote by A the shape operator of a real hypersurface in the non flat complex space form $M^n(c)$ with constant holomorphic sectional curvature. Recently in [5] and [6] Hamada applied such a notion of recurrent tensor to a shape operator or a Ricci tensor for a real hypersurface M in the complex projective space $\mathbb{C}P^n$, and proved the following :

Theorem 1.1. *The complex projective space $\mathbb{C}P^n$ does not admit any real hypersurface with recurrent shape operator or recurrent Ricci tensor.*

In [12] and [13], Suh studied the real hypersurfaces in complex two planes Grassmannians with recurrent shape operator and explained the geometrical

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meaning of the recurrent shape operator A by the relation

$$[\nabla_X A, A] = \omega(X)[A, A] = 0$$

for any tangent vector field X on M, which means that the eignespaces of the shape operator A of M are parallel along any curve in M. These eigenspaces are said to be parallel along a curve γ, if they are invariants under any parallel translation along γ.

Finally, Kim and et. al. in [8], completed the study of real hypersurfaces in the complex two plane Grassmannians with recurrent shape operator.

Milijevic studied CR submanifolds of maximal CR dimension of a complex space form with recurrent shape operator. Also . Ryan considered hypersurfaces of real space forms in [11]. He gave a complete classification of hypersurfaces in the sphere which satisfy a certain condition and proved the following:

Theorem 1.2. *Let M be a hypersurface of S^{n+1} whose shape operator has exactly two distinct eigenvalues, then M is locally a product of two spheres.*

This paper considers hypersurfaces of a Sasakian space form $\widetilde{M}^{2n+1}(c)$ with constant ϕ-sectional curvature c with recurrent shape operator. Note that in the case of $c = 1$, the Sasakian space form is the sphere itself. It is shown that, if M be a hypersurfaces of a Sasakian space form $\widetilde{M}^{2n+1}(c)$, where the structure vector field of ξ is tangent to M and the shape oparator of M is recurrent, then M is locally a product of M_1 and M_2 or a product of M' and γ , where M_1, M_2 and M' are constant ϕ-sectional curvature totally geodesic submanifolds and γ is a geodesic curve in M.

2. **Preliminaries**

A differentiable manifold $\widetilde{M}^{2n+1}$ is said to have an almost contact structure if it admits a (non-vanishing) vector field ξ , a one-form η and a $(1,1)-$tensor field ϕ satisfying

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi,$$

where I denotes the field of identity transformations of the tangent spaces at all points. These conditions imply that $\phi\xi = 0$ and $\eta \circ \phi = 0$, and that the endomorphism ϕ has rank $2n$ at every point in $\widetilde{M}^{2n+1}$. A manifold $\widetilde{M}^{2n+1}$, equipped with an almost contact structure (ϕ, ξ, η), is called an almost contact manifold and shall be denoted by $(\widetilde{M}^{2n+1}, (\phi, \xi, \eta))$.

Suppose that $\widetilde{M}^{2n+1}$ is a manifold carrying an almost contact structure. A Riemannian metric $\widetilde{g}$ on $\widetilde{M}^{2n+1}$ satisfying

$$\widetilde{g}(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields X and Y, is called compatible with the almost contact structure, and $(\widetilde{M}^{2n+1}, (\phi, \xi, \eta, g))$ is said to be an almost contact metric structure on $\widetilde{M}^{2n+1}$. It is known that an almost contact manifold always admits at least one compatible metric. Note that

$$\eta(X) = \widetilde{g}(X, \xi),$$

for all vector fields X tangent to $\widetilde{M}^{2n+1}$, which means that η is the metric dual of the characteristic vector field ξ.

A manifold $\widetilde{M}^{2n+1}$ is said to be a contact manifold if it carries a global one-form η such that

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere on M. The one-form η is called a contact form.

A submanifold M of a Riemannian contact manifold $\widetilde{M}^{2n+1}$ tangent to ξ is called an invariant (resp. anti-invariant) submanifold if $\phi(T_pM) \subset T_pM$, for each $p \in M$ (resp. $\phi(T_pM) \subset T_p^{\perp}M$, for each $p \in M$).

A submanifold M tangent to ξ of a contact manifold $\widetilde{M}^{2n+1}$ is called a contact CR-submanifold if there exists a pair of orthogonal differentiable distributions D and $D^{\perp}$ on M such that

(1) $TM = D \oplus D^{\perp} \oplus \mathbb{R}\xi$, where $\mathbb{R}\xi$ is the $1-$dimensional distribution spanned by ξ;
(2) D is invariant by ϕ, i.e., $\phi(D_p) \subset D_p$ for each $p \in M$;
(3) $D^{\perp}$ is anti-invariant by ϕ, i.e., $\phi(D_p^{\perp}) \subset T_p^{\perp}M$ for each $p \in M$.

Let $(\widetilde{M}, \phi, \xi, \eta, \widetilde{g})$ be a $(2n+1)$-dimensional contact manifold such that

$$\widetilde{\nabla}_X \xi = -\phi X, \qquad (\widetilde{\nabla}_X \phi)Y = \widetilde{g}(X, Y)\xi - \eta(Y)X,$$

where $\widetilde{\nabla}$ is the Levi-Chivita connection of $\widetilde{M}$, then $\widetilde{M}$ is called a Sasakian manifold. The plane section π of $T\widetilde{M}$ is called a $\phi-$section if $\phi\pi_x \subseteq \pi_x$, for each $x \in \widetilde{M}$. Also $\widetilde{M}$ is called of constant $\phi-$sectional curvature if the sectional curvature of $\phi-$sections are constant. A Sasakian space form is a Sasakian manifold of constant $\phi-$ sectional curvature. In this case the Riemannian curvature tensor field $\widetilde{R}$ is given by

$$\begin{aligned}\widetilde{R}(X,Y)Z = &\frac{c+3}{4}\{\widetilde{g}(Y,Z)X - \widetilde{g}(X,Z)Y\} \\ &-\frac{c-1}{4}\{\eta(Z)[\eta(Y)X - \eta(X)Y] + [\widetilde{g}(Y,Z)\eta(X) - \widetilde{g}(X,Z)\eta(Y)]\xi \\ &-\widetilde{g}(\phi Y, Z)\phi X + \widetilde{g}(\phi X, Z)\phi Y + 2\widetilde{g}(\phi X, Y)\phi Z\}\end{aligned}$$

for each $X, Y, Z \in \chi(\widetilde{M})$.

Definition 2.1. *Let T be a $(1,1)$ tensor field on a Riemannian manifold M. Then T is called a recurrent tensor field if $(\nabla_X T)Y = \omega(X)TY$, one form ω and vector fields X, Y on M.*

3. Hypersurfaces of a Sasakian space form with recurrent shape operator

Let (M^{2n}, g) be a real hypersurface of Sasakian space form $\widetilde{M}^{2n+1}(c)$ with recurrent shape operator and tangent to ξ. Assume moreover that N is a local unit normal vector field on M. Clearly $D^{\perp}$ is unidimensional distribution spanned by $U = -\phi(N)$.

Lemma 3.1. *Let M^{2n} be a hypersurface of a Sasakian space form $\widetilde{M}^{2n+1}(c)$ tangent to the vector field of ξ and A be the shape operator of M. Then $A\xi = -U$.*

Proof. Let $\widetilde{\nabla}$ and ∇ be the Levi-Chivita connections of $\widetilde{M}$ and M, respectively. Then, by the Gauss formula and Sasakian conditions,

$$\nabla_U \xi + g(AU, \xi)N = \widetilde{\nabla}_U \xi = -\phi U = -N.$$

Considering the tangential and normal parts of the above relation, we have

$$\nabla_U \xi = 0, \quad g(AU, \xi) = -1. \tag{3.1}$$

Also, since

$$\nabla_\xi \xi + g(A\xi, \xi)N = -\phi\xi = 0,$$

we have

$$\nabla_\xi \xi = 0, \quad g(A\xi, \xi) = 0. \tag{3.2}$$

On the other hand,

$$\nabla_X \xi + g(AX, \xi)N = \widetilde{\nabla}_X \xi = -\phi X$$

for each X in TM. Now if $X \in D$, then by the above relation we have

$$g(A\xi, X) = g(AX, \xi) = 0, \quad \nabla_X \xi = -\phi X,$$

which implies that $A\xi = -U$. □

Now let $AU = \alpha U + \beta\xi + (AU)_D$ where $(AU)_D$ is the projected part of AU on D. Since $g(AU, \xi) = -1$ we have

$$AU = -\xi + \alpha U + (AU)_D. \tag{3.3}$$

Lemma 3.2. *Let (M, g) be a real hypersurface tangent to ξ of the Sasakian space form $\widetilde{M}^{2n+1}(c)$. If the shape operator A of M is the recurrent operator then it is parallel.*

Proof. By Codazzi equation, Sasakian conditions together with recurrent assumption for the shape operator imply

$$\omega(X)AY - w(Y)AX = \frac{c-1}{4}\{g(X, U)FY - g(Y, U)FX - 2g(FX, Y)U\}, \tag{3.4}$$

for each X, Y in TM where FX is the tangent part of ϕX on TM. Choosing $X = \xi$ and $Y = U$ in (3.4) we have

$$\omega(\xi)AU - w(U)A\xi = \frac{c-1}{4}\{g(\xi, U)FU - g(U, U)F\xi - 2g(F\xi, U)U\} = 0.$$

Now from Lemma (3.1) and equation (3.3) we have

$$-\omega(\xi)\xi + \omega(\xi)(AU)_D + (w(U) + \alpha\omega(\xi))U = 0.$$

Since ξ, U and $(AU)_D$ are linearly independent,

$$\omega(\xi) = 0, \omega(U) = 0.$$

Setting $X \in D$ and $Y = \xi$ in (3.4), one obtains

$$\omega(X)A\xi - w(\xi)AX = \frac{c-1}{4}\{g(X, U)F\xi - g(\xi, U)FX - 2g(FX, \xi)U\} = 0.$$

Therefore $\omega(X) = 0$, for all X in D and $\nabla A = 0$. □

If the shape operator A of the hypersurface M is parallel, then

$$\begin{aligned} R(X, Y)(AZ) &= \nabla_X\nabla_Y(AZ) - \nabla_Y\nabla_X(AZ) - \nabla_{[X,Y]}(AZ) \\ &= AR(X, Y)Z \end{aligned} \tag{3.5}$$

for all X, Y, Z in TM.

Lemma 3.3. *Let (M, g) be a real hypersurface tangent to ξ of the Sasakian space form $\widetilde{M}^{2n+1}(c)$. If the shape operator A of M is parallel then the subspace of D is invariant under A.*

Proof. By the Gauss formula, we have

$$R(\xi, U)U = -\xi - AU - g(AU, U)U \tag{3.6}$$

and

$$R(U, \xi)\xi = U + g(A\xi, \xi)AU - g(AU, \xi)A\xi = U - U = 0. \tag{3.7}$$

Now, choosing $X = U$ and $Y = Z = \xi$ in (3.5), from Lemma (3.1), we conclude that

$$AR(U, \xi)\xi = R(U, \xi)A\xi = -R(U, \xi)U. \tag{3.8}$$

Therefore by (3.6),(3.7) and (3.8)

$$AU = -\xi - g(AU, U)U.$$

Hense $AU \in span\{\xi, U\}$. This shows that $AD = D$. □

Since A_p is self adjoint and D and $span\{\xi, U\}$ are invariant subspaces under A_p, for any $p \in M$, there exists a locally orthonormal frame

$$X_1, \ldots, X_{2n-2}$$

for D. Also there a frame $\{W_1, W_2\}$ for $span\{\xi, U\}$, where

$$AX_i = \mu_i X_i, \qquad i = 1, \dots, 2n-2,$$
$$AW_1 = \gamma_1 W_1 \ , \ AW_2 = \gamma_2 W_2.$$

We set

$$\begin{aligned} W_1 &= \xi \cos\theta + U \sin\theta, \\ W_2 &= -\xi \sin\theta + U \cos\theta. \end{aligned}$$

for some $0 < \theta < \pi/2$. Note that ξ and U can not be eigenvectors of A and hence $\cos\theta$ and $\sin\theta$ are not vanishing at any $p \in M$.

Lemma 3.4. *Under the above conditions,* $\gamma_1 = -\tan\theta$ *and* $\gamma_2 = \cot\theta$.

Proof. Using the structures of W_1, W_2 one has

$$U = W_1 \sin\theta + W_2 \cos\theta.$$

On the other hand, by Lemma 3.1 we get

$$\begin{aligned} AW_1 &= A\xi \cos\theta + AU \sin\theta \ = -U \cos\theta + AU \sin\theta, \\ AW_2 &= -A\xi \sin\theta + AU \cos\theta = U \sin\theta + AU \cos\theta. \end{aligned}$$

Therefore,

$$U = AW_2 \sin\theta - AW_1 \cos\theta = \gamma_2 W_2 \sin\theta - \gamma_1 W_1 \cos\theta.$$

Comparing the above values of U we have

$$(\gamma_2 \sin\theta - \cos\theta) W_2 - (\gamma_1 \cos\theta + \sin\theta) W_1 = 0.$$

Since W_1 and W_2 are linearly independent,

$$\gamma_1 = -\tan\theta, \quad \gamma_2 = \cot\theta.$$

□

Using the Gauss equation, the structure of curvature tensor of the a Sasakian space form and (3.5) , we get

$$\begin{aligned} &\frac{c+3}{4}\{g(Y,AZ)X - g(X,AZ)Y\} \\ &\quad -\frac{c-1}{4}\{\eta(AZ)[\eta(Y)X - \eta(X)Y] + [g(Y,AZ)\eta(X) - g(X,AZ)\eta(Y)]\xi \\ &\qquad -g(FY,AZ)FX + g(FX,AZ)FY + 2g(FX,Y)FAZ\} \\ &\quad +g(AY,AZ)AX - g(AX,AZ)AY \\ &= \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad -\frac{c-1}{4}\{\eta(Z)[\eta(Y)AX - \eta(X)AY] + [g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]A\xi \\ &\qquad -g(FY,Z)AFX + g(FX,Z)AFY + 2g(FX,Y)AFZ\} \\ &\quad +g(AY,AZ)A^2X - g(AX,AZ)A^2Y, \end{aligned} \tag{3.9}$$

for each X, Y, Z in TM .

Now from (3.9) setting $X = X_i$ and $Y = Z = X_j$, where X_j is normal to $span\{X_i, \phi X_i\}$, we have

$$(\lambda_i - \lambda_j)(\frac{c+3}{4} + \lambda_i\lambda_j) = 0. \tag{3.10}$$

Furthermore, setting $X = W_1$ and $Y = Z = X_i$ in (3.9), we get

$$(\frac{c+3}{4}\lambda_i + \lambda_i^2\gamma_1)W_1 - \frac{c-1}{4}\lambda_i \cos\theta\xi = (\frac{c+3}{4}\gamma_1 + \lambda_i\gamma_1^2)W_1 + \frac{c-1}{4}\cos\theta U.$$

Therefore

$$\begin{aligned}(\frac{c+3}{4}\lambda_i &+ \lambda_i^2\gamma_1)(\xi\cos\theta + U\sin\theta) - \frac{c-1}{4}\lambda_i\cos\theta\xi \\ &= (\frac{c+3}{4}\gamma_1 + \lambda_i\gamma_1^2)(\xi\cos\theta + U\sin\theta) + \frac{c-1}{4}\cos\theta U.\end{aligned}$$

Since ξ and U are linearly independent,

$$\begin{aligned}\lambda_i\cos\theta + \lambda_i^2\gamma_1\cos\theta &= \frac{c+3}{4}\gamma_1\cos\theta + \lambda_i\gamma_1^2\cos\theta, \\ \frac{c+3}{4}\lambda_i\sin\theta + \lambda_i^2\gamma_1\sin\theta &= \frac{c+3}{4}\gamma_1\sin\theta + \lambda_i\gamma_1^2\sin\theta + \frac{c-1}{4}\cos\theta\end{aligned}$$

and by the structure of W_1, W_2 and (3.4), we have

$$\begin{aligned}(\frac{c+3}{4} + \lambda_i\gamma_1)(\lambda_i - \gamma_1) &= \frac{c-1}{4}\lambda_i, \\ (\frac{c+3}{4} + \lambda_i\gamma_1)(\lambda_i - \gamma_1) &= \frac{c-1}{4}\gamma_2.\end{aligned} \tag{3.11}$$

Therefore, for $\lambda_i \neq \gamma_1$,

$$\frac{c-1}{4}\lambda_i = \frac{c-1}{4}\gamma_2. \tag{3.12}$$

Lemma 3.5. *For any $x \in M$, rank $A_x = 2n$.*

Proof. Assume that rank $A_x \neq 2n$. First, suppose that $c \neq 1$ for some $i \in \{1, \ldots, 2n-2\}$ or $j \in \{1, 2\}$, then since $\gamma_j \neq 0$ in (3.12) we have $\lambda_i \neq 0$ for all i. In this case, rank $A_x = 2n$. Now suppose that $c = 1$ for some $i \in \{1, \ldots, 2n-2\}$ or $j \in \{1, 2\}$, then since $\gamma_j \neq 0$ in (3.11) we have $\lambda_i = \gamma_1$ or $\lambda_i = \gamma_2$ ehich imply $\lambda_i \neq 0$ for all i. Again, we have rank $A_x = 2n$. □

Lemma 3.6. *A_x has exactly two distinct eigenvalues.*

Proof. If $c \neq 1$ the relations (3.11) and (3.12) imply that, for all $i = 1 \ldots 2n-2$, $\lambda_i = \delta_1$ or $\lambda_i = \delta_2$ hence A_x has at most two distinct eigenvalues. Otherwise, if $c = 1$, then the relation (3.10), for $i = 1$, reads as $(\lambda_i - \lambda_1)(1 + \lambda_i\lambda_1) = 0$. If $\lambda_i \neq \lambda_1$ then $\lambda_i = -\frac{1}{\lambda_1}$, but from (3.11) we have $\lambda_i = \gamma_1$ or $\lambda_i = \gamma_2$ which shows that in this case A_x has at most two distinct eigenvalues. Now if A_x has one eigenvalue then $\gamma_1 = \gamma_2$. On the other hand, $\gamma_1 = -\frac{1}{\gamma_2}$ implies $\gamma_2^2 = -1$ which is impossible. Hence A_x has exactly two distinct eigenvalues. $\square$

Corollary 3.7. *If $c \neq 1$ then multiplicities of the eigenvalues for the shape operator A at $x \in M$ are $2n-1$ and 1.*

Proof. If $c \neq 1$ the relations (3.11) and (3.12) imply $\lambda_i = \delta_1$ or $\lambda_i = \delta_2$ for all $i = 1, \ldots, 2n-2$. $\square$

Let us denote the two eigenvalues of A_x by λ and μ.

Lemma 3.8. *The multiplicities of the eigenvalues are constant for the shape operator A.*

Proof. Let λ be an eigenvalue of A of multiplicity p at $x \in M$ and multiplicity q at $y \in M$. Then μ has the multiplicity $n-p$ at x and $n-q$ at y. Therefore,

$$\begin{aligned}(\text{trace } A)(x) - (\text{trace } A)(y) &= p\lambda(x) - q\lambda(y) + (n-p)\mu(x) + (n-p)\mu(y)\\ &= (p-q)(\lambda(x) - \mu(x)) + q(\lambda(x) - \lambda(y)) - (n-q)(\mu(x) - \mu(y)).\end{aligned}$$

Since the trace map is continuous, this implies $p = q$.

For eigenvalues λ and μ of A we put

$$T_\lambda(x) = \{X_x \in T_x(M) | A_x X_x = \lambda X_x\}, \tag{3.13}$$

$$T_\mu(x) = \{X_x \in T_x(M) | A_x X_x = \mu X_x\}. \tag{3.14}$$

Then, using Lemma 3.8, we get two distributions T_λ and T_μ. $\square$

Lemma 3.9. *The distributions T_λ and T_μ are both involutive.*

Proof. Let us choose $X, Y \in T_\lambda$. Then, using Codazzi equation, it follows that

$$\begin{aligned}A[X,Y] &= A\nabla_X Y - A\nabla_Y X\\ &= \nabla_X(AY) - (\nabla_X A)Y - \nabla_Y(AX) + (\nabla_Y A)X\\ &= (X\lambda)Y - (Y\lambda)X + \lambda[X,Y].\end{aligned}$$

Hence,

$$(A - \lambda I)[X,Y] = (X\lambda)Y - (Y\lambda)X. \tag{3.15}$$

However, the left-hand sides of (3.15) belong to T_μ. In fact, $[X,Y] = [X,Y]_\lambda + [X,Y]_\mu$ implies that

$$\begin{aligned}(A - \lambda I)[X,Y] &= (A - \lambda I)([X,Y]_\lambda + [X,Y]_\mu)\\ &= A[X,Y]_\lambda + A[X,Y]_\mu - \lambda[X,Y]_\lambda - \lambda[X,Y]_\mu\\ &= (\mu - \lambda)[X,Y]_\mu \in T_\mu.\end{aligned}$$

On the other hand, the right-hand sides of (3.15) belong to T_λ and therefore,

$$A[X,Y] = \lambda[X,Y] \quad , \quad (X\lambda)Y - (Y\lambda)X = 0. \tag{3.16}$$

This shows that the distribution T_λ is involutive. Similarly, one can see that the distribution T_μ is also involutive. $\square$

Theorem 3.10. *Let M^{2n} be a hypersurface tangent to ξ of a Sasakian space form $\widetilde{M}^{2n+1}(c)$ with recurrent shape operator. If $c \neq 1$ then M is locally a product $M' \times \gamma$, where M' is a constant ϕ-sectional curvature totally geodesic submanifold and γ is a geodesic of M. If $c = 1$ then M is locally a product $M_1 \times M_2$ or $M' \times \gamma$, where M_1, M_2 and M' are constant ϕ-sectional curvature totally geodesic submanifolds and γ is a geodesic of M.*

Proof. Let T_λ and T_μ be as in the proof of Lemma 3.8. If $X \in T_\lambda, Y \in T_\mu$, the Codazzi equation yields

$$\nabla_X(\mu Y) - \nabla_Y(\lambda X) = A\nabla_X Y - A\nabla_Y X.$$

Since λ and μ are constant, we get $(A - \lambda I)\nabla_Y X = (A - \mu I)\nabla_X Y$. The left-hand side of the equation is in T_μ while the right-hand side is in T_λ. Hence both sides are zero. That is, $\nabla_Y X \in T_\lambda, \nabla_X Y \in T_\mu$, and for $Z \in T_\lambda$,

$$g(\nabla_Z X, Y) + g(X, \nabla_Z Y) = \nabla_Z(g(X, Y)) = 0.$$

On the other hand,$\nabla_Z Y \in T_\mu$ implies $g(X, \nabla_Z Y) = 0$. Thus, $\nabla_Z X \in T_\mu^\perp$ for all Z and $X \in T_\lambda$. Since $T_\mu^\perp = T_\lambda$, we may write $\nabla_{T_\lambda} T_\lambda \subseteq T_\lambda$ and $\nabla_{T_\lambda} T_\mu \subseteq T_\mu$. This means that T_λ is a totally geodesic parallel distribution. The same conclusion can be drawn for T_μ, namely, T_μ is also a totally geodesic parallel distribution. Hence, by de Rham decomposition theorem [10], M is locally isometric to the Riemannian product of the maximal integral manifolds M_λ and M_μ.

Now we consider the integral submanifold M_λ. Let ι_λ be the immersion of M_λ into M and $j = \iota \circ \iota_\lambda$, that is, j is the immersion of M_λ into $\widetilde{M}$ via M. Denoting by h_λ and h_λ^M the second fundamental forms of M_λ in $\widetilde{M}$ and M, respectively, we get for any $X', Y' \in T_\lambda$ the covariant derivative ∇^λ of M_λ as follows:

$$\begin{aligned}
\overline{\nabla}_{X'} jY' &= j\nabla^\lambda_{X'} Y' + h_\lambda(X', Y') \\
&= \overline{\nabla}_{X'} \iota \circ \iota_\lambda Y' \\
&= \iota\nabla_{X'} \iota_\lambda Y' + h(\iota_\lambda X', \iota_\lambda Y') \\
&= \iota\{\iota_\lambda \nabla^\lambda_{X'} Y' + h^M_\lambda(X', Y')\} + h(\iota_\lambda X', \iota_\lambda Y') \\
&= j\nabla^\lambda_{X'} Y' + \iota h^M_\lambda(X', Y') + h(\iota_\lambda X', \iota_\lambda Y').
\end{aligned}$$

Since M_λ is totally geodesic in M, $h_\lambda^M = 0$. One cased easily show that $h_\lambda(X', Y') = h(X, Y) = g(AX, Y)N = \lambda g(X, Y)N$. By the Gauss equation, the curvature tensor R^λ of M_λ satisfies

$$\begin{aligned} g^\lambda(R^\lambda(X',Y')Z',W') &= g(jY',jZ')g(jY',jW') - g(jX',jZ')g(jY',jW') \\ &\quad +h^\lambda(Y',Z')h^\lambda(X',W') - h^\lambda(X',Z')h^\lambda(Y',W') \\ &= (\frac{c+3}{4})[g(Y',Z')g(X',W') - g(X',Z')g(Y',W')] \\ &\quad +(\frac{c-1}{4})[g(X',\phi Z')g(\phi Y',W') - g(Y',\phi Z')g(\phi X',W') \\ &\qquad +2g(X',\phi Y')g(\phi Z',W')] \\ &\quad +\lambda^2 g(Y',Z')g(X',W') - \lambda^2 g(X',Z')g(Y',W') \end{aligned}$$

Thus

$$H(X') = R^\lambda(X',\phi X') = g^\lambda(R^\lambda(X',\phi X')\phi X',X') = c + \lambda^2.$$

This shows that the integral manifold M_λ is a Riemannian manifold of $\phi-$invariant constant curvature $c+\lambda^2$. In the same way we obtain that M_μ is a Riemannian manifold of $\phi-$invariant constant curvature $c+\mu^2$. Thus, M is locally a product of two constant $\phi-$sectional curvature spaces.

Now, if $c \neq 1$ from Corollary 3.7 one of the multiplicities of T_λ or T_μ is $2n-1$ and the other multiplicity is 1. Hence an integral manifold M' and a curve γ exist so that M is locally a product of $M' \times \gamma$. Moreover, M' is constant ϕ-sectional curvature totally geodesic submanifold and γ is a geodesic of M. If $c = 1$, the multiplicities of T_λ and T_μ are both greater than one or one of them is $2n-1$ and the other is 1. If the multiplicities are $2n-1$ and 1, then, similar to the previous case, M is locally a product $M' \times \gamma$, where M' is a constant ϕ-sectional curvature totally geodesic submanifold and γ is a geodesic of M. If both multiplicities are greater than one then the integral manifolds $M_1 = M_\lambda$ and $M_2 = M_\mu$ exist so that M is locally a product of $M_1 \times M_2$, where M_1 and M_2 are two constant ϕ-sectional curvature totally geodesic submanifolds of M. This completes the proof. □

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