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ON A CLASS OF KIRCHHOFF TYPE SYSTEMS WITH NONLINEAR BOUNDARY CONDITION

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Abstract. A class of Kirchhoff type systems with nonlinear boundary conditions considered in this paper. By using the method of Nehari manifold, it is proved that the system possesses two nontrivial nonnegative solutions if the parameters are small enough.
Keywords: Kirchhoff type systems, Nonlinear boundary condition, Nehari manifold.
MSC(2010): Primary: 35J50; Secondary: 35J60, 35J65.

1. **Introduction**

This paper is devoted to the study of the following Kirchhoff type systems with nonlinear boundary conditions:

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda f(x)|u|^{q-2}u, & x \in \Omega, \\ -M\left(\int_\Omega |\nabla v|^2 dx\right)\Delta v = \mu g(x)|v|^{q-2}v, & x \in \Omega, \\ \frac{\partial u}{\partial n} = \frac{\alpha}{\alpha+\beta} h(x)|u|^{\alpha-2}u|v|^{\beta}, & x \in \partial\Omega, \\ \frac{\partial v}{\partial n} = \frac{\beta}{\alpha+\beta} h(x)|u|^{\alpha}|v|^{\beta-2}v, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $2 < \alpha + \beta < 2^*(2^* = \frac{2N}{N-2}$ if $N \geq 3, 2^* = \infty$ if $N = 2)$, $1 < q < 2$, $M(s) = as + b$, $a, b > 0$, $(\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and the weight functions f, g, h are satisfying the following conditions:

(i) $f, g \in C(\overline{\Omega})$ with $\|f\|_\infty = \|g\|_\infty = 1$, and either $f^+ = \max\{f, 0\} \neq 0$ or $g^+ = \max\{g, 0\} \neq 0$;
(ii) $h \in C(\partial\Omega)$ with $\|h\|_\infty = 1$ and $h^+ = \max\{h, 0\} \neq 0$.

Problem (1.1) is called nonlocal because of the presence of the term $-M(\int_\Omega |\nabla u|^2 dx)$ which implies that the equation in (1.1) is no longer a pointwise identity.

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This phenomenon causes some mathematical difficulties which makes the study of such a class of problem particularly interesting. Beside, such a problem has physical motivation. Moreover, problem (1.1) is related to the stationary version of Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=0 \tag{1.2}$$

presented by Kirchhoff [16]. This equation extends the classical D. Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. The parameters in (1.2) have the following interpretation: L is the length of the string, h is the area of cross section, E is the Young's modulus of the material, ρ is the mass density, and P_0 is the initial tension.

When an elastic string with fixed ends is subjected to transverse vibrations, its length varies with the time: this introduces changes of the tension in the string. This inspired Kirchhoff to propose a nonlinear correction of the classical D'Alembert's equation. Later on, Woinowsky-Krieger (Nash-Modeer) incorporated this correction in the classical Euler–Bernoulli equation for the beam (plate) with hinged ends. See, for example, [4] and [5] and the references therein.

Moreover, nonlocal problems also appear in other fields as, for example, biological systems, where (u, v) describes a process which depends on the average of itself (for instance, population density). See, for example, [2,3,13,18] and [20] and the references therein. In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to [12,14,17,23], in which the authors have used different methods to get the existence of solutions.

In recent years, the existence of solutions for the semilinear / quasilinear elliptic equations with nonlinear boundary conditions have been widely studied (see, eg., [6, 9, 15] and the references therein). Motivated by these works, we are interested in the existence of multiple nontrivial nonnegative solutions to the Kirchhoff type systems with nonlinear boundary conditions. In fact we will prove that:

Theorem 1.1. *Suppose that the weight functions f, g and h satisfy conditions (i) and (ii) and $\alpha+\beta\geq 4$. Then there exists a positive number $C_0 = C_0(\alpha, \beta, b, q, S, \bar{S})$ such that if the parameters λ, μ satisfy*

$$0<|\lambda|^{\frac{2}{2-q}}+|\mu|^{\frac{2}{2-q}}<C_0,$$

then problem (1) has at least two nontrivial nonnegative solutions (u_0^+, v_0^+) and (u_0^-, v_0^-).

Proof. The proof is based on the method of Nehari manifold, which was first introduced by Nehari in [19], and the method turned out to be very useful in critical point theory (see, e.g., [1, 7, 8, 10, 11, 21]) and eventually came to bear his name. □

The rest of this work is organized as follows. In Section 2, we introduce some preliminaries including definitions and some lemmas for later use. In Section 3, the proof of the main result is given.

2. **Preliminaries**

Throughout this section, we denote by S, $\overline{S}$ the best Sobolev embedding and trace constant for the operators $H^1(\Omega) \hookrightarrow L^q(\Omega)$, $H^1(\Omega) \hookrightarrow L^{\alpha+\beta}(\partial\Omega)$, respectively. Define the Sobolev space $H = H^1(\Omega) \times H^1(\Omega)$ with the standard norm

$$\|(u,v)\| = \Big(\int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla v|^2 dx\Big)^{\frac{1}{2}}.$$

Moreover, a pair of functions $(u,v) \in H$ is said to be a weak solution of (1.1) if

$$\begin{aligned}
&M\Big(\int_\Omega |\nabla u|^2 dx\Big)\nabla u \nabla \phi_1 dx + M\Big(\int_\Omega |\nabla v|^2 dx\Big)\nabla v \nabla \phi_2 dx\\
&-\int_\Omega \lambda f(x)|u|^{q-2}u\phi_1 dx - \int_\Omega \mu g(x)|v|^{q-2}v\phi_2 dx\\
&-\frac{\alpha}{\alpha+\beta}\int_{\partial\Omega} h(x)|u|^{\alpha-2}u|v|^{\beta}\phi_1 ds - \frac{\beta}{\alpha+\beta}\int_{\partial\Omega} h(x)|u|^{\alpha}|v|^{\beta-2}v\phi_2 ds = 0,
\end{aligned}$$

for all $(\phi_1,\phi_2) \in H$. Thus, the corresponding energy functional of (1) is defined by

$$J_{\lambda,\mu}(u,v) = \frac{1}{2}\Big(\hat{M}(\|u\|^2_{H^1}) + \hat{M}(\|v\|^2_{H^1})\Big) - \frac{1}{\alpha+\beta}\int_{\partial\Omega} h|u|^\alpha|v|^\beta ds - \frac{1}{q}K_{\lambda,\mu}(u,v),$$

where $\hat{M}(s) = \int_0^s M(t)dt$, $K_{\lambda,\mu}(u,v) = \int_\Omega \lambda f|u|^q dx + \int_\Omega \mu g|v|^q dx$ and $M : \mathbb{R} \to \mathbb{R}^+$ is any function that is differentiable everywhere except at some finite points. It is well known that solutions of (1.1) are the critical points of the energy functional $J_{\lambda,\mu}$.

Now, we define the Nehari manifold

$$N_{\lambda,\mu} = \big\{(u,v) \in H - \{(0,0)\} | \langle J'_{\lambda,\mu}(u,v),(u,v)\rangle = 0\big\}.$$

Thus $(u,v) \in N_{\lambda,\mu}$, if and only if

$$\begin{aligned}
\langle J'_{\lambda,\mu}(u,v),(u,v)\rangle &= M(\|u\|^2_{H^1})\|u\|^2_{H^1}\\
&\quad + M(\|v\|^2_{H^1})\|v\|^2_{H^1} - K_{\lambda,\mu}(u,v) - \int_{\partial\Omega} h|u|^\alpha|v|^\beta ds\\
&= 0.
\end{aligned}$$

Define

$$\psi_{\lambda,\mu}(u,v) = \langle J'_{\lambda,\mu}(u,v),(u,v)\rangle.$$

Then for

$$\begin{aligned}(u,v)\in N_{\lambda,\mu}, \langle \psi'_{\lambda,\mu}(u,v),(u,v)\rangle &= 4a(\|u\|^4_{H^1}+\|v\|^4_{H^1})+2b(\|(u,v)\|^2)\\ &\quad -qK_{\lambda,\mu}(u,v)-(\alpha+\beta)\int_{\partial\Omega} h|u|^{\alpha}|v|^{\beta}ds.\end{aligned}$$

Now, we split $N_{\lambda,\mu}$ into three parts:

$$\begin{aligned}N^{+}_{\lambda,\mu} &= \left\{(u,v)\in N_{\lambda,\mu}|\langle \psi'_{\lambda,\mu}(u,v),(u,v)\rangle > 0\right\},\\ N^{0}_{\lambda,\mu} &= \left\{(u,v)\in N_{\lambda,\mu}|\langle \psi'_{\lambda,\mu}(u,v),(u,v)\rangle = 0\right\},\\ N^{-}_{\lambda,\mu} &= \left\{(u,v)\in N_{\lambda,\mu}|\langle \psi'_{\lambda,\mu}(u,v),(u,v)\rangle < 0\right\}.\end{aligned}$$

Then, we have the following results.

Lemma 2.1. *We have:*

(i) *if* $(u,v)\in N^{+}_{\lambda,\mu}$, *then* $K_{\lambda,\mu}(u,v)>0$;
(ii) *if* $(u,v)\in N^{-}_{\lambda,\mu}$, *then* $\int_{\partial\Omega} h|u|^{\alpha}|v|^{\beta}ds>0$;
(iii) *if* $(u,v)\in N^{0}_{\lambda,\mu}$, *then* $K_{\lambda,\mu}(u,v)>0, \int_{\partial\Omega} h|u|^{\alpha}|v|^{\beta}ds>0$.

Proof. (i) For $(u,v)\in N^{+}_{\lambda,\mu}$, we have

$$\begin{aligned}\langle \psi'_{\lambda,\mu}(u,v),(u,v)\rangle &= 4a(\|u\|^4_{H^1}+\|v\|^4_{H^1})+2b(\|(u,v)\|^2)-qK_{\lambda,\mu}(u,v)\\ &\quad -(\alpha+\beta)\int_{\partial\Omega} h|u|^{\alpha}|v|^{\beta}ds\\ &= a(4-\alpha-\beta)(\|u\|^4_{H^1}+\|v\|^4_{H^1})+b(2-\alpha-\beta)\|(u,v)\|^2\\ &+ (\alpha+\beta-q)(K_{\lambda,\mu}(u,v))>0.\end{aligned}$$

Then

$$K_{\lambda,\mu}(u,v) > \frac{b(\alpha+\beta-2)}{\alpha+\beta-q}(\|(u,v)\|^2)+\frac{a(\alpha+\beta-4)}{\alpha+\beta-q}(\|u\|^4_{H^1}+\|v\|^4_{H^1})>0.$$

(ii) For $(u,v)\in N^{-}_{\lambda,\mu}$, we have

$$\begin{aligned}\langle \psi'_{\lambda,\mu}(u,v),(u,v)\rangle &= 4a(\|u\|^4_{H^1}+\|v\|^4_{H^1})+2b(\|(u,v)\|^2)\\ &\quad -qK_{\lambda,\mu}(u,v)-(\alpha+\beta)\int_{\partial\Omega} h|u|^{\alpha}|v|^{\beta}ds\\ &= a(4-q)(\|u\|^4_{H^1}+\|v\|^4_{H^1})+b(2-q)\|(u,v)\|^2\\ &\quad +(q-\alpha-\beta)\int_{\partial\Omega} h|u|^{\alpha}|v|^{\beta}ds<0.\end{aligned}$$

Then

$$\int_{\partial\Omega} h|u|^{\alpha}|v|^{\beta}ds > \frac{b(2-q)}{\alpha+\beta-q}\|(u,v)\|^2 > 0.$$

(iii) For $(u,v) \in N^0_{\lambda,\mu}$, we have

$$K_{\lambda,\mu}(u,v) = \frac{b(\alpha+\beta-2)}{\alpha+\beta-q}(\|(u,v)\|^2) + \frac{a(\alpha+\beta-4)}{\alpha+\beta-q}(\|u\|^4_{H^1} + \|v\|^4_{H^1}) > 0,$$

$$\int_{\partial\Omega} h|u|^{\alpha}|v|^{\beta}ds = \frac{b(2-q)}{\alpha+\beta-q}\|(u,v)\|^2 > 0.$$

This complete the proof. □

Lemma 2.2. *There exists a positive* $C = C(\alpha,\beta,b,S,\bar{S})$*, such that if*

$$0 < |\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}} < C,$$

then $N^0_{\lambda,\mu} = \emptyset$.

Proof. We consider the following two cases.
case(a): $(u,v) \in N_{\lambda,\mu}$ and $\int_{\partial\Omega} h|u|^{\alpha}|v|^{\beta}ds \le 0$. We have

$$\begin{aligned}
\langle \psi'_{\lambda,\mu}(u,v),(u,v)\rangle &= 4a(\|u\|^4_{H^1} + \|v\|^4_{H^1}) + 2b(\|(u,v)\|^2) - qK_{\lambda,\mu}(u,v)\\
&- (\alpha+\beta)\int_{\partial\Omega} h|u|^{\alpha}|v|^{\beta}ds\\
&= a(4-q)(\|u\|^4_{H^1} + \|v\|^4_{H^1}) + b(2-q)\|(u,v)\|^2\\
&\quad +(q-\alpha-\beta)\int_{\partial\Omega} h|u|^{\alpha}|v|^{\beta}ds > 0
\end{aligned}$$

So $(u,v) \notin N^0_{\lambda,\mu}$.
Case(b): $(u,v) \in N_{\lambda,\mu}$ and $\int_{\partial\Omega} h|u|^{\alpha}|v|^{\beta}ds > 0$. Suppose that $N^0_{\lambda,\mu} \neq \emptyset$, by lemma (2.1), we have

$$\begin{aligned}
(\alpha+\beta-q)\int_{\partial\Omega} h|u|^{\alpha}|v|^{\beta}ds &= a(4-q)(\|u\|^4_{H^1} + \|v\|^4_{H^1}) + b(2-q)\|(u,v)\|^2\\
&\ge b(2-q)\|(u,v)\|^2,
\end{aligned}$$

Moreover, by the Hölder and sobolev inequalities,

$$\begin{aligned}
\|(u,v)\|^2 &\le \frac{\alpha+\beta-q}{b(2-q)}\int_{\partial\Omega} h|u|^{\alpha}|v|^{\beta}ds\\
&\le \frac{\alpha+\beta-q}{b(2-q)}\big(\int_{\partial\Omega}|u|^{\alpha+\beta}\big)^{\frac{\alpha}{\alpha+\beta}}\big(\int_{\partial\Omega}|v|^{\alpha+\beta}\big)^{\frac{\beta}{\alpha+\beta}}\\
&\le \frac{\alpha+\beta-q}{b(2-q)}\bar{S}^{\alpha+\beta}\|(u,v)\|^{\alpha+\beta}.
\end{aligned}$$

It follows that

$$\|(u,v)\| \ge \big(\frac{\alpha+\beta-q}{b(2-q)}\bar{S}^{\alpha+\beta}\big)^{\frac{1}{2-\alpha-\beta}},$$

and by $(u,v) \in N^0_{\lambda,\mu}$ and $\alpha+\beta>4$ we have

$$(\alpha+\beta-q)(K_{\lambda,\mu}(u,v))+b(2-\alpha-\beta)\|(u,v)\|^2 = a(\alpha+\beta-4)(\|u\|^4_{H^1}+\|v\|^4_{H^1})>0.$$

Thus

$$\begin{aligned} b(\alpha+\beta-2)\|(u,v)\|^2 &< (\alpha+\beta-q)(K_{\lambda,\mu}(u,v)) \\ &\leq (\alpha+\beta-q)|\lambda|\|u\|^q_{L^{\alpha+\beta}}+|\mu|\|v\|^q_{L^{\alpha+\beta}} \\ &\leq (\alpha+\beta-q)S^q\big(|\lambda|^{\frac{2}{2-q}}+|\mu|^{\frac{2}{2-q}}\big)^{\frac{2-q}{2}}\|(u,v)\|^q, \end{aligned}$$

and hence

$$\|(u,v)\| \leq \big(\frac{\alpha+\beta-q}{b(\alpha+\beta-2)}\big)^{\frac{1}{2-q}}S^{\frac{q}{2-q}}\big(|\lambda|^{\frac{2}{2-q}}+|\mu|^{\frac{2}{2-q}}\big)^{\frac{1}{2}}.$$

This implies

$$\big(|\lambda|^{\frac{2}{2-q}}+|\mu|^{\frac{2}{2-q}}\big)^{\frac{2-q}{2}} \geq C.$$

This is contradiction by choose

$$C=C(\alpha,\beta,b,S,\bar{S})=\big(\frac{\alpha+\beta-q}{b(2-q)}\bar{S}^{\alpha+\beta}\big)^{\frac{2}{2-\alpha-\beta}}\big(\frac{b(\alpha+\beta-2)}{\alpha+\beta-q}S^{-q}\big)^{\frac{1}{2-q}}.$$

□

Lemma 2.3. *Suppose that $M(s)=as+b$, then the energy functional $J_{\lambda,\mu}(u,v)$ is coercive and bounded below on $N_{\lambda,\mu}$.*

Proof. For $(u,v)\in N_{\lambda,\mu}$, we have

$$M(\|u\|^2_{H^1})\|u\|^2_{H^1}+M(\|v\|^2_{H^1})\|v\|^2_{H^1}=K_{\lambda,\mu}(u,v)+\int_{\partial\Omega}h|u|^\alpha|v|^\beta ds.$$

By the Sobolev inequality,

$$\begin{aligned} J_{\lambda,\mu}(u,v) &= \frac{1}{2}\big(\hat{M}(\|u\|^2_{H^1})+\hat{M}(\|v\|^2_{H^1})\big)-\frac{1}{q}K_{\lambda,\mu}(u,v)-\frac{1}{\alpha+\beta}\int_\Omega h|u|^\alpha|v|^\beta ds \\ &= \frac{1}{2}\big(\hat{M}(\|u\|^2_{H^1})+\hat{M}(\|v\|^2_{H^1})\big)-\frac{1}{q}K_{\lambda,\mu}(u,v)-\frac{1}{\alpha+\beta}\big(M(\|u\|^2_{H^1})\|u\|^2_{H^1} \\ &\quad + M(\|v\|^2_{H^1})\|v\|^2_{H^1}\big)+\frac{1}{\alpha+\beta}K_{\lambda,\mu}(u,v) \\ &= \frac{\|u\|^2_{H^1}}{2(\alpha+\beta)}\big(\frac{a(\alpha+\beta-4)}{2}\|u\|^2_{H^1}+b(\alpha+\beta-2)\big) \\ &\quad + \frac{\|v\|^2_{H^1}}{2(\alpha+\beta)}\big(\frac{a(\alpha+\beta-4)}{2}\|v\|^2_{H^1}+b(\alpha+\beta-2)\big)-\big(\frac{\alpha+\beta-q}{q(\alpha+\beta)}\big)K_{\lambda,\mu}(u,v) \\ &\geq \big(\frac{b(\alpha+\beta-2)}{2(\alpha+\beta)}\big)\|(u,v)\|^2-\big(\frac{\alpha+\beta-q}{q(\alpha+\beta)}\big)K_{\lambda,\mu}(u,v) \\ &\geq \big(\frac{b(\alpha+\beta-2)}{2(\alpha+\beta)}\big)\|(u,v)\|^2-\big(\frac{\alpha+\beta-q}{q(\alpha+\beta)}\big)S^q\big((|\lambda|^{\frac{2}{2-q}})+(|\mu|^{\frac{2}{2-q}})\big)^{\frac{2-q}{2}}\|(u,v)\|^q. \end{aligned}$$

Since $1<q<2$, $J_{\lambda,\mu}$ is coercive and bounded below on $N_{\lambda,\mu}$. □

The following lemma shows that the minimizers on $N_{\lambda,\mu}$ are usually critical points for $J_{\lambda,\mu}$.

Lemma 2.4. *Suppose that* (u_0,v_0) *is local minimizer for* $J_{\lambda,\mu}$ *on* $N_{\lambda,\mu}$*, and* $(u_0,v_0)\notin N^0_{\lambda,\mu}$*, then* $J'_{\lambda,\mu}(u_0,v_0)=0$.

Proof. Let (u_0,v_0) be a local minimizer for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}$. Then (u_0,v_0) is a solution of the following optimization problem:

$$\text{minimize } J_{\lambda,\mu}(u,v) \text{ subject to } \psi_{\lambda,\mu}(u,v)=0.$$

Hence, by the theory of Lagrange multipliers, there exists a $\Lambda\in\mathbb{R}$ such that

$$J'_{\lambda,\mu}(u_0,v_0)=\Lambda\psi'_{\lambda,\mu}(u_0,v_0) \text{ in } H^{-1}.$$

Thus

$$\langle J'_{\lambda,\mu}(u_0,v_0),(u_0,v_0)\rangle=\Lambda\langle\psi'_{\lambda,\mu}(u_0,v_0),(u_0,v_0)\rangle=0.$$

Since $(u_0,v_0)\in N_{\lambda,\mu}$, $\langle J'_{\lambda,\mu}(u_0,v_0),(u_0,v_0)\rangle=0$. On the other hand, $\langle\psi'_{\lambda,\mu}(u_0,v_0),(u_0,v_0)\rangle\neq 0$. Hence $\Lambda=0$, and this completes the proof. $\square$

Lemma 2.2 suggests that we introduce the set

$$\Theta=\left\{(\lambda,\mu)\in\mathbb{R}^2-\{(0,0)\}\,|\,0<\left(|\lambda|^{\frac{2}{2-q}}+|\mu|^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}<C_0\right\},$$

where $C_0=(\frac{q}{2})^{\frac{2}{2-q}}C<C$. If $(\lambda,\mu)\in\Theta$, we have $N_{\lambda,\mu}=N^+_{\lambda,\mu}\cup N^-_{\lambda,\mu}$ and define

$$\theta^+_{\lambda,\mu}=\inf_{(u,v)\in N^+_{\lambda,\mu}}J_{\lambda,\mu}(u,v);\quad \theta^-_{\lambda,\mu}=\inf_{(u,v)\in N^-_{\lambda,\mu}}J_{\lambda,\mu}(u,v).$$

We have the following lemma:

Lemma 2.5. *If* $(\lambda,\mu)\in\Theta$*, then:*

(i) $\theta^+_{\lambda,\mu}<0$*;*
(ii) *there exists* $d_0=d_0(\alpha,\beta,b,S,\bar{S},\lambda,\mu)>0$*, such that* $\theta^-_{\lambda,\mu}>d_0$*.*

Proof. (i) Let $(u,v)\in N^+_{\lambda,\mu}$. Since

$$(\alpha+\beta-q)K_{\lambda,\mu}(u,v)>a(\alpha+\beta-4)(\|u\|^4_{H^1}+\|v\|^4_{H^1})+b(\alpha+\beta-2)\|(u,v)\|^2,$$

and so

$$\begin{aligned}
J_{\lambda,\mu}(u,v) &= \frac{1}{2}\big(\hat{M}(\|u\|^2_{H^1})+\hat{M}(\|v\|^2_{H^1})\big)-\frac{1}{q}K_{\lambda,\mu}(u,v)-\frac{1}{\alpha+\beta}\int_\Omega h|u|^\alpha|v|^\beta ds\\
&= \frac{1}{2}\big(\hat{M}(\|u\|^2_{H^1})+\hat{M}(\|v\|^2_{H^1})\big)-\frac{1}{q}K_{\lambda,\mu}(u,v)\\
&\quad -\frac{1}{\alpha+\beta}\big(M(\|u\|^2_{H^1})\|u\|^2_{H^1}+M(\|v\|^2_{H^1})\|v\|^2_{H^1}-K_{\lambda,\mu}(u,v)\big)\\
&= a(\frac{1}{4}-\frac{1}{\alpha+\beta})(\|u\|^4_{H^1}+\|v\|^4_{H^1})+b(\frac{1}{2}-\frac{1}{\alpha+\beta})\|(u,v)\|^2\\
&\quad -\frac{\alpha+\beta-q}{q(\alpha+\beta)}K_{\lambda,\mu}(u,v)\\
&\le a(\alpha+\beta-4)(\|u\|^4_{H^1}+\|v\|^4_{H^1})\big(\frac{1}{4(\alpha+\beta)}-\frac{1}{q(\alpha+\beta)}\big)\\
&\quad +b(\alpha+\beta-2)(\|(u,v)\|^2)\big(\frac{1}{2(\alpha+\beta)}-\frac{1}{q(\alpha+\beta)}\big)<0.
\end{aligned}$$

Thus, $\theta^+_{\lambda,\mu}<0$.

(ii) Let $(u,v)\in N^-_{\lambda,\mu}$. By Lemma 2.1

$$\int_{\partial\Omega} h|u|^\alpha|v|^\beta ds>0,$$

by Lemma 2.2

$$\|(u,v)\|\ge\big(\frac{\alpha+\beta-q}{b(2-q)}\bar{S}^{\alpha+\beta}\big)^{\frac{1}{2-\alpha-\beta}}. \tag{2.1}$$

Moreover, by Lemma 2.3, we have

$$\begin{aligned}
&J_{\lambda,\mu}(u,v)\\
&\ge \big(\frac{b(\alpha+\beta-2)}{2(\alpha+\beta)}\big)\|(u,v)\|^2-\big(\frac{\alpha+\beta-q}{q(\alpha+\beta)}\big)S^q\big(|\lambda|^{\frac{2}{2-q}}+|\mu|^{\frac{2}{2-q}}\big)^{\frac{2-q}{2}}\|(u,v)\|^q.\\
&= \|(u,v)\|^q\big[\frac{b(\alpha+\beta-2)}{2(\alpha+\beta)}\|(u,v)\|^{2-q}-\big(\frac{\alpha+\beta-q}{q(\alpha+\beta)}\big)S^q\big)\big(|\lambda|^{\frac{2}{2-q}}+|\mu|^{\frac{2}{2-q}}\big)^{\frac{2-q}{2}}\big]\\
&\ge \big(\frac{\alpha+\beta-q}{b(2-q)}\bar{S}^{\alpha+\beta}\big)^{\frac{q}{2-\alpha-\beta}}\big[\frac{b(\alpha+\beta-2)}{2(\alpha+\beta)}\big(\frac{\alpha+\beta-q}{b(2-q)}\bar{S}^{\alpha+\beta}\big)^{\frac{2-q}{2-\alpha-\beta}}\\
&- \quad \frac{\alpha+\beta-q}{q(\alpha+\beta)}\big)S^q\big(|\lambda|^{\frac{2}{2-q}}+|\mu|^{\frac{2}{2-q}}\big)^{\frac{2-q}{2}}\big].
\end{aligned}$$

Thus if

$$0<\big(|\lambda|^{\frac{2}{2-q}}+|\mu|^{\frac{2}{2-q}}\big)^{\frac{2-q}{2}}<C_0,$$

for each $(u,v)\in N^-_{\lambda,\mu}$, we have

$$J_{\lambda,\mu}(u,v),(u,v)\ge d_0=d_0(\alpha,\beta,b,S,\bar{S},\lambda,\mu)>0,$$

for some $d_0>0$. This completes the proof. □

Lemma 2.6. *For each $(u,v)\in N^-_{\lambda,\mu}$, we have:*

(i) *if* $K_{\lambda,\mu}(u,v)\leq 0$, *then there is a unique* $t^->t_{a,\max}$ *such that* $(t^-u,t^-v)\in N^-_{\lambda,\mu}$ *and*
$$J_{\lambda,\mu}(t^-u,t^-v)=\sup_{t\geq 0}J_{\lambda,\mu}(tu,tv);$$

(ii) *if* $K_{\lambda,\mu}(u,v)>0$, *then there are unique* t^+,t^- *with* $0<t^+<t_{a,\max}<t^-$ *such that* $(t^+u,t^+v)\in N^+_{\lambda,\mu}$, $(t^-u,t^-v)\in N^-_{\lambda,\mu}$ *and*
$$J_{\lambda,\mu}(t^+u,t^+v)=\inf_{0\leq t\leq t_{a,\max}}J_{\lambda,\mu}(tu,tv) \text{ and } J_{\lambda,\mu}(t^-u,t^-v)=\sup_{t\geq 0}J_{\lambda,\mu}(tu,tv).$$

Proof. (i) Fix $(u,v)\in N^-_{\lambda,\mu}$, let
$$m_a(t)=at^{4-q}(\|u\|^4_{H^1}+\|v\|^4_{H^1})+bt^{2-q}(\|(u,v)\|^2)-t^{\alpha+\beta-q}\int_{\partial\Omega}h|u|^\alpha|v|^\beta ds$$
for $a,t\geq 0$. Clearly, $m_a(0)=0$, $m_a(t)\longrightarrow-\infty$ as $t\longrightarrow-\infty$. Since $\int_{\partial\Omega}h|u|^\alpha|v|^\beta ds>0$ and
$$\begin{aligned}&m_a'(t)=\\&t^{1-q}\big(a(4-q)(\|u\|^4_{H^1}+\|v\|^4_{H^1})t^2+b(2-q)\|(u,v)\|^2\big)\\&-(\alpha+\beta-q)t^{\alpha+\beta-q-1}\int_{\partial\Omega}h|u|^\alpha|v|^\beta ds\end{aligned}$$
there is a unique $t_{a,\max}>0$ such that $m_a(t)$ achieves its maximum at $t_{a,\max}$, increasing for $t\in[0,t_{a,\max})$ and decreasing for $t\in(t_{a,\max},\infty)$ with $\lim_{t\to\infty}m_a(t)=-\infty$. Moreover
$$t_{0,\max}=\Big(\frac{b(2-q)\|(u,v)\|^2}{(\alpha+\beta-q)\int_{\partial\Omega}h|u|^\alpha|v|^\beta ds}\Big)^{\frac{1}{\alpha+\beta-2}},$$
and
$$\begin{aligned}&m_0(t_{0,\max})\\
=\;& b\Big(\frac{b(2-q)\|(u,v)\|^2}{(\alpha+\beta-q)\int_{\partial\Omega}h|u|^\alpha|v|^\beta ds}\Big)^{\frac{2-q}{\alpha+\beta-2}}\|(u,v)\|^2\\
-\;& \Big(\frac{b(2-q)\|(u,v)\|^2}{(\alpha+\beta-q)\int_{\partial\Omega}h|u|^\alpha|v|^\beta ds}\Big)^{\frac{\alpha+\beta-q}{\alpha+\beta-2}}\int_{\partial\Omega}h|u|^\alpha|v|^\beta ds\\
=\;& b\|(u,v)\|^q\Big[\Big(\frac{2-q}{\alpha+\beta-q}\Big)^{\frac{2-q}{\alpha+\beta-2}}-\Big(\frac{2-q}{\alpha+\beta-q}\Big)^{\frac{\alpha+\beta-q}{\alpha+\beta-2}}\Big]\Big(\frac{b\|(u,v)\|^{\alpha+\beta}}{\int_{\partial\Omega}h|u|^\alpha|v|^\beta ds}\Big)^{\frac{2-q}{\alpha+\beta-2}}\\
\geq\;& b\|(u,v)\|^q\Big(\frac{\alpha+\beta-2}{\alpha+\beta-q}\Big)\Big(\frac{b(2-q)}{\bar{S}^{\alpha+\beta}(\alpha+\beta-q)}\Big)^{\frac{2-q}{\alpha+\beta-2}}.\end{aligned}$$

(i) $K_{\lambda,\mu}(u,v)\leq 0$. There is a unique $t^->t_{a,\max}$ such that $m_a(t^-)=K_{\lambda,\mu}(u,v)$ and $m_a'(t^-)<0$. Now
$$\begin{aligned}\langle\psi'_{\lambda,\mu}(t^-u,t^-v),(t^-u,t^-v)\rangle\;=\;& a(t^-)^4(4-q)(\|u\|^4_{H^1}+\|v\|^4_{H^1})+b(t^-)^2(2-q)\|(u,v)\|^2\\&-(t^-)^{\alpha+\beta}(\alpha+\beta-q)\int_{\partial\Omega}h|u|^\alpha|v|^\beta ds\\&=(t^-)^{1+q}m_a'(t^-)<0,\end{aligned}$$

and

$$\langle J'_{\lambda,\mu}(t^- u, t^- v), (t^- u, t^- v)\rangle = (t^-)^q \big[m_a(t^-) - K_{\lambda,\mu}(u,v)\big] = 0.$$

Thus $(t^- u, t^- v) \in N^-_{\lambda,\mu}$, $t^- = 1$. Since for $t > t_{a,max}$, we have

$$a(4-q)(\|tu\|^4_{H^1} + \|tv\|_{H^1})^4 + b(2-q)\|(tu,tv)\|^2 - (\alpha+\beta)\int_{\partial\Omega} h|tu|^\alpha |tv|^\beta ds < 0$$

and $\frac{d^2}{dt^2} J_{\lambda,\mu}(tu,tv) < 0$, moreover for $t = t^-$

$$\begin{aligned}\frac{d}{dt} J_{\lambda,\mu}(tu,tv) &= at^3(\|u\|^4_{H^1} + \|v\|^4_{H^1}) - t^{q-1}(K_{\lambda,\mu}(u,v)) - t^{\alpha+\beta-1}\int_{\partial\Omega} h|u|^\alpha |v|^\beta ds \\ &= t^{q-1}(m_a(t) - K_{\lambda,\mu}(u,v)) = 0.\end{aligned}$$

Thus, $J_{\lambda,\mu}(t^- u, t^- v) = \sup_{t\geq 0} J_{\lambda,\mu}(tu,tv)$.

(ii) $K_{\lambda,\mu}(u,v) > 0$. For $0 < \big(|\lambda|^{\frac{2}{2-q}} + |\mu|^{\frac{2}{2-q}}\big)^{\frac{2-q}{2}} < C_0 < C$, we have

$$\begin{aligned} m_a(0) = 0 &< K_{\lambda,\mu}(u,v) \\ &\leq b\|(u,v)\|^q \Big(\frac{\alpha+\beta-2}{\alpha+\beta-q}\Big)\Big(\frac{b(2-q)}{\bar{S}^{\alpha+\beta}(\alpha+\beta-q)}\Big)^{\frac{2-q}{\alpha+\beta-2}}. \\ &\leq m_0(t_{0,\max}) < m_a(t_{a,\max}).\end{aligned}$$

There are unique t^+ and t^- such that $0 < t^+ < t_{a,\max} < t^-$,

$$m_a(t^+) = K_{\lambda,\mu}(u,v)) = m_a(t^-),$$

and

$$m'_a(t^+) > 0 > m'_a(t^-).$$

We have $(t^+ u, t^+ v) \in N^+_{\lambda,\mu}$, $(t^- u, t^- v) \in N^-_{\lambda,\mu}$ and

$$J_{\lambda,\mu}(t^- u, t^- v) \geq J_{\lambda,\mu}(tu,tv) \geq J_{\lambda,\mu}(t^+ u, t^+ v) \ \text{ for each } t \in [t^+, t^-],$$

Thus

$$J_{\lambda,\mu}(t^+ u, t^+ v) = \inf_{0\leq t\leq t_{a,\max}} J_{\lambda,\mu}(tu,tv) \text{ and } J_{\lambda,\mu}(t^- u, t^- v) = \sup_{t\geq 0} J_{\lambda,\mu}(tu,tv).$$

This complete the proof. □

3. **Proof of Theorem 1.1**

First we establish the existence of local minimum for $J_{\lambda,\mu}$ on $N^+_{\lambda,\mu}$.

Theorem 3.1. *If $(\lambda,\mu) \in \Theta$, then $J_{\lambda,\mu}$ has a minimizer (u_0^+, v_0^+) in $N^+_{\lambda,\mu}$ and satisfies:*

(i) $J_{\lambda,\mu}(u_0^+, v_0^+) = \theta^+_{\lambda,\mu}$;

(ii) (u_0^+, v_0^+) *is a solution of* (1.1) *such that* $u_0^+ \geq 0, v_0^+ \geq 0$ *in* Ω.

Proof. By there is a minimizing sequence (u_n, v_n) for $J_{\lambda,\mu}$ on $N^+_{\lambda,\mu}$ such that

$$J_{\lambda,\mu}(u_n, v_n) = \theta^+_{\lambda,\mu} + o(1) \text{ and } J'_{\lambda,\mu}(u_n, v_n) = o(1) \text{ in } H^*(\Omega).$$

Then by Lemma 2.3 and compact embedding theorem, there exist a subsequence (u_n, v_n) and $(u_0^+, v_0^+) \in H$ is solution of (1.1) and

$$\begin{aligned}
&u_n \rightharpoonup u_0^+, \text{ weakly in } H^1(\Omega),\\
&u_n \to u_0^+, \text{ strongly in } L^q(\Omega) \text{ and in } L^{\alpha+\beta}(\partial\Omega),\\
&v_n \rightharpoonup v_0^+, \text{ weakly in } H^1(\Omega),\\
&v_n \to v_0^+, \text{ strongly in } L^q(\Omega) \text{ and in } L^{\alpha+\beta}(\partial\Omega).
\end{aligned}$$

This implies

$$K_{\lambda,\mu}(u_n, v_n) \to K_{\lambda,\mu}(u_0^+, v_0^+), \text{ as } n \to \infty,$$

$$\int_{\partial\Omega} h|u_n|^\alpha |v_n|^\beta ds \to \int_{\partial\Omega} h|u_0^+|^\alpha |v_0^+|^\beta ds, \text{ as } n \to \infty. \tag{3.1}$$

First, we claim that $K_{\lambda,\mu}(u_0^+, v_0^+) > 0$. Suppose otherwise, by (3.1) we can conclude that

$$K_{\lambda,\mu}(u_n, v_n) \to K_{\lambda,\mu}(u_0^+, v_0^+) = 0, \text{ as } n \to \infty,$$

and so

$$M(\|u_n\|^2_{H^1}) + M(\|v_n\|^2_{H^1}) = \int_{\partial\Omega} h|u_n|^\alpha |v_n|^\beta ds + o(1).$$

Thus

$$\begin{aligned}
&J_{\lambda,\mu}(u_n, v_n)\\
&= \frac{1}{2}\big(\hat{M}(\|u_n\|^2_{H^1}) + \hat{M}(\|v_n\|^2_{H^1})\big) - \frac{1}{q}K_{\lambda,\mu}(u_n, v_n) - \frac{1}{\alpha+\beta}\int_\Omega h|u_n|^\alpha |v_n|^\beta ds\\
&= \frac{a}{4}(\|u_n\|^4_{H^1} + \|v_n\|_{H^1})^4) + \frac{b}{2}\|(u_n, v_n)\|^2 - \frac{a}{\alpha+\beta}(\|u_n\|^4_{H^1} + \|v_n\|_{H^1})^4)\\
&- \frac{b}{\alpha+\beta}\|(u_n, v_n)\|^2 + o(1)\\
&= (\frac{a}{4} - \frac{a}{\alpha+\beta})(\|u_0^+\|^4_{H^1} + \|v_0^+\|^4_{H^1}) + (\frac{b}{2} - \frac{b}{\alpha+\beta})\|(u_0^+, v_0^+)\|^2 \text{ as } n \to \infty,
\end{aligned}$$

which contradicts $J_{\lambda,\mu}(u_n, v_n) \to \theta^+_{\lambda,\mu} < 0$ as $n \to \infty$. Now we prove that

$$\begin{aligned}
&u_n \to u_0^+, \text{ strongly in } H^1(\Omega),\\
&v_n \to v_0^+, \text{ strongly in } H^1(\Omega).
\end{aligned}$$

Suppose otherwise, then either $\|u_0^+\| < \liminf_{n\to\infty} \|u_n\|_{H^1}$ or $\|v_0^+\| < \liminf_{n\to\infty} \|v_n\|_{H^1}$ and so

$$\begin{aligned}
&a(\|u_0^+\|^4_{H^1} + \|v_0^+\|_{H^1})^4) + b\|(u_0^+, v_0^+)\|^2 - K_{\lambda,\mu}(u_0^+, v_0^+) - \int_{\partial\Omega} h|u_0^+|^\alpha |v_0^+|^\beta ds\\
&< \liminf_{n\to\infty}\big(a(\|u_n\|^4_{H^1} + \|v_n\|^4_{H^1}) + b\|(u_n, v_n)\|^2 - K_{\lambda,\mu}(u_n, v_n) - \int_\Omega h|u_n|^\alpha |v_n|^\beta ds\big)\\
&< \quad 0,
\end{aligned}$$

which is contradicts $(u_0^+, v_0^+) \in N_{\lambda,\mu}$. Moreover, we have $(u_0^+, v_0^+) \in N_{\lambda,\mu}^+$. In fact, if $(u_0^+, v_0^+) \in N_{\lambda,\mu}^-$. By Lemma 2.6, there are unique t_0^+ and t_0^- such that $(t_0^+ u_0^+, t_0^+ v_0^+) \in N_{\lambda,\mu}^+$,$(t_0^- u_0^+, t_0^- v_0^+) \in N_{\lambda,\mu}^-$ and $t_0^+ < t_0^- = 1$. since

$$\frac{d}{dt} J_{\lambda,\mu}(t_0^+ u_0^+, t_0^+ v_0^+) = 0 \text{ and } \frac{d^2}{dt^2} J_{\lambda,\mu}(t_0^+ u_0^+, t_0^+ v_0^+) > 0.$$

There exists $t_0^+ < \bar{t} \le t_0^-$ such that $J_{\lambda,\mu}(t_0^+ u^+, t_0^+ v_0^+) < J_{\lambda,\mu}(\bar{t} u_0^+, \bar{t} v_0^+)$. by Lemma 2.6 , we have

$$\begin{aligned}
J_{\lambda,\mu}(t_0^+ u_0^+, t_0^+ v_0^+) &< J_{\lambda,\mu}(\bar{t} u_0^+, \bar{t} v_0^+) \\
&\le J_{\lambda,\mu}(t_0^- u_0^+, t_0^- v_0^+) \\
&= J_{\lambda,\mu}(u_0^+, v_0^+) \\
&< \lim_{n\to\infty} J_{\lambda,\mu}(u_n, v_n) \\
&= \theta_{\lambda,\mu}^+.
\end{aligned}$$

Which is contradicts $\theta_{\lambda,\mu}^+ = \inf_{(u,v)\in N_{\lambda,\mu}^+} J_{\lambda,\mu}(u,v)$. Since $J_{\lambda,\mu}(u_0^+, v_0^+) = J_{\lambda,\mu}(|u_0^+|, |v_0^+|)$ and $(|u_0^+|, |v_0^+|) \in N_{\lambda,\mu}^+$, by Lemma 2.4 we may assume that (u_0^+, v_0^+) is a nonnegative solution of (1.1). □

Next we establish the existence of local minimum for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}^-$.

Theorem 3.2. *If $(\lambda,\mu) \in \Theta$, then $J_{\lambda,\mu}$ has a minimizer (u_0^-, v_0^-) in $N_{\lambda,\mu}^-$ and satisfies:*

(i) $J_{\lambda,\mu}(u_0^-, v_0^-) = \theta_{\lambda,\mu}^-$;
(ii) (u_0^-, v_0^-) *is a solution of* (1.1) *such that* $u_0^- \ge 0, v_0^- \ge 0$ *in* Ω.

Proof. Let (u_n, v_n) be a minimizing sequence for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}^-$. Then by Lemma 2.3 and compact embedding theorem, there exist a subsequence (u_n, v_n) and $(u_0^-, v_0^-) \in H$ is solution of (1.1) and

$$\begin{aligned}
&u_n \rightharpoonup u_0^-, \text{ weakly in } H^1(\Omega), \\
&u_n \to u_0^-, \text{ strongly in } L^q(\Omega) \text{ and in } L^{\alpha+\beta}(\partial\Omega), \\
&v_n \rightharpoonup v_0^-, \text{ weakly in } H^1(\Omega), \\
&v_n \to v_0^-, \text{ strongly in } L^q(\Omega) \text{ and in } L^{\alpha+\beta}(\partial\Omega).
\end{aligned}$$

This implies

$$K_{\lambda,\mu}(u_n, v_n) \to K_{\lambda,\mu}(u_0^-, v_0^-), \text{ as } n \to \infty,$$

$$\int_{\partial\Omega} h|u_n|^\alpha |v_n|^\beta ds \to \int_{\partial\Omega} h|u_0^-|^\alpha |v_0^-|^\beta ds, \text{ as } n \to \infty.$$

Since

$$\begin{aligned}
&\langle \psi'_{\lambda,\mu}(u_n,v_n),(u_n,v_n)\rangle \\
&= 4a(\|u_n\|_{H^1}^4+\|v_n\|_{H^1}^4)+2b(\|(u_n,v_n)\|^2)-qK_{\lambda,\mu}(u_n,v_n) \\
&- (\alpha+\beta)\int_{\partial\Omega} h|u_n|^{\alpha}|v_n|^{\beta}ds \\
&= a(4-\alpha-\beta)(\|u_n\|_{H^1}^4+\|v_n\|_{H^1}^4)+b(2-\alpha-\beta)\|(u_n,v_n)\|^2 \\
&+ (\alpha+\beta-q)(K_{\lambda,\mu}(u_n,v_n))<0.
\end{aligned}$$

We have

$$K_{\lambda,\mu}(u_n,v_n) > \frac{a(4-\alpha-\beta)}{q-\alpha-\beta}(\|u_n\|_{H^1}^4+\|v_n\|_{H^1}^4)+\frac{b(2-\alpha-\beta)}{q-\alpha-\beta}\|(u_n,v_n)\|^2. \tag{3.2}$$

By (2.1) and (3.2) there exists a positive number $\bar{C}$ such that

$$K_{\lambda,\mu}(u_n,v_n) > \bar{C}.$$

This implies

$$K_{\lambda,\mu}(u_0^-,v_0^-) \geq \bar{C}.$$

Now we prove that

$$\begin{aligned}
&u_n \to u_0^-, \text{ strongly in } H^1(\Omega),\\
&v_n \to v_0^-, \text{ strongly in } H^1(\Omega).
\end{aligned}$$

Suppose otherwise, then either $\|u_0^-\| < \liminf_{n\to\infty}\|u_n\|_{H^1}$ or $\|v_0^-\| < \liminf_{n\to\infty}\|v_n\|_{H^1}$. By Lemma 2.6, there is a unique t_0^- such that $(t_0^-u_0^-, t_0^-v_0^-) \in N_{\lambda,\mu}^-$. Since $(u_n,v_n)\in N_{\lambda,\mu}^-$, $J_{\lambda,\mu}(u_n,v_n) \geq J_{\lambda,\mu}(t_0^-u_n,t_0^-v_n)$ for all $t\geq 0$, we have

$$J_{\lambda,\mu}(t_0^-u_0^-,t_0^-v_0^-) < \lim_{n\to\infty} J_{\lambda,\mu}(t_0^-u_n,t_0^-v_n) \leq \lim_{n\to\infty} J_{\lambda,\mu}(u_n,v_n) = \theta_{\lambda,\mu}^-,$$

and this is contradiction. Hence

$$\begin{aligned}
&u_n \to u_0^-, \text{ strongly in } H^1(\Omega),\\
&v_n \to v_0^-, \text{ strongly in } H^1(\Omega).
\end{aligned}$$

This implies

$$J_{\lambda,\mu}(u_n,v_n) \to J_{\lambda,\mu}(u_0^-,v_0^-) = \theta_{\lambda,\mu}^- \text{ as } n\to\infty.$$

Since $J_{\lambda,\mu}(u_0^-,v_0^-) = J_{\lambda,\mu}(|u_0^-|,|v_0^-|)$ and $(|u_0^-|,|v_0^-|) \in N_{\lambda,\mu}^-$, by Lemma 2.4 we may assume that (u_0^-,v_0^-) is a nonnegative solution of (1). □

Now, we complete the proof of Theorem 1.1: By Theorem 3.1 and 3.2, problem (1.1) has two nonnegative solutions $(u_0^+,v_0^+)\in N_{\lambda,\mu}^+$ and $(u_0^-,v_0^-)\in N_{\lambda,\mu}^-$. Since $N_{\lambda,\mu}^+ \cap N_{\lambda,\mu}^- = \emptyset$, this implies (u_0^+,v_0^+) and (u_0^-,v_0^-) are distinct.

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