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ON THE UNIQUENESS THEORY OF ALGEBROID FUNCTIONS IN SOME PROPER DOMAIN

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Abstract. We consider the uniqueness problem of algebroid functions on an angular domain. Several theorems are established to extend the uniqueness theory of meromorphic functions to algebroid functions.
Keywords: Algebroid functions, Nevanlinna theory, angular domain, uniqueness.
MSC(2010): Primary: 30D1; Secondary: 30D20, 30B10, 34M05.

1. **Introduction and main results**

We assume that the readers are familiar with the fundamental results and standard notations of the Nevanlinna theory in the unit disk $\Delta = \{z : |z| < 1\}$ and in the complex plane $\mathbb{C}$ (see [2,17]). The value distribution theory of meromorphic functions introduced by Nevanlinna was extended to the corresponding theory of algebroid functions by Selberg [13], Ullrich [10] and Valiron [11, 12] around 1930. For the uniqueness theory of algebroid functions, Valiron [11], Baganas [1], He [4], Sun [14], Xuan [16] and others have already done a lot of work.

Zheng [19, 20] was the first one who considered the uniqueness of two meromorphic functions dealing with shared values in a proper subset of $\mathbb{C}$. After Zheng's work, many authors have investigated the uniqueness of meromorphic functions in angular domains, such as Lin, Mori and Tohge [5], Lin, Mori and Yi [6], Liu and Sun [7], Mao and Liu [9]. In 2010, Liu and Sun [8] studied the uniqueness theorem of algebroid functions in an angular domain.

In this paper, we will get some uniqueness theorems of algebroid functions defined on an angular domain and extend some uniqueness theorems of meromorphic functions given by Yi [18] and Zheng [19] to algebroid functions.

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Before stating the results, we give some notations and definitions of an algebroid function. Let $A_0(z), \cdots, A_\nu(z)$ be analytic functions with no common zeros in the complex plane. Then the following equation

$$A_\nu(z)W^\nu + A_{\nu-1}(z)W^{\nu-1} + \cdots + A_1(z)W + A_0(z) = 0 \tag{1.1}$$

defines a ν-valued algebroid function $W(z)$.

Let $W(z)$ be a ν-valued algebroid function and $a \in \widehat{\mathbb{C}}$ be a finite or infinite complex number. $\overline{E}_{t)}(W = a)$ denotes the set of zeros of $W(z) - a$, whose multiplicities are not greater than t. Denote by $\overline{n}_{t)}(W = a)$ the number of distinct zeros of $W(z) - a$ in $\{z : |z| \le r\}$, whose multiplicities are not greater than t and are counted only once. Similarly, we define the functions $\overline{n}_{(t+1}(W = a)$, $\overline{N}_{t)}(W = a)$ and $\overline{N}_{(t+1}(W = a)$. Denote by $\overline{E}(a, X, W)$ the set of zeros of $W(z) - a$ in a subset X of the complex plane $\mathbb{C}$.

Next we recall some definitions of a ν-valued algebroid function.

Definition 1.1. *Let $W(z)$ be a ν-valued algebroid function in the disc $\{z : |z| < R\}, 0 < R \le \infty$. Define*

$$m(r, W) = \frac{1}{2\pi\nu} \sum_{j=1}^{\nu} \int_0^{2\pi} \log^+ |W_j(re^{i\theta})| d\theta,$$
$$N(r, W) = \frac{1}{\nu} \int_0^r \frac{n(t, W) - n(0, W)}{t} dt + \frac{n(0, W)}{\nu} \log r,$$
$$T(r, W) = m(r, W) + N(r, W).$$

Definition 1.2. *Let $W(z)$ be a ν-valued algebroid function in the unit disc $\triangle$. We call $W(z)$ an admissible ν-valued algebroid function in the unit disc, if*

$$\limsup_{r \to 1-} \frac{T(r, W)}{\log \frac{1}{1-r}} = +\infty.$$

Recently, Sun and Gao [14, 15] gave the definitions of $+, \times$ and the inverse of $+, \times$ of two algebroid functions as follows:

Definition 1.3. *Let*

$$W(z) = \{w_j(z), b\}_{j=1}^{\nu}$$

and

$$M(z) = \{m_j(z), b\}_{j=1}^{k}$$

be ν-valued and k-valued algebroid functions respectively. Define the operations of $W(z)$ and $M(z)$ as follows:

$$-W(z) = \{-w_j(z), b\}_{j=1}^{\nu};$$
$$W^{-1}(z) = \{w_j^{-1}(z), b\}_{j=1}^{\nu};$$
$$W(z) \pm M(z) = \{(w \pm m)_j(z), b\}_{j=1}^{\nu k} = \{(w_j(z) \pm m_s(z), b); j = 1, 2, \cdots, \nu, s = 1, 2, \cdots, k\};$$
$$W(z) \cdot M(z) = \{(w \cdot m)_j(z), b\}_{j=1}^{\nu k} = \{(w_j(z) \cdot m_s(z), b); j = 1, 2, \cdots, \nu, s = 1, 2, \cdots, k\}.$$

The single-valued domain $\widetilde{R}_z$ of definition of $W(z)$ is a ν-valued covering of the z-plane and it is a Riemann surface. A point in $\widetilde{R}_z$ is denoted by $\widetilde{z}$ if its projection in the $z-$plane is z. The open set which lies over $|z|<r$ is denoted by $|\widetilde{z}|<r$. Given an angular domain $X=\{z:\alpha<\arg z<\beta\}$, and write $\widetilde{X}$ for the part of $\widetilde{R}_z$ on X, we can define the Ahlfors-Shimizu characteristic function of a ν-valued algebroid function as follows:

$$\mathcal{S}(r,X,W(z))=\frac{1}{\pi}\int_0^r\int_\alpha^\beta\left(\frac{|W'(z)|}{1+|W(z)|^2}\right)^2 tdtd\theta, z=te^{i\theta},$$

$$\mathcal{T}(r,X,W(z))=\frac{1}{\nu}\int_0^r\frac{\mathcal{S}(t,X,W(z))}{t}dt.$$

We have known that $\mathcal{T}(r,\mathbb{C},W(z))=T(r,W(z))+O(1)$.

In this article, we mainly obtain the following theorems.

Theorem 1.4. *Let $W(z)$ and $M(z)$ be two ν-valued algebroid functions in the complex plane and let $a_1,a_2,\cdots,a_{4\nu+1}$ be distinct complex numbers. Assume that on an angular domain $X=\{z:\alpha<\arg z<\beta\}$ with $0\le\alpha<\beta\le 2\pi$,*

$$\overline{E}(a_j,X,W(z))=\overline{E}(a_j,X,M(z))(j=1,2,\cdots,4\nu+1),$$

and for some $0<\varepsilon<(\beta-\alpha)/2$,

$$\limsup_{r\to\infty}\frac{\mathcal{S}(r,X_\varepsilon,W)}{r^\omega}=\infty \tag{1.2}$$

hold, where $X_\varepsilon=\{z:\alpha+\varepsilon<\arg z<\beta-\varepsilon\}$ and $\omega=\pi/(\beta-\alpha)$. Then $W(z)\equiv M(z)$.

Theorem 1.5. *Let $W(z)$ and $M(z)$ be two ν-valued algebroid functions in the complex plane and let $a_1,a_2,\cdots,a_{4\nu+1}$ be distinct complex numbers. Assume that on an angular domain $X=\{z:\alpha<\arg z<\beta\}$ with $0\le\alpha<\beta\le 2\pi$,*

$$\overline{E}(a_j,X,W(z))=\overline{E}(a_j,X,M(z))(j=1,2,\cdots,4\nu+1),$$

and for some $0<\varepsilon<(\beta-\alpha)/2$,

$$\limsup_{r\to\infty}\frac{\mathcal{T}(r,X_\varepsilon,W)}{r^\omega\log r}=\infty \tag{1.3}$$

hold, where $X_\varepsilon=\{z:\alpha+\varepsilon<\arg z<\beta-\varepsilon\}$ and $\omega=\pi/(\beta-\alpha)$. Then $W(z)\equiv M(z)$.

Theorem 1.6. *Let $W(z)$ and $M(z)$ be two ν-valued algebroid functions in the complex plane and let $a_1,a_2,\cdots,a_{4\nu+1}$ be distinct complex numbers. Assume that on an angular domain $X=\{z:\alpha<\arg z<\beta\}$ with $0\le\alpha<\beta\le 2\pi$,*

$$\overline{E}(a_j,X,W(z))=\overline{E}(a_j,X,M(z))\qquad(j=1,2,\cdots,4\nu+1)$$

and for some positive number $0<\varepsilon<(\beta-\alpha)/2$ and for some $a\in\widehat{\mathbb{C}}$

$$\limsup_{r\to\infty}\frac{n(r,X_\varepsilon,W=a)}{r^\omega\log r}=\infty \tag{1.4}$$

hold, where $n(r,X_\varepsilon,W=a)$ *is the number of the roots of* $W(z)=a$ *in* $\{|z|<r\}\cap X_\varepsilon$, $X_\varepsilon=\{z:\alpha+\varepsilon<\arg z<\beta-\varepsilon\}$ *and* $\omega=\pi/(\beta-\alpha)$. *Then* $W(z)\equiv M(z)$.

In the case of multiple values, we get the following:

Theorem 1.7. *Let* $W(z)$ *and* $M(z)$ *be two* ν*-valued algebroid functions in the complex plane and let* $a_1,a_2,\cdots,a_q$ *be* q *distinct complex numbers. Assume that on an angular domain* X, $W(z)$ *satisfies the condition* (1.2) , (1.3) *or* (1.4), *and* $t_j(j=1,2,\cdots,q)$ *are* q *positive integers or* ∞, *such that* $t_1\geq t_2\geq\cdots\geq t_q$, *and*

$$\overline{E}_{t_j)}(a_j,X,W(z))=\overline{E}_{t_j)}(a_j,X,M(z))(j=1,2,\cdots,q),$$

and

$$\sum_{j=2\nu+1}^{q}\frac{t_j}{t_j+1}>2\nu.$$

Then $W(z)\equiv M(z)$.

2. Some lemmas

We need some lemmas for the proofs of the theorems.

Lemma 2.1. *[14, 15] Let* $W(z)$ *be a* ν*-valued algebroid function and* $M(z)$ *be a* k*-valued algebroid function. Suppose that there are no poles included in the sets* $W(0)$ *and* $M(0)$, *and* $a\in\widehat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$. *Then*

$$T(r,W\pm M)\leq T(r,W)+T(r,M)+\log 2,$$
$$T\left(r,\frac{1}{W-a}\right)=T(r,W)+O(1).$$

Lemma 2.2. *[3] Let* $W(z)$ *be a* ν*-valued algebroid function in the unit disc* $\triangle$ *and* $a_j(j=1,2,\cdots,q)$ *be* q *distinct complex numbers (finite or infinite). Then*

$$(q-2\nu)T(r,W)<\sum_{j=1}^{q}\overline{N}(r,W=a_j)+S(r,W),$$

where $S(r,W)=O(\log(\frac{1}{1-r}T(r,W)))$ *outside a possible exceptional set* E *with* $\int_E\frac{dr}{1-r}<\infty$.

Lemma 2.3. *[14] Let $W(z)$ be a ν-valued algebroid function in the unit disc $\triangle$ and $a_j (j = 1, 2, \cdots, q)$ be q distinct complex numbers, $t_j (j = 1, 2, \cdots, q)$ be q positive integers. Then*

$$(q-2\nu)T(r,W) < \sum_{j=1}^{q} \frac{t_j}{t_j+1} \overline{N}_{t_j)}(r, W = a_j) + \sum_{j=1}^{q} \frac{1}{t_j+1} N(r, W = a_j) + S(r, W),$$

$$(q - 2\nu - \sum_{j=1}^{q} \frac{1}{t_j+1})T(r,W) < \sum_{j=1}^{q} \frac{t_j}{t_j+1} \overline{N}_{t_j)}(r, W = a_j) + S(r, W),$$

where $S(r, W) = O(\log(\frac{1}{1-r}T(r, W)))$ outside a possible exceptional set E with $\int_E \frac{dr}{1-r} < \infty$.

Now, we are in the position to give the main lemma of this paper.

Lemma 2.4. *Let $W(z)$ and $M(z)$ be two ν-valued admissible algebroid functions in the unit disc $\triangle$, and $a_1, a_2, \cdots, a_q$ be q distinct complex numbers. If $t_j (j = 1, 2, \cdots, q)$ are positive integer numbers or ∞, such that*

$$t_1 \geq t_2 \geq \cdots \geq t_q, \tag{2.1}$$

and

$$\overline{E}_{t_j)}(a_j, \Delta, W(z)) = \overline{E}_{t_j)}(a_j, \Delta, M(z))(j = 1, 2, \cdots, q), \tag{2.2}$$

and

$$\sum_{j=2\nu+1}^{q} \frac{t_j}{t_j+1} > 2\nu, \tag{2.3}$$

then $W(z) \equiv M(z)$.

Proof. Firstly, we suppose that $a_j (j = 1, 2, \cdots, q)$ are all finite complex numbers. Let $W(z) \not\equiv M(z)$. Note that

$$\sum_{j=1}^{q} \overline{n}_{t_j)}(r, W = a_j) = \sum_{j=1}^{q} \overline{n}_{t_j)}(r, M = a_j) \leq n(r, W - M = 0).$$

Hence

$$\sum_{j=1}^{q} \overline{N}_{t_j)}(r, W = a_j) \leq \nu N(r, W - M = 0),$$

$$\sum_{j=1}^{q} \overline{N}_{t_j)}(r, M = a_j) \leq \nu N(r, W - M = 0).$$

We can assume that 0 is not the pole of $W(z)$ and $M(z)$. Otherwise, we can multiply by a proper factor of z^n. By using Lemma 2.1, we get

$$\begin{aligned}\sum_{j=1}^{q}\overline{N}_{t_j)}(r,W=a_j)+\sum_{j=1}^{q}\overline{N}_{t_j)}(r,M=a_j) &< 2\nu N(r,W-M=0)\\ &\le 2\nu T(r,W-M)+O(1)\\ &\le 2\nu\{T(r,W)+T(r,M)\}+O(1).\end{aligned}$$

Applying Lemma 2.3 to $W(z)$, we obtain

$$\left(q-2\nu-\sum_{j=1}^{q}\frac{1}{t_j+1}\right)T(r,W)<\sum_{j=1}^{q}\frac{t_j}{t_j+1}\overline{N}_{t_j)}(r,W=a_j)+S(r,W). \tag{2.4}$$

Now, (2.1) implies that

$$1\ge\frac{t_1}{t_1+1}\ge\frac{t_2}{t_2+1}\ge\cdots\ge\frac{t_q}{t_q+1}\ge\frac{1}{2}.$$

Combining (2.1) with (2.4), we have

$$\begin{aligned}\left(\sum_{j=1}^{q}\frac{t_j}{t_j+1}-2\nu\right)T(r,W) &< \frac{t_{2\nu}}{t_{2\nu}+1}\sum_{j=1}^{q}\overline{N}_{t_j)}(r,W=a_j)\\ &\quad+\sum_{j=1}^{q}\left(\frac{t_j}{t_j+1}-\frac{t_{2\nu}}{t_{2\nu}+1}\right)\overline{N}_{t_j)}(r,W=a_j)+S(r,W)\\ &< \frac{t_{2\nu}}{t_{2\nu}+1}\sum_{j=1}^{q}\overline{N}_{t_j)}(r,W=a_j)\\ &\quad+T(r,W)\sum_{j=1}^{2\nu-1}\left(\frac{t_j}{t_j+1}-\frac{t_{2\nu}}{t_{2\nu}+1}\right)+S(r,W).\end{aligned}$$

Then we get

$$\left(\sum_{j=2\nu+1}^{q}\frac{t_j}{t_j+1}+\frac{2\nu t_{2\nu}}{t_{2\nu}+1}-2\nu\right)T(r,W)<\frac{t_{2\nu}}{t_{2\nu}+1}\sum_{j=1}^{q}\overline{N}_{t_j)}(r,W=a_j)+S(r,W). \tag{2.5}$$

We can also obtain

$$\left(\sum_{j=2\nu+1}^{q}\frac{t_j}{t_j+1}+\frac{2\nu t_{2\nu}}{t_{2\nu}+1}-2\nu\right)T(r,M)<\frac{t_{2\nu}}{t_{2\nu}+1}\sum_{j=1}^{q}\overline{N}_{t_j)}(r,M=a_j)+S(r,M). \tag{2.6}$$

Combining (2.5) with (2.6), we obtain

$$\left(\sum_{j=2\nu+1}^{q}\frac{t_j}{t_j+1}-2\nu\right)(T(r,W)+T(r,M))<S(r,W)+S(r,M). \tag{2.7}$$

But (2.7) contradicts (2.3). Thus $W(z)\equiv M(z)$.

Secondly, suppose one of the complex numbers $a_j(j=1,\dots,q)$ equals ∞. Without loss of generality, we suppose that $a_q=\infty$.

Take a finite number

$$c\neq a_j \quad (j=1,\dots,q),$$

and set

$$F(z)=\frac{1}{W(z)-c},\quad G(z)=\frac{1}{M(z)-c},$$

$$b_j=\frac{1}{a_j-c}\quad (j=1,\dots,q-1),\quad b_q=0.$$

Then we have

$$\overline{E}_{t_j)}(a_j,\Delta,F(z))=\overline{E}_{t_j)}(a_j,\Delta,G(z)),\quad (j=1,2,\cdots,q).$$

From the above proof, we have $F(z)\equiv G(z)$. Therefore, $W(z)\equiv M(z)$. now the proof is completed $\square$

Letting $t_1=t_2=\ldots=t_{4\nu+1}=+\infty$, $q=4\nu+1$ in Lemma 2.4, we have the following result.

Lemma 2.5. *Let $W(z)$ and $M(z)$ be two ν-valued admissible algebroid functions in the unit disc $\triangle$. Suppose that $a_1,\cdots,a_{4\nu+1}$ are distinct complex numbers. If*

$$\overline{E}(a_j,\Delta,W(z))=\overline{E}(a_j,\Delta,M(z))$$

holds in the unit disc, then $W(z)\equiv M(z)$.

3. Proof of Theorem ??.

Proof. Put $\theta_0=\frac{\alpha+\beta}{2}$. The function

$$\zeta(z)=\frac{(ze^{-i\theta_0})^{\pi/(\beta-\alpha)}-1}{(ze^{-i\theta_0})^{\pi/(\beta-\alpha)}+1} \tag{3.1}$$

maps X conformally onto the unit disc $|\zeta|<1$ and maps $z=e^{i\theta_0}$ to $\omega=0$. Let $z=pe^{i\theta}\in\{z:z\in X_\varepsilon,1\leq|z|\leq r\}$. Then

$$|\zeta|=\sqrt{1-\frac{4p^{\frac{\pi}{\beta-\alpha}}\cos\frac{\pi}{\beta-\alpha}(\theta-\theta_0)}{p^{\frac{2\pi}{(\beta-\alpha)}}+2p^{\frac{\pi}{\beta-\alpha}}\cos\frac{\pi}{\beta-\alpha}(\theta-\theta_0)+1}}. \tag{3.2}$$

Notice that

$$p^{\frac{2\pi}{(\beta-\alpha)}}+2p^{\frac{\pi}{\beta-\alpha}}\cos\frac{\pi}{\beta-\alpha}(\theta-\theta_0)+1\leq 4p^{\frac{2\pi}{\beta-\alpha}},$$

$$4p^{\frac{\pi}{\beta-\alpha}}\cos\frac{\pi}{\beta-\alpha}(\theta-\theta_0)\geq 4p^{\frac{\pi}{\beta-\alpha}}\cos\frac{\pi}{\beta-\alpha}(\alpha+\varepsilon-\theta_0)\geq\frac{8\varepsilon}{\beta-\alpha}p^{\frac{\pi}{\beta-\alpha}}.$$

Then we have

$$|\zeta| \leq \sqrt{1-\frac{\frac{8\varepsilon}{\beta-\alpha}p^{\frac{\pi}{\beta-\alpha}}}{4p^{\frac{2\pi}{(\beta-\alpha)}}}} = \sqrt{1-2\frac{\varepsilon}{\beta-\alpha}p^{-\frac{\pi}{\beta-\alpha}}} \leq 1-\frac{\varepsilon}{\beta-\alpha}r^{-\frac{\pi}{\beta-\alpha}}. \tag{3.3}$$

Put $\eta = \frac{\varepsilon}{\beta-\alpha}, \omega = \frac{\pi}{\beta-\alpha}$. Then

$$\mathcal{S}(1-\eta r^{-\omega}, X, W(z(\zeta))) \geq \mathcal{S}(r, X_\varepsilon, W(z)).$$

Write $t = 1-\eta r^{-\omega}$. Then it follows that

$$\begin{aligned}\limsup_{t\to 1-}\frac{\mathcal{S}(t,\mathbb{C},W(z(\zeta)))}{\frac{1}{1-t}} &= \limsup_{r\to\infty}\frac{\eta\mathcal{S}(1-\eta r^{-\omega},\mathbb{C},W(z(\zeta)))}{r^{\omega}}\\ &\geq \limsup_{r\to\infty}\frac{\eta\mathcal{S}(r,X_\varepsilon,W(z))}{r^{\omega}} = +\infty.\end{aligned}$$

It is obvious that the relation $\limsup_{t\to 1-}\frac{\mathcal{S}(t,\mathbb{C},W(z(\zeta)))}{\frac{1}{1-t}} = +\infty$ implies

$$\limsup_{t\to 1-}\frac{\mathcal{T}(t,\mathbb{C},W(z(\zeta)))}{\log\frac{1}{1-t}} = +\infty.$$

Since $\mathcal{T}(t,\mathbb{C},W(z(\zeta))) = T(t,W(z(\zeta)))) + O(1)$, $W(z(\zeta))$ is admissible. It follows from

$$\overline{E}(a_j, W) = \overline{E}(a_j, M)$$

that

$$\sum_{j=1}^{4\nu+1}\overline{n}(r, W=a_j) = \sum_{j=1}^{4\nu+1}\overline{n}(r, M=a_j).$$

Hence

$$\sum_{j=1}^{4\nu+1}\overline{N}(r, W=a_j) = \sum_{j=1}^{4\nu+1}\overline{N}(r, M=a_j).$$

From Lemma 2.2, we have

$$\begin{aligned}(2\nu+1)T(r,W) &< \sum_{j=1}^{4\nu+1}N(r,W=a_j) + O\left(\log\frac{1}{1-r}+\log T(r,W)\right)\\ &= \sum_{j=1}^{4\nu+1}N(r,M=a_j) + O\left(\log\frac{1}{1-r}+\log T(r,W)\right)\\ &\leq (4\nu+1)T(r,M) + O\left(\log\frac{1}{1-r}+\log T(r,W)\right).\end{aligned} \tag{3.4}$$

Consequently, $M(z(\zeta))$ is admissible in $|\zeta|<1$. Since

$$\overline{E}(a_j, X, W(z)) = \overline{E}(a_j, X, M(z)),$$

it follows that

$$\overline{E}(a_j, \Delta, W(z(\zeta))) = \overline{E}(a_j, \Delta, M(z(\zeta))).$$

By Lemma 2.5, we obtain that $W(z(\zeta)) = M(z(\zeta))$. Thus the equality $W(z) \equiv M(z)$ holds in X. Then $W(z) \equiv M(z)$ in $\mathbb{C}$. □

4. **Proof of Theorem ??.**

Proof. We can apply the method as in Theorem 1.1 to prove Theorem 1.2. We only prove that $W(z(\zeta))$ is an admissible algebroid function in the unit disc. From (3.3), we have

$$\mathcal{S}(1-\eta r^{-\omega}, W(z(\zeta))) \geq \mathcal{S}(r, X_\varepsilon, W(z))$$

and

$$\begin{aligned}\int_1^r \frac{\mathcal{S}(t, X_\varepsilon, W(z))}{t} dt &\leq \int_1^r \frac{\mathcal{S}(1-\eta t^{-\omega}, \mathbb{C}, W(z(\zeta)))}{t} dt \\ &= \frac{1}{\omega} \int_{1-\eta}^{1-\eta r^{-\omega}} \frac{\mathcal{S}(x, \mathbb{C}, W(z(\zeta)))}{1-x} dx \\ &\leq \frac{r^\omega}{\omega\eta} \int_{1-\eta}^{1-\eta r^{-\omega}} \mathcal{S}(x, \mathbb{C}, W(z(\zeta))) dx \\ &\leq \frac{r^\omega}{\omega\eta} \int_{1-\eta}^{1-\eta r^{-\omega}} \frac{\mathcal{S}(x, \mathbb{C}, W(z(\zeta)))}{x} dx.\end{aligned} \tag{4.1}$$

Hence

$$\frac{\mathcal{T}(r, X_\varepsilon, f)}{r^\omega} \leq \frac{\mathcal{T}(1-\eta r^{-\omega}, \mathbb{C}, W(z(\zeta)))}{\omega\eta}.$$

As a result, we can obtain the following

$$\begin{aligned}\limsup_{t\to 1-} \frac{T(t, W)}{\log(1-t)^{-1}} &= \limsup_{t\to 1-} \frac{\mathcal{T}(t, \mathbb{C}, W)}{\log(1-t)^{-1}} = \limsup_{r\to\infty} \frac{\mathcal{T}(1-\eta r^{-\omega}, \mathbb{C}, W)}{\omega \log r - \log \omega\eta} \\ &\geq \limsup_{r\to\infty} \frac{\mathcal{T}(1-\eta r^{-\omega}, \mathbb{C}, W)}{\omega \log r - \log \omega\eta} \\ &\geq \limsup_{r\to\infty} \frac{\eta \mathcal{T}(r, X_\varepsilon, W)}{r^\omega(\omega \log r - \log \omega\eta)} = +\infty.\end{aligned} \tag{4.2}$$

Thus $T(t, W(z(\zeta)))$ is admissible in the unit disc. □

5. **Proof of Theorem 1.6.**

Proof. We can prove this theorem by the same method used in Theorem 1.1. Here we only prove that $W(z(\zeta))$ is an admissible algebroid function in the unit disc. By (3.3), we have

$$n(1-\eta r^{-\omega}, W(z(\zeta)) = a) \geq n(r, X_\varepsilon, W(z) = a).$$

Let $y = 1 - \eta r^{-\omega}, y' = 1 - \eta(2r)^{-\omega}$. Then

$$\begin{aligned} T(y', W(z(\zeta))) &> N(y', W(z(\zeta)) = a) + O(1) \\ &> \int_y^{y'} \frac{n(t, W(z(\zeta)) = a)}{t} dt + O(1) \\ &= n(y, W(z(\zeta)) = a) \log \frac{y'}{y} + O(1) \\ &= \frac{n(r, X_\varepsilon, W(z(\zeta)) = a)}{r^\omega}. \end{aligned} \tag{5.1}$$

We can prove that

$$\begin{aligned} \limsup_{y \to 1-} \frac{T(y, W(z(\zeta)))}{\log(1-y)^{-1}} &\geq \limsup_{y' \to 1-} \frac{T(y', W(z(\zeta)))}{\log(1-y')^{-1}} \\ &\geq \limsup_{r \to \infty} \frac{n(r, X_\varepsilon, W(z) = a)}{r^\omega[\omega \log(2r) - \log \eta]} = +\infty. \end{aligned}$$

Therefore, $W(z(\zeta))$ is an admissible algebroid function in the unit disc. □

6. **Proof of Theorem 1.7.**

Proof. First, by a similar argument as in Theorem 1.1 and Theorem 1.2, we can prove that the function $W(z(\omega))$ is admissible in the unit disc. Since

$$\overline{E}_{t_j)}(a_j, \Delta, W(z(\zeta))) = \overline{E}_{t_j)}(a_j, \Delta, M(z(\zeta))), (j = 1, 2, \cdots, q),$$

we have

$$\begin{aligned} \sum_{j=1}^q \overline{N}_{t_j)}(r, W(z(\zeta)) = a_j) &= \sum_{j=1}^q \overline{N}_{t_j)}(r, M(z(\zeta)) = a_j) \\ &\leq 2\nu T(r, M(z(\zeta))) + O(1). \end{aligned}$$

Combining the above with (2.5), we get

$$\begin{aligned} &\left(\sum_{j=2\nu+1}^q \frac{t_j}{t_j + 1} + \frac{2\nu t_{2\nu}}{t_{2\nu} + 1} - 2\nu \right) T(r, W) \\ &\qquad < \frac{t_{2\nu}}{t_{2\nu} + 1} \sum_{j=1}^q \overline{N}_{t_j)}(r, W = a_j) + S(r, W) \\ &\qquad \leq \frac{2\nu t_{2\nu}}{t_{2\nu} + 1} T(r, M(z(\zeta))) + S(r, W). \end{aligned}$$

Thus

$$\frac{2\nu t_{2\nu}}{t_{2\nu} + 1} T(r, W(z(\zeta))) < \frac{2\nu t_{2\nu}}{t_{2\nu} + 1} T(r, M(z(\zeta))) + S(r, W).$$

Hence $M(z(\omega))$ is also admissible in the unit disc. By using Lemma 2.4, we get $W(z) \equiv M(z)$. This completes the proof. □

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