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AN UPPER BOUND TO THE SECOND HANKEL FUNCTIONAL FOR THE CLASS OF GAMMA-STARLIKE FUNCTIONS

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Abstract. The objective of this paper is to obtain an upper bound to the second Hankel determinant $|a_2a_4 - a_3^2|$ for the function f, belonging to the class of Gamma-starlike functions, using Toeplitz determinants. The result presented here include two known results as their special cases.
Keywords: Analytic function, gamma-starlike function, second Hankel functional, positive real function, Toeplitz determinants.
MSC(2010): Primary: 30C45, Secondary: 30C50.

1. Introduction

Let A denote the class of analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions.

The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [23] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.2}$$

This determinant has been considered by several authors in the literature. For example, Noonan and Thomas [20] studied the second Hankel determinant of areally mean p-valent functions. In [21], Noor determined the rate of growth of

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$H_q(n)$ as $n \to \infty$ for the functions in S with a bounded boundary. Ehrenborg [7] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [13]. One can easily observe that the Fekete-Szegö functional is $H_2(1)$. Fekete-Szegö then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real and $f \in S$. Ali [2] found sharp bounds for the first four coefficients and sharp estimate for the Fekete-Szegö functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f defined as $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ to the class of strongly starlike functions of order α $(0 < \alpha \leq 1)$ denoted by $\widetilde{ST}(\alpha)$. In this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$, known as the second Hankel determinant, given by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2. \tag{1.3}$$

Janteng, Halim and Darus [12] considered the functional $|a_2a_4 - a_3^2|$ and found a sharp upper bound for the function f in the subclass RT of S, consisting of functions whose derivative has a positive real part studied by Mac Gregor [16]. In their work, they have shown that if $f \in RT$ then $|a_2a_4 - a_3^2| \leq \frac{4}{9}$. Janteng, Halim and Darus [11] also obtained the second Hankel determinant and sharp bounds for the familiar subclasses of S, namely, starlike and convex functions denoted by ST and CV and have shown that $|a_2a_4 - a_3^2| \leq 1$ and $|a_2a_4 - a_3^2| \leq \frac{1}{8}$ respectively. Similarly, the same coefficient inequality was calculated for certain subclasses of univalent and multivalent analytic functions by many authors ([1, 3, 4, 10, 17–19]) in the literature.

Motivated by the above mentioned results obtained by different authors in this direction, in the present paper, we consider the subclass β-convex of S and obtain an upper bound to the functional $|a_2a_4 - a_3^2|$ for the function f which belongs to this class.

Definition 1.1. A function $f \in A$ is said to be Gamma-starlike function, denoted by $f \in ST_\gamma$ $(0 \leq \gamma \leq 1)$, if and only if

$$Re\left\{\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\gamma}\right\} \geq 0, \; for\ all\ z \in E, \tag{1.4}$$

where the powers are meant for principal values. This class was defined and studied by Lewandowski, Miller and Zlotkiewicz [14]. It is observed that for $\gamma = 0$ and $\gamma = 1$, we get $ST_0 = ST$ and $ST_1 = CV$, respectively. Furthermore they have obtained the Fekete-Szegö inequality for the function f belonging to this class. Darus and Thomas [5] investigated this class and proved that the functions in it are starlike.

Some preliminary lemmas required in proving our result are as follows:

2. Preliminary results

Let $\mathscr{P}$ denote the class of functions consisting of p, such that

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... = \left[1 + \sum_{n=1}^{\infty} c_n z^n\right], \tag{2.1}$$

which are regular in the open unit disc E and satisfies $\mathrm{Re}\{p(z)\} > 0$ for any $z \in E$. Here $p(z)$ is called the Carathe`odory function [6].

Lemma 2.1. *([22, 24]) If $p \in \mathscr{P}$, then $|c_k| \leq 2$ for each $k \geq 1$ and the inequality is sharp for the function $\left(\frac{1+z}{1-z}\right)$.*

Lemma 2.2. *([9]) The power series for $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ given in (2.1) converges in the open unit disc E to a function in $\mathscr{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, n = 1, 2, 3, \cdots$$

are all non-negative $c_{-k} = \bar{c}_k$. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k P_0(e^{it_k} z)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $P_0(z) = \left(\frac{1+z}{1-z}\right)$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

This necessary and sufficient condition found in [9] is due to Carathe`odory and Toeplitz. We may assume without restriction that $c_1 > 0$. On using Lemma 2.2, for $n = 2$, we have

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \bar{c}_1 & 2 & c_1 \\ \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix} = [8 + 2Re\{c_1^2 c_2\} - 2 \mid c_2 \mid^2 - 4|c_1|^2] \geq 0,$$

which is equivalent to

$$2c_2 = \{c_1^2 + x(4 - c_1^2)\}, \text{ for some } x, |x| \leq 1. \tag{2.2}$$

For $n = 3$,

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \bar{c}_1 & 2 & c_1 & c_2 \\ \bar{c}_2 & \bar{c}_1 & 2 & c_1 \\ \bar{c}_3 & \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix} \geq 0$$

and it is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{2.3}$$

From the relations (2.2) and (2.3), after simplifying, we get

$$4c_3 = \{c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z\} \tag{2.4}$$

for some real value of z, with $|z| \leq 1$.

To obtain our result, we refer to the classical method initiated by Libera and Zlotkiewicz [15] and used by several authors in the literature.

3. Main result

Theorem 3.1. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in ST_\gamma$ $(0 \leq \gamma \leq 1)$ *then*

$$|a_2a_4 - a_3^2| \leq \left[\frac{(112\gamma^5 + 768\gamma^4 + 2263\gamma^3 + 1700\gamma^2 + 372\gamma - 4)}{(1 + 2\gamma)^2(1 + 3\gamma)(37\gamma^4 + 253\gamma^3 + 603\gamma^2 + 263\gamma - 4)}\right].$$

Proof. For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in ST_\gamma$, there exists an analytic function $p \in \mathscr{P}$ in the open unit disc E with $p(0) = 1$ and $\operatorname{Re}\{p(z)\} > 0$ such that

$$\left[\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\gamma}\right] = p(z). \tag{3.1}$$

Using the series representations for $f(z)$, $f'(z)$ and $f''(z)$, we have

$$\left(\frac{zf'(z)}{f(z)}\right) = \left[z\left\{1 + \sum_{n=2}^{\infty} na_n z^{n-1}\right\} \times \left\{z + \sum_{n=2}^{\infty} a_n z^n\right\}^{-1}\right]. \tag{3.2}$$

$$\left(1 + \frac{zf''(z)}{f'(z)}\right) = \left[\left\{1 + \sum_{n=2}^{\infty} n^2 a_n z^{n-1}\right\} \times \left\{1 + \sum_{n=2}^{\infty} na_n z^{n-1}\right\}^{-1}\right]. \tag{3.3}$$

Applying the binomial expansion on the right-hand side of (3.2) and (3.3) subject to the conditions $|\sum_{n=2}^{\infty} a_n z^n| < 1$ and $\left|\sum_{n=2}^{\infty} na_n z^{n-1}\right| < 1$, gives

$$\left(\frac{zf'(z)}{f(z)}\right) = \left\{1 + a_2 z + (2a_3 - a_2^2)z^2 + (3a_4 - 3a_2a_3 + a_2^3)z^3 + \cdots\right\} \tag{3.4}$$

and

$$\left(1 + \frac{zf''(z)}{f'(z)}\right) = \left\{1 + 2a_2 z + (6a_3 - 4a_2^2)z^2 + (12a_4 - 18a_2a_3 + 8a_2^3)z^3 + \cdots\right\}. \tag{3.5}$$

Similarly, using the binomial expansion and simplifying the expressions (3.4) and (3.5), we obtain

$$\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma} = \Big[1 + (1-\gamma)a_2 z + \left\{2(1-\gamma)a_3 - \frac{(1-\gamma)(2+\gamma)}{2}a_2^2\right\} z^2 + \left\{3(1-\gamma)a_4 - (1-\gamma)(3+2\gamma)a_2a_3 + \frac{(1-\gamma)(1+\gamma)(6+\gamma)}{6}a_2^3\right\} z^3 + \cdots\Big] \tag{3.6}$$

and

$$\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma} = \Big[1 + 2a_2\gamma z + \left\{6a_3\gamma + 2(\gamma^2 - 3\gamma)a_2^2\right\} z^2 + \left\{12a_4\gamma + 3(4\gamma^2 - 10\gamma)a_2a_3 + \frac{4}{3}\gamma(\gamma-2)(\gamma-7)a_2^3\right\} z^3 + \cdots\Big]. \tag{3.7}$$

Since $f \in ST_\gamma$, there exists $p(z)$ of the form (2.1) whereby the product of (3.6) and (3.7) equates to $p(z)$. Thus, we have

$$\Big[1 + (1-\gamma)a_2 z + \left\{2(1-\gamma)a_3 - \frac{(1-\gamma)(2+\gamma)}{2}a_2^2\right\} z^2 + \left\{3(1-\gamma)a_4 - (1-\gamma)(3+2\gamma)a_2a_3 + \frac{(1-\gamma)(1+\gamma)(6+\gamma)}{6}a_2^3\right\} z^3 + \cdots\Big] \times \Big[1 + 2a_2\gamma z + \left\{6a_3\gamma + 2(\gamma^2 - 3\gamma)a_2^2\right\} z^2 + \left\{12a_4\gamma + 3(4\gamma^2 - 10\gamma)a_2a_3 + \frac{4}{3}\gamma(\gamma-2)(\gamma-7)a_2^3\right\} z^3 + \cdots\Big] = \Big[1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots\Big].$$

Upon simplification, we obtain

$$\Big[1 + (1+\gamma)a_2 z + \left\{2(1+2\gamma)a_3 + \frac{(\gamma^2 - 7\gamma - 2)}{2}a_2^2\right\} z^2 + \left\{3(1+3\gamma)a_4 + (4\gamma^2 - 19\gamma - 3)a_2a_3 + \frac{(\gamma^3 - 24\gamma^2 + 65\gamma + 6)}{6}a_2^3\right\} z^3 + \cdots\Big] = \Big[1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots\Big]. \tag{3.8}$$

Equating the coefficients of equal powers of z, z^2 and z^3, respectively, on both sides of (3.8), we get

$$a_2 = \frac{c_1}{(1+\gamma)};$$
$$a_3 = \frac{1}{4(1+\gamma)^2(1+2\gamma)}\left\{2(1+\gamma)^2 c_2 - (\gamma^2 - 7\gamma - 2)c_1^2\right\};$$
$$a_4 = \frac{1}{36(1+\gamma)^3(1+2\gamma)(1+3\gamma)} \times \left\{12(1+\gamma)^3(1+2\gamma)c_3\right.$$
$$\left. - 6(1+\gamma)^2(4\gamma^2 - 19\gamma - 3)c_1c_2 + (8\gamma^4 - 47\gamma^3 + 154\gamma^2 + 23\gamma + 6)c_1^3\right\}. \tag{3.9}$$

Considering the second Hankel functional $|a_2a_4 - a_3^2|$ for the function $f \in ST_\gamma$ and substituting the values of $a_2,\ a_3$ and a_4 from (3.9), we have

$$|a_2a_4 - a_3^2| = \left| \frac{c_1}{(1+\gamma)} \times \frac{1}{36(1+\gamma)^3(1+2\gamma)(1+3\gamma)} \times \left\{12(1+\gamma)^3(1+2\gamma)c_3\right.\right.$$
$$\left. -6(1+\gamma)^2(4\gamma^2 - 19\gamma - 3)c_1c_2 + (8\gamma^4 - 47\gamma^3 + 154\gamma^2 + 23\gamma + 6)c_1^3\right\}$$
$$\left. - \frac{1}{16(1+\gamma)^4(1+2\gamma)^2}\left\{2(1+\gamma)^2c_2 - (\gamma^2 - 7\gamma - 2)c_1^2\right\}^2 \right|.$$

Further simplification gives

$$|a_2a_4 - a_3^2| = \frac{1}{144(1+\gamma)^4(1+2\gamma)^2(1+3\gamma)} \times$$
$$|48(1+\gamma)^3(1+2\gamma)^2c_1c_3 + 12\gamma(1+\gamma)^2(-7\gamma^2 + 8\gamma + 11)c_1^2c_2$$
$$- 36(1+\gamma)^4(1+3\gamma)c_2^2 + (37\gamma^5 + 25\gamma^4 - 45\gamma^3 - 361\gamma^2 - 220\gamma - 12)c_1^4|,$$

which is equivalent to

$$|a_2a_4 - a_3^2| = \frac{1}{144(1+\gamma)^4(1+2\gamma)^2(1+3\gamma)} \times \left|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\right|, \tag{3.10}$$

where
$$d_1 = 48(1+\gamma)^3(1+2\gamma)^2;$$
$$d_2 = 12\gamma(1+\gamma)^2(-7\gamma^2 + 8\gamma + 11);$$
$$d_3 = -36(1+\gamma)^4(1+3\gamma);$$
$$d_4 = (37\gamma^5 + 25\gamma^4 - 45\gamma^3 - 361\gamma^2 - 220\gamma - 12). \tag{3.11}$$

Substituting the values of c_2 and c_3 given in (2.2) and (2.4) respectively from Lemma 2.2 on the right-hand side of (3.10), we have

$$\begin{aligned}&|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4|\\ &= |d_1c_1 \times \frac{1}{4}\{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\} +\\ &\qquad d_2c_1^2 \times \frac{1}{2}\{c_1^2 + x(4-c_1^2)\} + d_3 \times \frac{1}{4}\{c_1^2 + x(4-c_1^2)\}^2 + d_4c_1^4|.\end{aligned}$$

Using triangle inequality and the fact that $|z| < 1$, we get

(3.12)

$$\begin{aligned}4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| &\le |(d_1 + 2d_2 + d_3 + 4d_4)c_1^4 + 2d_1c_1(4-c_1^2) +\\ 2(d_1+d_2+d_3)c_1^2(4-c_1^2)|x| &- \left\{(d_1+d_3)c_1^2 + 2d_1c_1 - 4d_3\right\}(4-c_1^2)|x|^2|.\end{aligned}$$

From (3.11), we can now write

$$\begin{aligned}(d_1 + 2d_2 + d_3 + 4d_4) &= 4(16\gamma^5 + 64\gamma^4 + 177\gamma^3 - 115\gamma^2 - 133\gamma - 9);\\ d_1 &= 48(1+\gamma)^3(1+2\gamma)^2;\\ (d_1 + d_2 + d_3) &= 12(1+\gamma)^2(19\gamma^2 + 16\gamma + 1). \qquad (3.13)\end{aligned}$$

$$\text{and } \left\{(d_1+d_3)c_1^2 + 2d_1c_1 - 4d_3\right\} = 12(1+\gamma)^3 \times \left\{(7\gamma^2 + 4\gamma + 1)c_1^2 + 8(1+2\gamma)^2c_1 + 12(1+\gamma)(1+3\gamma)\right\}. \qquad (3.14)$$

Consider

$$\begin{aligned}&\left\{(7\gamma^2 + 4\gamma + 1)c_1^2 + 8(1+2\gamma)^2c_1 + 12(1+\gamma)(1+3\gamma)\right\}\\ &= (7\gamma^2 + 4\gamma + 1) \times \left[c_1^2 + \frac{8(1+2\gamma)^2}{(7\gamma^2+4\gamma+1)}c_1 + \frac{12(1+\gamma)(1+3\gamma)}{(7\gamma^2+4\gamma+1)}\right].\\ &= (7\gamma^2 + 4\gamma + 1)\times\\ &\left[\left\{c_1 + \frac{4(1+2\gamma)^2}{(7\gamma^2+4\gamma+1)}\right\}^2 - \frac{16(1+2\gamma)^4}{(7\gamma^2+4\gamma+1)^2} + \frac{12(1+\gamma)(1+3\gamma)}{(7\gamma^2+4\gamma+1)}\right].\\ &= (7\gamma^2 + 4\gamma + 1)\times\\ &\left[\left\{c_1 + \frac{4(1+2\gamma)^2}{(7\gamma^2+4\gamma+1)}\right\}^2 - \left\{\frac{2\sqrt{\gamma^4 + 8\gamma^3 + 18\gamma^2 + 8\gamma} + 1}{(7\gamma^2+4\gamma+1)}\right\}^2\right].\\ &= (7\gamma^2 + 4\gamma + 1)\times\\ &\left[c_1 + \left\{\frac{4(1+2\gamma)^2}{(7\gamma^2+4\gamma+1)} + \frac{2\sqrt{\gamma^4 + 8\gamma^3 + 18\gamma^2 + 8\gamma} + 1}{(7\gamma^2+4\gamma+1)}\right\}\right]\\ &\times \left[c_1 + \left\{\frac{4(1+2\gamma)^2}{(7\gamma^2+4\gamma+1)} - \frac{2\sqrt{\gamma^4 + 8\gamma^3 + 18\gamma^2 + 8\gamma} + 1}{(7\gamma^2+4\gamma+1)}\right\}\right]. \qquad (3.15)\end{aligned}$$

Since $c_1 \in [0,2]$, using the result $(c_1+a)(c_1+b) \geq (c_1-a)(c_1-b)$, where $a, b \geq 0$ on the right-hand side of (3.15), upon simplification, we obtain

$$\begin{aligned}(3.16)\quad &\left\{(7\gamma^2+4\gamma+1)c_1^2+8(1+2\gamma)^2c_1+12(1+\gamma)(1+3\gamma)\right\}\\ &\geq \left\{(7\gamma^2+4\gamma+1)c_1^2-8(1+2\gamma)^2c_1+12(1+\gamma)(1+3\gamma)\right\}.\end{aligned}$$

From the expressions (3.14) and (3.16), we get

$$\begin{aligned}(3.17)\quad &-\left\{(d_1+d_3)c_1^2+2d_1c_1-4d_3\right\}\\ &\leq -\left\{(7\gamma^2+4\gamma+1)c_1^2-8(1+2\gamma)^2c_1+12(1+\gamma)(1+3\gamma)\right\}.\end{aligned}$$

Substituting the calculated values from (3.13) and (3.17) on the right-hand side of (3.12), after simplifying, we obtain

$$\begin{aligned}&|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4|\\ &\leq |(16\gamma^5+64\gamma^4+177\gamma^3-115\gamma^2-133\gamma-9)c_1^4+\\ &24(1+\gamma)^3(1+2\gamma)^2c_1(4-c_1^2)+6(1+\gamma)^2(19\gamma^2+16\gamma+1)c_1^2(4-c_1^2)|x|\\ &-3(1+\gamma)^3\left\{(7\gamma^2+4\gamma+1)c_1^2-8(1+2\gamma)^2c_1+12(1+\gamma)(1+3\gamma)\right\}(4-c_1^2)|x|^2|.\end{aligned}$$

Choosing $c_1 = c \in [0,2]$, applying triangle inequality and replacing $|x|$ by μ on the right-hand side of the above inequality, we have

$$\begin{aligned}(3.18)\quad &|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4|\\ &\leq [(-16\gamma^5-64\gamma^4-177\gamma^3+115\gamma^2+133\gamma+9)c^4+\\ &24(1+\gamma)^3(1+2\gamma)^2c(4-c^2)+6(1+\gamma)^2(19\gamma^2+16\gamma+1)c^2(4-c^2)\mu\\ &+3(1+\gamma)^3\left\{(7\gamma^2+4\gamma+1)c^2-8(1+2\gamma)^2c+12(1+\gamma)(1+3\gamma)\right\}(4-c^2)\mu^2]\\ &= F(c,\mu),\ \text{for } 0\leq \mu = |x| \leq 1.\end{aligned}$$

We next maximize the function $F(c,\mu)$ on the closed region $[0,2]\times[0,1]$. Differentiating $F(c,\mu)$ partially with respect to μ, we get

$$\begin{aligned}(3.19)\quad \frac{\partial F}{\partial \mu} &= [6(1+\gamma)^2(19\gamma^2+16\gamma+1)c^2(4-c^2)\\ &+6(1+\gamma)^3\left\{(7\gamma^2+4\gamma+1)c^2-8(1+2\gamma)^2c+12(1+\gamma)(1+3\gamma)\right\}(4-c^2)\mu].\end{aligned}$$

For $0<\mu<1$, for any fixed c with $0<c<2$ and $o \leq \gamma \leq 1$, from (3.19), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c,\mu)$ is an increasing function of μ and hence it cannot have maximum value at any point in the interior of the closed region $[0,2]\times[0,1]$. Moreover, for fixed $c \in [0,2]$, we have

$$(3.20)\qquad \max_{0\leq\mu\leq 1} F(c,\mu) = F(c,1) = G(c).$$

Therefore, replacing μ by 1 in $F(c,\mu)$ on the right-hand side of (3.18), upon simplification, we obtain

$$G(c) = F(c,1) = \left\{-\gamma(37\gamma^4 + 253\gamma^3 + 603\gamma^2 + 263\gamma - 4)c^4 + 24\gamma(1+\gamma)^2(-\gamma^2 + 14\gamma + 11)c^2 + 144(1+\gamma)^4(1+3\gamma)\right\} \tag{3.21}$$

$$G'(c) = \left\{-4\gamma(37\gamma^4 + 253\gamma^3 + 603\gamma^2 + 263\gamma - 4)c^3 + 48\gamma(1+\gamma)^2(-\gamma^2 + 14\gamma + 11)c\right\}. \tag{3.22}$$

$$G''(c) = \left\{-12\gamma(37\gamma^4 + 253\gamma^3 + 603\gamma^2 + 263\gamma - 4)c^2 + 48\gamma(1+\gamma)^2(-\gamma^2 + 14\gamma + 11)\right\}. \tag{3.23}$$

For optimum value of c, consider $G'(c) = 0$. From (3.22), we get

$$-4\gamma c\left\{37\gamma^4 + 253\gamma^3 + 603\gamma^2 + 263\gamma - 4)c^2 - 12(1+\gamma)^2(-\gamma^2 + 14\gamma + 11)\right\} = 0. \tag{3.24}$$

We now discuss the following cases.
Case 1) If $c = 0$ and $\gamma \neq 0$, then, we have $G'(c) = 0$ and

$$G''(c) = 48(1+\gamma)^2(-\gamma^2 + 14\gamma + 11) > 0, \quad \text{for} \quad 0 < \gamma \leq 1.$$

From the second derivative test, $G(c)$ has minimum value at $c = 0$.
Case 2) If $c \neq 0$ and $\gamma = 0$, then, we have $G'(c) = 0$ and $G''(c) = 0$. Therefore, $G(c)$ is a constant and the constant value is 144, i.e., $G(c) = 144$.
Case 3) If $c = 0$ and $\gamma = 0$, then, we have $G'(c) = 0$ and $G''(c) = 0$. In this case also, we have $G(c) = 144$, which is a constant.
From cases 2 and 3, we conclude that $G(c) = 144$, a constant, for every $c \in [0,2]$, provided $\gamma = 0$.
Case 4) If $c \neq 0$ and $\gamma \neq 0$, from (3.24), on applying the Division algorithm for polynomials, we obtain

$$c^2 = \left[\frac{1}{37}\left\{-12 + \frac{(8364\gamma^3 + 24108\gamma^2 + 19140\gamma + 4836)}{(37\gamma^4 + 253\gamma^3 + 603\gamma^2 + 263\gamma - 4)}\right\}\right] > 0, \quad \text{for } 0 < \gamma \leq 1. \tag{3.25}$$

Substituting the c^2 value given by (3.25) in (3.23), it can be shown that

$$G''(c) = \left\{-\frac{12}{37}(-296\gamma^4 + 3552\gamma^3 + 11248\gamma^2 + 10658\gamma + 3256)\right\} < 0, \quad \text{for } 0 < \gamma \leq 1.$$

Therefore, by the second derivative test, $G(c)$ has maximum value at c, where c^2 is given by (3.25). Substituting the c^2 value in (3.21), after simplifying, we get

$$G_{max} = 144(1+\gamma)^4 \times \left[\frac{(112\gamma^5 + 768\gamma^4 + 2263\gamma^3 + 1700\gamma^2 + 372\gamma - 4)}{(37\gamma^4 + 253\gamma^3 + 603\gamma^2 + 263\gamma - 4)}\right]. \tag{3.26}$$

Considering, only the maximum value of $G(c)$ at c, where c^2 is given by (3.25), from the relations (3.18) and (3.26), we have

$$|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \le 144(1+\gamma)^4 \times \left[\frac{(112\gamma^5 + 768\gamma^4 + 2263\gamma^3 + 1700\gamma^2 + 372\gamma - 4)}{(37\gamma^4 + 253\gamma^3 + 603\gamma^2 + 263\gamma - 4)}\right]. \tag{3.27}$$

From the expressions (3.10) and (3.27), upon simplification, we obtain

$$|a_2a_4 - a_3^2| \le \left[\frac{(112\gamma^5 + 768\gamma^4 + 2263\gamma^3 + 1700\gamma^2 + 372\gamma - 4)}{(1+2\gamma)^2(1+3\gamma)(37\gamma^4 + 253\gamma^3 + 603\gamma^2 + 263\gamma - 4)}\right]. \tag{3.28}$$

This completes the proof of our Theorem. □

Remark 3.2. Choosing $\gamma = 0$, we have $ST_0 = ST$, from (3.28), we obtain $|a_2a_4 - a_3^2| \le 1$ and this inequality is sharp.

Remark 3.3. For the choice of $\gamma = 1$, we have $ST_1 = CV$, for which, from (3.28), we get $|a_2a_4 - a_3^2| \le \frac{1}{8}$ and is sharp.

Both the results coincide with those of Jateng, Halim and Darus [11].

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