Title:

On Ads-modules with the SIP

Author(s):

F. Takil Mutlu
ON ADS-MODULES WITH THE SIP

F. TAKIL MUTLU

(Communicated by Omid Ali S. Karamzadeh)

Abstract. The class of ads modules with the SIP (briefly, SA-modules) is studied. Various conditions for a module to be SA-module are given. It is proved that for a quasi-continuous module $M$, $M$ is a UC-module if and only if $M$ is an SA-module. Also, it is proved that the direct sum of two SA-modules as $R$-modules is an SA-module when $R$ is the sum of the annihilators of these modules.

Keywords: Ads-modules, summand intersection property, extending modules.

MSC(2010): Primary: 16D70; Secondary: 16D50, 16D60.

1. Introduction

The purpose of this paper is to study the class of ads-modules with the SIP. Fuchs [7] calls a module $M$ to have the absolute direct summand property (ads), if for every decomposition $M = A \oplus B$ of $M$ and every complement $C$ of $A$ in $M$ we have $M = A \oplus C$. We note that every quasi-continuous module is an ads-module, but not conversely. However, an ads-module which is also extending is quasi-continuous.

Wilson [15] calls a module $M$ to have the summand intersection property (SIP), if the intersection of every pair of direct summands of $M$ is a direct summand of $M$.

The motivation of the current study comes from the following question: “Do the absolute direct summand property and the summand intersection property necessitate the other?”

Example 2.3 and Example 2.4 show that the class of ads-modules and the class of modules with the SIP are different. Therefore, we say a right $R$-module $M$ is an SA-module if $M$ is an ads-module with the SIP. It is clear that indecomposable modules and semisimple modules are SA-modules. In this paper, we provide various conditions for a module to be an SA-module. We...
prove that for a quasi-continuous module $M$, $M$ is a UC-module if and only if $M$ is an SA-module. We also provide a condition for the direct sum of two SA-modules to be an SA-module.

Throughout the paper all rings are associative with unity and $R$ always denotes such a ring. All modules are unital right $R$-modules unless indicated otherwise. $N \leq M$ will mean $N$ is a submodule of $M$.

A module $M$ is called extending (or CS) if every submodule of $M$ is essential in a direct summand of $M$. A module $M$ is called quasi-continuous if it satisfies extending and $(C_3)$ condition: the sum of two direct summands of $M$ with zero intersection is again a direct summand of $M$.

For two modules $A$ and $B$, we say that $A$ is $B$-injective if any homomorphism from a submodule $X$ of $B$ to $A$ can be extended to a homomorphism from $B$ to $A$.

For any module $M$, $E(M_R)$, $End(M_R)$ and $r(X)$ (resp. $r(x)$) will denote the injective hull of $M$, the ring of endomorphisms of $M$ and the right annihilator of a subset $X$ (resp. an element $x$) in $M$, respectively. The notions which are not explained here can be found in [16].

2. SA-Modules

We begin with two lemmas which are useful in determining the ads property and the SIP property of a module. The first lemma appears in [5] and the second lemma appears in [8].

Lemma 2.1. A module $M$ is an ads-module if and only if for any decomposition $M = A \oplus B$, $B$ is $A$-injective.

Lemma 2.2. A module $M$ has the SIP if and only if for every decomposition $M = A \oplus B$ and every homomorphism $f$ from $A$ to $B$, the kernel of $f$ is a direct summand.

The following examples show that the class of ads-modules and the class of modules with the SIP are different.

Example 2.3. Let $K$ be a field, $R = [K 0]$. Then $N = [0 K]$ and $L = [K 0]$ are right $R$-modules. Let $M = R/L$ and $U = M \oplus N$. By ([8], Remark on page 81), $U$ does not have the SIP. However, since $R/L \cong [0 K] \cong K$, $R/L$ is injective by the Baer criteria. Then $M = R/L$ is $N$-injective. On the other hand, since $M = R/L$ is field, its submodules are only the trivial ones. So, $N$ is $M$-injective. Thus $U$ is an ads-module.

Example 2.4. Let $p$ be a prime integer and $M = \mathbb{Z}/Zp \oplus \mathbb{Q}$. Since all direct summands of $M$ are $\mathbb{Z}/Zp \oplus 0$, $0 \oplus \mathbb{Q}$, $0 \oplus 0$ and $M$, clearly $M$ has the SIP. Now, we show that $M$ is not an ads-module. Since $\mathbb{Q}$ is injective, $\mathbb{Q}$ is $\mathbb{Z}/Zp$-injective. Now suppose that $\mathbb{Z}/Zp$ is $\mathbb{Q}$-injective. Let $\pi : \mathbb{Z} \rightarrow \mathbb{Z}/Zp$ denote the canonical epimorphism, defined by $\pi(n) = n + Zp$ ($n \in \mathbb{Z}$).
Then there exists a homomorphism \( \alpha : \mathbb{Q} \rightarrow \mathbb{Z}/p \mathbb{Z} \) which extends \( \pi \). Now \( \alpha(1/p) = x + \mathbb{Z}p \) for some \( x \in \mathbb{Z} \). Thus \( p\alpha(1/p) = \alpha(1) = \pi(1) = 1 + \mathbb{Z}p \). It follows that \( px + \mathbb{Z}p = 1 + \mathbb{Z}p \) and hence \( 1 \equiv 0 \pmod{p} \), a contradiction. Thus \( \mathbb{Z}/p \mathbb{Z} \) is not \( \mathbb{Q} \)– injective. Hence \( M \) is not an \( \text{ads-module} \) by Lemma 2.1.

**Definition 2.5.** We say that a module \( M \) is an \( \text{SA-set} \) if \( M \) is an \( \text{ads-module} \) with the SIP.

The next proposition gives a characterization of \( \text{SA-modules} \). We remark that, the second part of this proposition also appears in [1] as Proposition 3.2. as one of the equivalent conditions for a module to be an \( \text{ads-module} \).

**Proposition 2.6.** A module \( M \) is an \( \text{SA-set} \) if and only if the following statements are satisfied:

for any decomposition \( M = A \oplus B \),

i) for every homomorphism \( f \) from \( A \to B \), the kernel of \( f \) is a direct summand.

ii) for any complement \( C \) of \( A \) in \( M \) and the projection map \( \pi : M \to B \), the restricted map \( \pi|_C : C \to B \) is an isomorphism.

**Proof.** Suppose \( M \) is an \( \text{SA-set} \). The first part is Lemma 2.2. We show the second part. Let \( C \) be a complement of \( A \). Take \( x \in \text{Ker}(\pi|_C) \). Then \( x \in C \cap A = 0 \), so \( \text{Ker}(\pi|_C) = 0 \). Since \( A \oplus C = (A \oplus C) \cap M = (A \oplus C) \cap (A \oplus B) = ((A \oplus C) \cap B) + A \), we have

\[
\pi(C) = \pi(A \oplus C) = \pi((A \oplus C) \cap B) = (A \oplus C) \cap B.
\]

Since \( M \) is an \( \text{ads-module} \), \( A \oplus C = M \) and \( \pi(C) = B \). Therefore, \( \pi|_C \) is an isomorphism from \( C \) to \( B \).

Conversely, let \( M = A \oplus B \) and \( C \) be a complement of \( A \) in \( M \). Since the first part is satisfied, \( M \) has the SIP. Since the second part is satisfied, \( \pi|_C(C) = B \) and \( M = A \oplus B = A \oplus C \). So \( M \) is an \( \text{ads-module} \). \( \square \)

**Lemma 2.7.** Every direct summand of an \( \text{SA-module} \) is an \( \text{SA-set} \).

**Proof.** Let \( M \) be an \( \text{SA-module} \) and \( N \) be a direct summand of \( M \). Let \( K \) and \( L \) be direct summands of \( N \). Since every direct summand of \( N \) is direct summand of \( M \) and \( M \) has the SIP, \( K \cap L \) is a direct summand of \( M \). Then there exists a submodule \( F \) of \( M \) such that \( M = (K \cap L) \oplus F \). Hence \( N = (K \cap L) \oplus (N \cap F) \) by modular law. Thus \( N \) has the SIP. Now we show that \( N \) is an \( \text{ads-module} \). Let \( N = N_1 \oplus N_2 \), \( U \subseteq N_2 \) and \( \phi : U \to N_1 \) be a homomorphism. Then there exists a submodule \( S \) of \( M \) such that \( M = N \oplus S = N_1 \oplus N_2 \oplus S \). Set \( \phi = \{ u - \phi(u) | u \in U \} \). Now \( X \cap N_1 = 0 \) and so \( X \) lies in a complement, say \( C \), of \( N_1 \) in \( M \). Hence \( X + S \) lies in the complement \( C + S \), of \( N_1 \) in \( M \). Since \( M \) is an \( \text{ads-module} \), \( M = N_1 \oplus (C + S) \). Then each \( n_2 \in N_2 \) has the form \( b - n_1 \), \( b \in (C + S) \) and \( n_1 \in N_1 \), so that \( b = n_1 + n_2 \). The composition of the projection \( \pi_{C+S} : M \to C + S \) along \( N_1 \) followed by the projection \( \pi_{N_1} : M \to N_1 \)
along \( N_2 \) and restricting to \( N_2 \) gives a homomorphism from \( N_2 \) to \( N_1 \) which extends \( \phi \). Hence \( N_1 \) is \( N_2 - \text{injective} \). Thus \( N \) is an \( \text{ads-module} \). \( \square \)

The direct sum of two \( \text{SA-modules} \) may not be an \( \text{SA-module} \).

**Example 2.8.** There are \( \text{SA-modules} \) such that their direct sum need not be an \( \text{SA-module} \).

(i) Consider \( Z \) as a right \( \mathbb{Z} \)-module. It is clear that \( Z \) is indecomposable and hence it is an \( \text{SA-module} \). Since \( Z \) is not \( \mathbb{Z} \)-injective, \( Z \oplus \mathbb{Z} \) is not an \( \text{ads-module} \) by Lemma 2.1 and hence it is not an \( \text{SA-module} \).

(ii) Consider the Prüfer \( p \)-group \( \mathbb{Z}_{p^\infty} \) as a right \( \mathbb{Z} \)-module. It is clear that \( \mathbb{Z}_{p^\infty} \) is indecomposable and hence, it is an \( \text{SA-module} \). Now define a homomorphism \( f \) from \( \mathbb{Z}_{p^\infty} \) to \( \mathbb{Z}_{p^\infty} \) as follows

\[
f \left( \frac{n}{p^t} + \mathbb{Z} \right) = \frac{n}{p^t-1} + \mathbb{Z} \quad \text{with} \quad n \in \mathbb{Z} \quad \text{and} \quad t \in \mathbb{N}.
\]

It is clear that \( \text{Ker} \ f = \left( \frac{1}{p} + \mathbb{Z} \right) \). But \( \mathbb{Z}_{p^\infty} \) is indecomposable and hence \( \text{Ker} \ f \) is not a direct summand of \( \mathbb{Z}_{p^\infty} \). By Proposition 2.6, \( \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty} \) is not an \( \text{SA-module} \).

**Lemma 2.9.** ([10], Proposition 3.9) Let \( M = M_1 \oplus M_2 \) be an \( R \)-module. If \( r(M_1) + r(M_2) = R \), then every submodule \( N \) of \( M \) can be written as \( N = N_1 \oplus N_2 \), where \( N_1 \leq M_1 \) and \( N_2 \leq M_2 \).

We now state a condition for which the direct sum of \( \text{SA-modules} \) is an \( \text{SA-module} \).

**Theorem 2.10.** Let \( M \) and \( N \) be two \( \text{SA-modules} \), such that \( r(M)+r(N) = R \). Then, \( M \oplus N \) is an \( \text{SA-module} \).

**Proof.** \( M \oplus N \) has the SIP by Theorem 3.10 in [10]. We show that \( M \oplus N \) is an \( \text{ads-module} \). Let \( A \) be a direct summand of \( M \oplus N \). Then there exists a submodule \( B \) such that \( M \oplus N = A \oplus B \). By Lemma 2.9, \( A = M_1 \oplus N_1 \) and \( B = M_2 \oplus N_2 \), where \( M_1 \) and \( M_2 \) are submodules of \( M \), \( N_1 \) and \( N_2 \) are submodules of \( N \). It is easy to show that \( M_1 \) and \( M_2 \) are direct summands of \( M \), and \( N_1 \) and \( N_2 \) are direct summands of \( N \). Let \( C \) be a complement of \( A \). Then \( C = M'_1 \oplus N'_1 \) for some submodules \( M'_1 \subseteq M \) and \( N'_1 \subseteq N \). Then

\[
M'_1 \cap M_1 \quad \text{and} \quad N'_1 \cap N_1
\]

Thus \( M \oplus N = (M_1 \oplus M'_1) \oplus (N_1 \oplus N'_1) = A \oplus C \).

Thus \( M \oplus N \) is an \( \text{ads-module} \) and hence an \( \text{SA-module} \). \( \square \)
Theorem 2.11. If $R$ is an injective right Ore domain, then $(R \oplus R)_R$ is an SA-module.

Proof. Since $R_R$ is injective, $R \oplus R$ is injective and hence it is an ads-module. However, $R \oplus R$ has the SIP by Proposition 4 in [3]. □

Theorem 2.12. Let $R$ be a noetherian domain and $M$ an injective $R$-module. The following conditions are equivalent:

(1) $M \oplus M$ is an SA-module.
(2) $M$ is torsion free.
(3) $\oplus M$ is an SA-module for any index set $\Lambda$.

Proof. The proof follows from Theorem 3.8 in [10]. □

Proposition 2.13. Let $M$ and $N$ be indecomposable modules such that $M$ is injective, and $\text{Hom}(M, N) \neq 0$. Let $C$ be a complement of $M$ in $M \oplus N$. If $M \oplus N$ is an SA-module, then $N \cong M \cong C$.

Proof. Let $0 \neq f \in \text{Hom}(M, N)$. Since $M \oplus N$ has the SIP, $M$ is isomorphic to a submodule $N_1$ of $N$ by Lemma 2.2. Since $M$ is injective, $N_1$ is injective submodule of $N$. By the injectivity of $N_1$, $N_1$ is a direct summand of $N$. By hypothesis, $N_1 = N$. Thus $M$ is isomorphic to $N$. Since $M \oplus N$ is an SA-module, for the projection map $\pi : M \oplus N \to N$, the restricted map $\pi|_C$ is an isomorphism by Proposition 2.6. Hence $M \cong N \cong C$. □

The classes of SA-modules and extending modules are different and demonstrated by the following example.

Example 2.14. (i) Let $F$ be a field, $V$ a vector space over $F$ such that $\dim V_F \geq 2$. Let $R = \left[ \begin{array}{c} F \ \ V \\ 0 \ \ F \end{array} \right] = \left\{ \left[ \begin{array}{c} f \\ v \\ \end{array} \right] : f \in F, \ v \in V \right\}$ a trivial extension of $V$ by $F$. Since $R$ is indecomposable as an $R$-module, $R_R$ is an SA-module. $R_R$ is extending when $\dim V_F = 1$ but not in other cases. For example, if $\dim V_F = 2$, i.e., $V = v_1 F \oplus v_2 F$, $v_1, v_2 \in V$, then $I = \left\{ \left[ \begin{array}{c} 0 \\ v_1 \end{array} \right] : f \in F \right\}$ is a complement of $J = \left\{ \left[ \begin{array}{c} f \\ 0 \end{array} \right] : f \in F \right\}$. But $I$ is not a direct summand of $R_R$.

(ii) This example is taken from [4]. Let $F$ be a field and

$$T = \left\{ \left[ \begin{array}{ccc} a & b & 0 \\ 0 & b & a \\ 0 & 0 & a \end{array} \right] : a, b, x, y \in F \right\}.$$ 

By [11], $T$ as a $T$-module does not have the SIP. Hence $T$ is not an SA-module. But $T$ is extending by [4].

A module $M$ is said to be principally quasi-injective (PQ-injective) if for every element $m \in M$ and $S = \text{End}(M_R)$, every homomorphism from $mR$ to $M$ can be extended to an endomorphism in $S$ (see, [13]). A module $M$ is said to be duo module if every submodule of $M$ is fully invariant. A module $M$ is said to satisfy the $(C_2)$ condition if every submodule of $M$ that is isomorphic
to a direct summand of $M$ is itself a direct summand of $M$. This condition is related to the $PQ$-injectivity as follows.

**Proposition 2.15.** Let $M$ be an extending, duo, $PQ$-injective module. Then $M$ is an $SA$-module.

**Proof.** Since $M$ is duo and $PQ$-injective, $M$ has the SIP by Proposition 3.3 in [13]. But $M$ has the $(C_2)$ condition by Proposition 2.3 in [13] and hence has the $(C_3)$ condition, and so $M$ is an $ads$-module. Thus $M$ is an $SA$-module. □

Recall that an $R$-module $M$ is called a prime module if $r(x) = r(y)$ for every non-zero elements $x$ and $y$ in $M$. The following proposition gives another condition under which an $R$-module is an $SA$-module.

**Proposition 2.16.** ([10], Proposition 2.1) Let $M$ be an injective and a prime module. Then $M$ is an $SA$-module.

The converse of the Proposition 2.16 is not always true.

**Example 2.17.** There exist $SA$-modules that are neither injective nor prime.

Consider $M = \mathbb{Z}/6\mathbb{Z}$ as a $\mathbb{Z}$-module. Then $M$ is semisimple and hence $M$ has the SIP. Since $\mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ are mutually injective, $\mathbb{Z}/6\mathbb{Z}$ is an $ads$-module. Thus $M$ is an $SA$-module. But, since $M$ is not a divisible abelian group, $M$ is not injective. Let $2, 3 \in M$. Then $r(2) = 3\mathbb{Z}$ and $r(3) = 2\mathbb{Z}$. Hence $M$ is not prime.

Let $N \leq M$. Whenever $N \leq K \leq M$ implies $N = K$, $N$ is called (essentially) closed in $M$ and we denote by $N \leq_c M$. A module $M$ is said to be a polyform module if for every $K \leq M$ and $f \in Hom(K, M)$, $Ker f$ is closed in $K$ (see, [6]). By Example 2.8 (ii), $\mathbb{Z}_p^\infty \oplus \mathbb{Z}_p^\infty$ as a $\mathbb{Z}$-module is not an $SA$-module. But it is injective and hence a quasi-continuous module. Therefore, a quasi-continuous module need not be an $SA$-module. Now we give the following lemmas which indicate when a quasi-continuous module is an $SA$-module.

**Lemma 2.18.** Let $M$ be a quasi-continuous polyform module. Then $M$ is an $SA$-module.

**Proof.** Let $M = A \oplus B$ and $f \in Hom(A, B)$. Since $M$ is polyform, $Ker f$ is closed in $A$. Since $M$ is quasi-continuous, $M$ is an extending $ads$-module. Hence $Ker f$ is a direct summand of $M$. So, $M$ has the SIP by Lemma 2.2. Thus $M$ is an $SA$-module. □

**Corollary 2.19.** Every quasi-continuous nonsingular module is an $SA$-module.

**Proof.** Immediate from Lemma 2.18. □

Let $M$ be a module. It is well known that for any submodule $N$ of $M$ there exists a closed submodule $K$ such that $N \leq_c K$ and $K$ is called a closure of $N$ in $M$. A module $M$ is called a $UC$-module if every submodule has a unique closure in $M$ (see, [14]).
Theorem 2.20. Let $M$ be a quasi-continuous module. Then $M$ is a UC-module if and only if $M$ is an SA-module.

Proof. Let $M$ be a UC-module, $N$ and $L$ direct summands of $M$. Then $N \cap L$ is closed in $M$ by Lemma 6 in [14]. Since $M$ is quasi-continuous, $M$ is an ads-module and by hypothesis, $N \cap L$ is a direct summand of $M$. Hence $M$ is an SA-module. For the converse, let $N \leq M$. Suppose that there are $K \leq M$ and $L \leq M$ such that $N \leq e K \leq e M$ and $N \leq e L \leq e M$. By quasi-continuity of $M$, $K$ and $L$ are direct summands of $M$ and by assumption, $(K \cap L) \oplus D = M$ for some $D \leq M$. Hence $K = (K \cap L) \oplus (K \cap D)$. Since $N \leq e K$ and $N \cap (K \cap D) = 0$, $K \cap D = 0$. Then $K = K \cap L$. Similarly, it is shown that $L = K \cap L$. Therefore $K = K \cap L = L$. Thus $M$ is a UC-module. □

The injective hull of an SA-module may not be an SA-module.

Example 2.21. Let $p$ be a prime integer and let $M_1$ denote the Z-module $\mathbb{Z}/\mathbb{Z}p$ and $M_2$ be the injective hull $E(M_1)$ of $M_1$. From ([12], Example 3.36), every submodule of $M_2$ is cyclic and generated by $1/p^n$ for some positive integer $n$. Therefore, it satisfies descending chain condition on submodules, $M_1$ and $M_2$ are uniform Z-modules, and they are indecomposable Z-modules. Then $M = M_1 \oplus M_2$ is an ads-module. Let $f$ be a nonzero homomorphism from $M_1$ to $M_2$. Since $Ker f$ is a submodule of $M_1$ and $M_1$ is simple, $Ker f = 0$, i.e., $Ker f$ is a direct summand of $M$. On the other hand, let $\alpha$ be a homomorphism from $M_2$ to $M_1$. Since $M_2$ is divisible and $M_1$ is not divisible, $\text{Hom}(M_2, M_1) = 0$. Then $\alpha$ must be a zero homomorphism. So $Ker \alpha = M_2$ is a direct summand of $M$. Thus $M$ has the SIP by Lemma 2.2 and so $M$ is an SA-module. But $E(M) = \mathbb{Z}/\mathbb{Z}p \oplus \mathbb{Z}/\mathbb{Z}p$, is not an SA-module by Example 2.8 (ii).

Now we give the following proposition showing that the injective hull of an SA-module $M$ is again an SA-module when $M$ is extending.

Proposition 2.22. Let $M$ be an extending SA-module. Then $E(M)$ is an SA-module.

Proof. Since $M$ is an extending ads-module, $M$ is quasi-continuous. Then $E(M)$ has the SIP by Proposition 18 in [2]. Since $E(M)$ is injective, $E(M)$ is an ads-module and hence an SA-module. □

Corollary 2.23. Let $M$ be a quasi-continuous module. Then $M$ is an SA-module if and only if $E(M)$ is an SA-module.

Proof. Assume that $E(M)$ is an SA-module. Then $M$ has the SIP by Proposition 18 in [2]. Hence $M$ is an SA-module. The converse is clear by Proposition 2.22. □

Proposition 2.24. Let $M = A \oplus B$ be an R-module, $f$ a homomorphism from $A$ to $B$, $E(M) = E_1 \oplus E_2$, where $E_1$ is injective hull of $Ker f$, and $\pi$ the projection of $E(M)$ onto $E_1$ along $E_2$. If $M$ is an SA-module, then $\pi(M) \subseteq M$. 
Suppose $M$ is an $SA$-module. Let $f$ be an homomorphism from $A$ to $B$. Since $M$ has the SIP, $\ker f$ is a direct summand of $M$. The submodule $K = E_2 \cap M$ is a complement of $\ker f$ in $M$. Indeed, if $C$ is a complement containing $K$ in $M$ and $c \in C$, we can write $c = e_1 + e_2$, $e_i \in E_i$. If $e_1 = 0$, then $c = e_2 \in E_2 \cap M = K$. If $e_1 \neq 0$, then there is $r \in R$ with $0 \neq re_1 \in \ker f$. Then $rc = re_1 + re_2$ and $re_1 \in \ker f \cap C = 0$, a contradiction. Now since $M$ is an $SA$-module, $M = \ker f \oplus K$ and $\pi(M) = \ker f$.

Note that the converse of Proposition 2.24 is not always true.

Example 2.25. Consider $M = \mathbb{Z} \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module and the homomorphism $f : \mathbb{Z} \rightarrow \mathbb{Z}$, defined by $f(n) = 2n$, $n \in \mathbb{Z}$. Then $\ker f = 0$ and the injective hull of $\ker f$ is $E_1 = 0$. Hence $E(M) = E_1 \oplus E_2 = E_2 = \mathbb{Q} \oplus \mathbb{Q}$. Consider the projection map $\pi$ of $E(M)$ onto $E_1$ along $E_2$. Then $\pi(M) = 0 \subseteq M$. But $\mathbb{Z} \oplus \mathbb{Z}$ is not an $SA$-module since $\mathbb{Z}$ is not $\mathbb{Z}$-injective by Lemma 2.1.

It is known that the sum of two closed submodules of a quasi-continuous module is closed (see, [9]). We prove that the sum of two closed submodules of an $SA$-module is again closed when both of them are direct summands.

Proposition 2.26. Let $A$ and $B$ be two closed submodules of an $SA$-module $M$ such that $A$ and $B$ are direct summands of $M$. Then $A + B$ is a closed submodule of $M$.

Proof. Since $M$ has the SIP, $A \cap B$ is a direct summand of $M$. Let $K$ be a complement of $A \cap B$. Since $M$ is an ads-module, $M = (A \cap B) \oplus K$. Hence, by modular law,

$$A + B = (A + B) \cap [(A \cap B) \oplus K] = A + [(A \cap B) \oplus (B \cap K)] = A \oplus (B \cap K)$$

Now let $C$ be a complement of $A$ containing $B \cap K$ in $M$. Since $M$ is an ads-module, $M = A \oplus C$. Let $x = a + c$ be in the closure of $A + B$, say $E$, in $M$, where $a \in A$ and $c \in C$. Since $a \in A \subseteq E$, we have $a \in E$. Hence there exists an essential right ideal $I$ of $R$ such that $cI \subseteq (A + B) \cap C = [A \oplus (B \cap K)] \cap C = B \cap K \subseteq B$.

The fact that $B$ is closed implies $c \in B$. Hence $x \in A + B$, as desired. □

Acknowledgments

The author is grateful for the thorough reading and useful suggestions by the referee.

References


(Figen Takl Mutlu) DEPARTMENT OF MATHEMATICS, ANADOLU UNIVERSITY, 26470, ESKIŞEHİR, TURKEY

E-mail address: figent@anadolu.edu.tr