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ON ADS-MODULES WITH THE SIP

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Abstract. The class of ads modules with the SIP (briefly, SA-modules) is studied. Various conditions for a module to be SA-module are given. It is proved that for a quasi-continuous module M, M is a UC-module if and only if M is an SA-module. Also, it is proved that the direct sum of two SA-modules as R-modules is an SA-module when R is the sum of the annihilators of these modules.
Keywords: Ads-modules, summand intersection property, extending modules.
MSC(2010): Primary: 16D70; Secondary: 16D50, 16D60.

1. **Introduction**

The purpose of this paper is to study the class of *ads*-modules with the SIP.

Fuchs [7] calls a module M to have the *absolute direct summand property (ads)*, if for every decomposition $M = A \oplus B$ of M and every complement C of A in M we have $M = A \oplus C$. We note that every quasi-continuous module is an ads-module, but not conversely. However, an ads-module which is also extending is quasi-continuous.

Wilson [15] calls a module M to have the *summand intersection property (SIP)*, if the intersection of every pair of direct summands of M is a direct summand of M.

The motivation of the current study comes from the following question: "Do the absolute direct summand property and the summand intersection property necessitate the other?"

Example 2.3 and Example 2.4 show that the class of ads-modules and the class of modules with the SIP are different. Therefore, we say a right R-module M is an SA-module if M is an ads-module with the SIP. It is clear that indecomposable modules and semisimple modules are SA-modules. In this paper, we provide various conditions for a module to be an SA-module. We

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prove that for a quasi-continuous module M, M is a UC-module if and only if M is an SA-module. We also provide a condition for the direct sum of two SA-modules to be an SA-module.

Throughout the paper all rings are associative with unity and R always denotes such a ring. All modules are unital right R-modules unless indicated otherwise. $N \leq M$ will mean N is a submodule of M.

A module M is called *extending* (or CS) if every submodule of M is essential in a direct summand of M. A module M is called *quasi-continuous* if it satisfies extending and (C_3) condition: the sum of two direct summands of M with zero intersection is again a direct summand of M.

For two modules A and B, we say that A is $B-injective$ if any homomorphism from a submodule X of B to A can be extended to a homomorphism from B to A.

For any module M, $E(M_R)$, $End(M_R)$ and $r(X)$ (resp. $r(x)$) will denote the injective hull of M, the ring of endomorphisms of M and the right annihilator of a subset X (resp. an element x) in M, respectively. The notions which are not explained here can be found in [16].

2. **SA-Modules**

We begin with two lemmas which are useful in determining the ads property and the SIP property of a module. The first lemma appears in [5] and the second lemma appears in [8].

Lemma 2.1. *A module M is an ads-module if and only if for any decomposition $M = A \oplus B$, B is $A-injective$.*

Lemma 2.2. *A module M has the SIP if and only if for every decomposition $M = A \oplus B$ and every homomorphism f from A to B, the kernel of f is a direct summand.*

The following examples show that the class of ads-modules and the class of modules with the SIP are different.

Example 2.3. *Let K be a field, $R = \left[\begin{smallmatrix} K & K \\ 0 & K \end{smallmatrix}\right]$. Then $N = \left[\begin{smallmatrix} 0 & K \\ 0 & K \end{smallmatrix}\right]$ and $L = \left[\begin{smallmatrix} K & K \\ 0 & 0 \end{smallmatrix}\right]$ are right R-modules. Let $M = R/L$ and $U = M \oplus N$. By ([8], Remark on page 81), U does not have the SIP. However, since $R/L \cong \left[\begin{smallmatrix} 0 & 0 \\ 0 & K \end{smallmatrix}\right] \cong K$, R/L is injective by the Baer criteria. Then $M = R/L$ is $N-injective$. On the other hand, since $M = R/L$ is field, its submodules are only the trivial ones. So, N is $M-injective$. Thus U is an ads-module.*

Example 2.4. *Let p be a prime integer and $M = \mathbb{Z}/\mathbb{Z}p \oplus \mathbb{Q}$. Since all direct summands of M are $(\mathbb{Z}/\mathbb{Z}p \oplus 0)$, $0 \oplus \mathbb{Q}$, $0 \oplus 0$ and M, clearly M has the SIP. Now, we show that M is not an ads-module. Since $\mathbb{Q}$ is injective, $\mathbb{Q}$ is $\mathbb{Z}/\mathbb{Z}p-injective$. Now suppose that $\mathbb{Z}/\mathbb{Z}p$ is $\mathbb{Q}-injective$. Let $\pi : \mathbb{Z} \longrightarrow \mathbb{Z}/\mathbb{Z}p$ denote the canonical epimorphism, defined by $\pi(n) = n + \mathbb{Z}p$ $(n \in \mathbb{Z})$.*

Then there exists a homomorphism $\alpha : \mathbb{Q} \longrightarrow \mathbb{Z}/\mathbb{Z}p$ *which extends* π*. Now* $\alpha(1/p) = x + \mathbb{Z}p$ *for some* $x \in \mathbb{Z}$*. Thus* $p\alpha(1/p) = \alpha(1) = \pi(1) = 1 + \mathbb{Z}p$*. It follows that* $px + \mathbb{Z}p = 1 + \mathbb{Z}p$ *and hence* $1 \equiv 0 \ (mod\ p)$*, a contradiction. Thus* $\mathbb{Z}/\mathbb{Z}p$ *is not* $\mathbb{Q}-injective$*. Hence* M *is not an ads-module by Lemma 2.1.*

Definition 2.5. *We say that a module* M *is an* SA*-module if* M *is an ads-module with the SIP.*

The next proposition gives a characterization of SA-modules. We remark that, the second part of this proposition also appears in [1] as Proposition 3.2. as one of the equivalent conditions for a module to be an ads-module.

Proposition 2.6. *A module* M *is an* SA*-module if and only if the following statements are satisfied:*
for any decomposition $M = A \oplus B$*,*

i) *for every homomorphism* f *from* A *to* B*, the kernel of* f *is a direct summand.*
ii) *for any complement* C *of* A *in* M *and the projection map* $\pi : M \longrightarrow B$*, the restricted map* $\pi_{|C} : C \longrightarrow B$ *is an isomorphism.*

Proof. Suppose M is an SA-module. The first part is Lemma 2.2. We show the second part. Let C be a complement of A. Take $x \in Ker\left(\pi_{|C}\right)$. Then $x \in C \cap A = 0$, so $Ker\left(\pi_{|C}\right) = 0$. Since $A \oplus C = (A \oplus C) \cap M = (A \oplus C) \cap (A \oplus B) = ((A \oplus C) \cap B) + A$, we have

$$\pi(C) = \pi(A \oplus C) = \pi((A \oplus C) \cap B) = (A \oplus C) \cap B.$$

Since M is an ads-module , $A \oplus C = M$ and $\pi(C) = B$. Therefore, $\pi_{|C}$ is an isomorphism from C to B.

Conversely, let $M = A \oplus B$ and C be a complement of A in M. Since the first part is satisfied, M has the SIP. Since the second part is satisfied, $\pi_{|C}(C) = B$ and $M = A \oplus B = A \oplus C$. So M is an ads-module. □

Lemma 2.7. *Every direct summand of an* SA*-module is an* SA*-module.*

Proof. Let M be an SA-module and N be a direct summand of M. Let K and L be direct summands of N. Since every direct summand of N is direct summand of M and M has the SIP, $K \cap L$ is a direct summand of M. Then there exists a submodule F of M such that $M = (K \cap L) \oplus F$. Hence $N = (K \cap L) \oplus (N \cap F)$ by modular law. Thus N has the SIP. Now we show that N is an ads-module. Let $N = N_1 \oplus N_2$, $U \subseteq N_2$ and $\phi : U \longrightarrow N_1$ be a homomorphism. Then there exists a submodule S of M such that $M = N \oplus S = N_1 \oplus N_2 \oplus S$. Set $X = \{u - \phi(u) | u \in U\}$. Now $X \cap N_1 = 0$ and so X lies in a complement, say C, of N_1 in M. Hence $X + S$ lies in the complement $C + S$, of N_1 in M. Since M is an ads-module, $M = N_1 \oplus (C + S)$. Then each $n_2 \in N_2$ has the form $b - n_1$, $b \in (C+S)$ and $n_1 \in N_1$, so that $b = n_1 + n_2$. The composition of the projection $\pi_{C+S} : M \longrightarrow C + S$ along N_1 followed by the projection $\pi_{N_1} : M \longrightarrow N_1$

along N_2 and restricting to N_2 gives a homomorphism from N_2 to N_1 which extends ϕ. Hence N_1 is $N_2 - injective$. Thus N is an ads-module. □

The direct sum of two SA-modules may not be an SA-module.

Example 2.8. *There are SA-modules such that their direct sum need not be an SA-module.*

(i) *Consider $\mathbb{Z}$ as a right $\mathbb{Z}$-module. It is clear that $\mathbb{Z}$ is indecomposable and hence it is an SA-module. Since $\mathbb{Z}$ is not $\mathbb{Z}$-injective, $\mathbb{Z} \oplus \mathbb{Z}$ is not an ads-module by Lemma 2.1 and hence it is not an SA-module.*

(ii) *Consider the Prüfer p-group $\mathbb{Z}_{p^\infty}$ as a right $\mathbb{Z}$-module. It is clear that $\mathbb{Z}_{p^\infty}$ is indecomposable and hence, it is an SA-module. Now define a homomorphism f from $\mathbb{Z}_{p^\infty}$ to $\mathbb{Z}_{p^\infty}$ as follows*

$$f\left(\frac{n}{p^t} + \mathbb{Z}\right) = \frac{n}{p^{t-1}} + \mathbb{Z} \text{ with } n \in \mathbb{Z} \text{ and } t \in \mathbb{N}.$$

It is clear that $Kerf = \left(\frac{1}{p} + \mathbb{Z}\right)$. But $\mathbb{Z}_{p^\infty}$ is indecomposable and hence $Kerf$ is not a direct summand of $\mathbb{Z}_{p^\infty}$. By Proposition 2.6, $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ is not an SA-module.

Lemma 2.9. *([10], Proposition 3.9) Let $M = M_1 \oplus M_2$ be an R-module. If $r(M_1) + r(M_2) = R$, then every submodule N of M can be written as $N = N_1 \oplus N_2$, where $N_1 \leq M_1$ and $N_2 \leq M_2$.*

We now state a condition for which the direct sum of SA-modules is an SA-module.

Theorem 2.10. *Let M and N be two SA-modules, such that $r(M)+r(N) = R$. Then, $M \oplus N$ is an SA-module.*

Proof. $M \oplus N$ has the SIP by Theorem 3.10 in [10]. We show that $M \oplus N$ is an ads-module. Let A be a direct summand of $M \oplus N$. Then there exists a submodule B such that $M \oplus N = A \oplus B$. By Lemma 2.9, $A = M_1 \oplus N_1$ and $B = M_2 \oplus N_2$, where M_1 and M_2 are submodules of M, N_1 and N_2 are submodules of N. It is easy to show that M_1 and M_2 are direct summands of M, and N_1 and N_2 are direct summands of N. Let C be a complement of A. Then $C = M_1^{'} \oplus N_1^{'}$ for some submodules $M_1^{'} \subseteq M$ and $N_1^{'} \subseteq N$. Then

$$\begin{aligned} M_1^{'} \cap M_1 &= (M_1^{'} \oplus N_1^{'}) \cap M_1 \leq (M_1^{'} \oplus N_1^{'}) \cap (M_1 \oplus N_1) = C \cap A = 0, \\ N_1^{'} \cap N_1 &= (M_1^{'} \oplus N_1^{'}) \cap N_1 \leq (M_1^{'} \oplus N_1^{'}) \cap (M_1 \oplus N_1) = C \cap A = 0, \end{aligned}$$

and hence we have $M_1^{'} \cap M_1 = 0$ and $N_1^{'} \cap N_1 = 0$. So $M_1^{'}$ is a complement of M_1 and $N_1^{'}$ is a complement of N_1. Since M and N are ads-modules, $M = M_1 \oplus M_1^{'}$ and $N = N_1 \oplus N_1^{'}$. Hence

$$M \oplus N = (M_1 \oplus M_1^{'}) \oplus (N_1 \oplus N_1^{'}) = A \oplus C.$$

Thus $M \oplus N$ is an ads-module and hence an SA-module. □

Theorem 2.11. *If R is an injective right Ore domain, then $(R \oplus R)_R$ is an SA-module.*

Proof. Since R_R is injective, $R \oplus R$ is injective and hence it is an ads-module. However, $R \oplus R$ has the SIP by Proposition 4 in [3]. □

Theorem 2.12. *Let R be a noetherian domain and M an injective R-module. The following conditions are equivalent:*

(1) *$M \oplus M$ is an SA-module.*
(2) *M is torsion free.*
(3) *$\underset{\Lambda}{\oplus} M$ is an SA-module for any index set Λ.*

Proof. The proof follows from Theorem 3.8 in [10]. □

Proposition 2.13. *Let M and N be indecomposable modules such that M is injective, and $Hom(M,N) \neq 0$. Let C be a complement of M in $M \oplus N$. If $M \oplus N$ is an SA-module, then $N \cong M \cong C$.*

Proof. Let $0 \neq f \in Hom(M,N)$. Since $M \oplus N$ has the SIP, M is isomorphic to a submodule N_1 of N by Lemma 2.2. Since M is injective, N_1 is injective submodule of N. By the injectivity of N_1, N_1 is a direct summand of N. By hypothesis, $N_1 = N$. Thus M is isomorphic to N. Since $M \oplus N$ is an SA-module, for the projection map $\pi : M \oplus N \longrightarrow N$, the restricted map $\pi_{|C}$ is an isomorphism by Proposition 2.6. Hence $M \cong N \cong C$. □

The classes of SA-modules and extending modules are different and demonstrated by the following example.

Example 2.14. ***(i)*** *Let F be a field, V a vector space over F such that $dimV_F \geq 2$. Let $R = \begin{bmatrix} F & V \\ 0 & F \end{bmatrix} = \left\{ \begin{bmatrix} f & v \\ 0 & f \end{bmatrix} | f \in F,\ v \in V \right\}$ a trivial extension of V by F. Since R is indecomposable as an R-module, R_R is an SA-module. R_R is extending when $dimV_F = 1$ but not in other cases. For example, if $dimV_F = 2$, i.e., $V = v_1F \oplus v_2F,\ v_1, v_2 \in V$, then $I = \left\{ \left[\begin{smallmatrix} 0 & v_1f \\ 0 & 0 \end{smallmatrix}\right] \mid f \in F \right\}$ is a complement of $J = \left\{ \left[\begin{smallmatrix} 0 & v_2f \\ 0 & 0 \end{smallmatrix}\right] \mid f \in F \right\}$. But I is not a direct summand of R_R.*
(ii) *This example is taken from [4]. Let F be a field and*

$$T = \left\{ \left[\begin{smallmatrix} a & x & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & y \\ 0 & 0 & 0 & a \end{smallmatrix}\right] \mid a,\ b,\ x,\ y \in F \right\}.$$

By [11], T as a T-module does not have the SIP. Hence T is not an SA-module. But T is extending by [4].

A module M is said to be *principally quasi-injective (PQ-injective)* if for every element $m \in M$ and $S = End(M_R)$, every homomorphism from mR to M can be extended to an endomorphism in S (see, [13]). A module M is said to be *duo module* if every submodule of M is fully invariant. A module M is said to satisfy the (C_2) condition if every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M. This condition is related to the PQ-injectivity as follows.

Proposition 2.15. *Let M be an extending, duo, PQ-injective module. Then M is an SA-module.*

Proof. Since M is duo and PQ-injective, M has the SIP by Proposition 3.3 in [13]. But M has the (C_2) condition by Proposition 2.3 in [13] and hence has the (C_3) condition, and so M is an ads-module. Thus M is an SA-module. □

Recall that an R-module M is called a *prime* module if $r(x) = r(y)$ for every non-zero elements x and y in M. The following proposition gives another condition under which an R-module is an SA-module.

Proposition 2.16. *([10], Proposition 2.1) Let M be an injective and a prime module. Then M is an SA-module.*

The converse of the Proposition 2.16 is not always true.

Example 2.17. *There exist SA-modules that are neither injective nor prime.*

Consider $M = \mathbb{Z}/6\mathbb{Z}$ as a $\mathbb{Z}$-module. Then M is semisimple and hence M has the SIP. Since $\mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ are mutually injective, $\mathbb{Z}/6\mathbb{Z}$ is an ads-module. Thus M is an SA-module. But, since M is not a divisible abelian group, M is not injective. Let $\overline{2}, \overline{3} \in M$. Then $r(\overline{2}) = 3\mathbb{Z}$ and $r(\overline{3}) = 2\mathbb{Z}$. Hence M is not prime.

Let $N \leq M$. Whenever $N \leq_e K \leq M$ implies $N = K$, N is called (essentially) *closed* in M and we denote by $N \leq_c M$. A module M is said to be a *polyform* module if for every $K \leq M$ and $f \in Hom(K, M)$, $Kerf$ is closed in K (see, [6]). By Example 2.8 (ii), $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module is not an SA-module. But it is injective and hence a quasi-continuous module. Therefore, a quasi-continuous module need not be an SA-module. Now we give the following lemmas which indicate when a quasi-continuous module is an SA-module.

Lemma 2.18. *Let M be a quasi-continuous polyform module. Then M is an SA-module.*

Proof. Let $M = A \oplus B$ and $f \in Hom(A, B)$. Since M is polyform, $Kerf$ is closed in A. Since M is quasi-continuous, M is an extending ads-module. Hence $Kerf$ is a direct summand of M. So, M has the SIP by Lemma 2.2. Thus M is an SA-module. □

Corollary 2.19. *Every quasi-continuous nonsingular module is an SA-module.*

Proof. Immediate from Lemma 2.18. □

Let M be a module. It is well known that for any submodule N of M there exists a closed submodule K such that $N \leq_e K$ and K is called a *closure* of N in M. A module M is called a UC-module if every submodule has a unique closure in M (see, [14]).

Theorem 2.20. *Let M be a quasi-continuous module. Then M is a UC-module if and only if M is an SA-module.*

Proof. Let M be a UC-module, N and L direct summands of M. Then $N \cap L$ is closed in M by Lemma 6 in [14]. Since M is quasi-continuous, M is an ads-module and by hypothesis, $N \cap L$ is a direct summand of M. Hence M is an SA-module. For the converse, let $N \leq M$. Suppose that there are $K \leq M$ and $L \leq M$ such that $N \leq_e K \leq_c M$ and $N \leq_e L \leq_c M$. By quasi-continuity of M, K and L are direct summands of M and by assumption, $(K \cap L) \oplus D = M$ for some $D \leq M$. Hence $K = (K \cap L) \oplus (K \cap D)$. Since $N \leq_e K$ and $N \cap (K \cap D) = 0$, $K \cap D = 0$. Then $K = K \cap L$. Similarly, it is shown that $L = K \cap L$. Therefore $K = K \cap L = L$. Thus M is a UC-module. □

The injective hull of an SA-module may not be an SA-module.

Example 2.21. *Let p be a prime integer and let M_1 denote the $\mathbb{Z}$-module $\mathbb{Z}/\mathbb{Z}p$ and M_2 be the injective hull $E(M_1)$ of M_1. From ([12], Example 3.36), every submodule of M_2 is cyclic and generated by $1/p^n$ for some positive integer n. Therefore, it satisfies descending chain condition on submodules, M_1 and M_2 are uniform $\mathbb{Z}$-modules, and they are indecomposable $\mathbb{Z}$-modules. Then $M = M_1 \oplus M_2$ is an ads-module. Let f be a nonzero homomorphism from M_1 to M_2. Since $Kerf$ is a submodule of M_1 and M_1 is simple, $Kerf = 0$, i.e, $Kerf$ is a direct summand of M. On the other hand, let α be a homomorphism from M_2 to M_1. Since M_2 is divisible and M_1 is not divisible, $Hom(M_2, M_1) = 0$. Then α must be a zero homomorphism. So $Ker\alpha = M_2$ is a direct summand of M. Thus M has the SIP by Lemma 2.2 and so M is an SA-module. But $E(M) = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ is not an SA-module by Example 2.8 (ii).*

Now we give the following proposition showing that the injective hull of an SA-module M is again an SA-module when M is extending.

Proposition 2.22. *Let M be an extending SA-module. Then $E(M)$ is an SA-module.*

Proof. Since M is an extending ads-module, M is quasi-continuous. Then $E(M)$ has the SIP by Proposition 18 in [2]. Since $E(M)$ is injective, $E(M)$ is an ads-module and hence an SA-module. □

Corollary 2.23. *Let M be a quasi-continuous module. Then M is an SA-module if and only if $E(M)$ is an SA-module.*

Proof. Assume that $E(M)$ is an SA-module. Then M has the SIP by Proposition 18 in [2]. Hence M is an SA-module. The converse is clear by Proposition 2.22. □

Proposition 2.24. *Let $M = A \oplus B$ be an R-module, f a homomorphism from A to B, $E(M) = E_1 \oplus E_2$, where E_1 is injective hull of $Kerf$, and π the projection of $E(M)$ onto E_1 along E_2. If M is an SA-module, then $\pi(M) \subseteq M$.*

Proof. Suppose M is an SA-module. Let f be an homomorphism from A to B. Since M has the SIP, $Ker\ f$ is a direct summand of M. The submodule $K = E_2 \cap M$ is a complement of $Kerf$ in M. Indeed, if C is a complement containing K in M and $c \in C$, we can write $c = e_1 + e_2$, $e_i \in E_i$. If $e_1 = 0$, then $c = e_2 \in E_2 \cap M = K$. If $e_1 \neq 0$, then there is $r \in R$ with $0 \neq re_1 \in Ker\ f$. Then $rc = re_1 + re_2$ and $re_1 \in Ker\ f \cap C = 0$, a contradiction. Now since M is an SA-module, $M = Ker\ f \oplus K$ and $\pi(M) = Ker\ f$. □

Note that the converse of Proposition 2.24 is not always true.

Example 2.25. *Consider $M = \mathbb{Z} \oplus \mathbb{Z}$ as a $\mathbb{Z}$-module and the homomorphism $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, defined by $f(n) = 2n$, $n \in \mathbb{Z}$. Then $Ker\ f = 0$ and the injective hull of $Kerf$ is $E_1 = 0$. Hence $E(M) = E_1 \oplus E_2 = E_2 = \mathbb{Q} \oplus \mathbb{Q}$. Consider the projection map π of $E(M)$ onto E_1 along E_2. Then $\pi(M) = 0 \subseteq M$. But $\mathbb{Z} \oplus \mathbb{Z}$ is not an SA-module since $\mathbb{Z}$ is not $\mathbb{Z}$-injective by Lemma 2.1.*

It is known that the sum of two closed submodules of a quasi-continuous module is closed (see, [9]). We prove that the sum of two closed submodules of an SA-module is again closed when both of them are direct summands.

Proposition 2.26. *Let A and B be two closed submodules of an SA-module M such that A and B are direct summands of M. Then $A + B$ is a closed submodule of M.*

Proof. Since M has the SIP, $A \cap B$ is a direct summand of M. Let K be a complement of $A \cap B$. Since M is an ads-module, $M = (A \cap B) \oplus K$. Hence, by modular law,

$$A + B = (A + B) \cap [(A \cap B) \oplus K] = A + [(A \cap B) \oplus (B \cap K)] = A \oplus (B \cap K)$$

Now let C be a complement of A containing $B \cap K$ in M. Since M is an ads-module, $M = A \oplus C$. Let $x = a + c$ be in the closure of $A + B$, say E, in M, where $a \in A$ and $c \in C$. Since $a \in A \subseteq E$, we have $a \in E$. Hence there exists an essential right ideal I of R such that

$$cI \subseteq (A + B) \cap C = [A \oplus (B \cap K)] \cap C = B \cap K \subseteq B.$$

The fact that B is closed implies $c \in B$. Hence $x \in A + B$, as desired. □

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