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REITER'S PROPERTIES FOR THE ACTIONS OF LOCALLY COMPACT QUANTUM GROUPS ON VON NEUMANN ALGEBRAS

M. RAMEZANPOUR AND H. R. EBRAHIMI VISHKI*

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Abstract. The notion of an action of a locally compact quantum group on a von Neumann algebra is studied from the amenability point of view. Various Reiter's conditions for such an action are discussed. Several applications to some specific actions related to certain representations and corepresentaions are presented.

1. Introduction

In order to extend some theories of harmonic analysis from abelian locally compact groups (especially, to restore the so-called Pontrjagin duality theorem) to non-abelian ones, a more general object—Kac algebra, which covers all locally compact groups— was constructed by some authors in early 70's, whose complete account can be found in [7]. The notion of quantum group was introduced by Drinfeld and improved by others using an operator algebraic approach. Their approach did not satisfy some axioms of the Kac algebras and instigated more efforts to

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*Corresponding author

introduce a more general theory. Some improvements in this direction were made by Woronowicz, Baaj, Skandalis and Van Daele. Finally, Kustermans and Vaes [9, 10] introduced the concept of locally compact quantum group along a comprehensive set of axioms which covers the notion of Kac algebras [7], and also quantum groups. Some of the most famous locally compact quantum groups (which have been extensively studied in abstract harmonic analysis) are $\mathrm{L}^\infty(G)$ and $\mathrm{VN}(G)$, in which G is a locally compact group. In spite of the fact that these two algebras have different feature in abstract harmonic analysis, they have mostly a unified framework from the locally compact quantum group point of view. Some aspects of abstract harmonic analysis on locally compact groups are intensively extended by Runde [15, 16] to the framework of locally compact quantum groups. The same discipline continued by the authors in [14]. Daws and Runde [17] introduced the Reiter's properties P_1 and P_2 for a general locally compact quantum group. Following [17], here we present a slightly different approach to give various Reiter's properties for the action of a locally compact quantum group on a von Neumann algebra. This method generalizes not only several results of [2, 3, 11, 12, 18] but also allows us to give shorter proofs for the main results of [17] .

We first fix some notations. If H is a Hilbert space, then $\mathcal{B}(H)$ and $\mathcal{K}(H)$ denote the algebra of all bounded and compact operators on H, respectively. For B, a $*$-algebra, $M(B)$ denotes the multiplier algebra of B. As usual, $\otimes$ denotes tensor product; depending on the context, it may be the algebraic tensor product of linear spaces, the tensor product of Hilbert spaces, the minimal tensor product of C*-algebras or the tensor product of von Neumann algebras. If E and F are operator spaces, as in [5], we denote the completely bounded operators from E to F by $\mathcal{CB}(E, F)$ and use $\check{\otimes}$ and $\hat{\otimes}$ to denote the injective and projective tensor product of operator spaces, respectively; it should mentioned that in the C*-algebraic setting the injective and minimal tensor products coincide. For $n \in \mathbb{N}$, the n-th matrix level of E is denoted by $\mathbb{M}_n(E)$ and for a linear map $T : E \to F$, we write $T^{(n)} : \mathbb{M}_n(E) \to \mathbb{M}_n(F)$ for the n-th amplification of T. If H is a Hilbert space, we denote the column operator space over H by H_c.

2. **Amenability of** $(\mathfrak{N}, \alpha)$

Here, we introduce the amenability of $(\mathfrak{N}, \alpha)$, where α is a left action of a Hopf-von Neumann algebra $(\mathfrak{M}, \Gamma)$ on a von Neumann algebra $\mathfrak{N}$. Then, we investigate some of its equivalent formulations. First, we formulate briefly the notions of Hopf-von Neumann algebras.

Definition 2.1. A Hopf-von Neumann algebra is a pair $(\mathfrak{M}, \Gamma)$, where $\mathfrak{M}$ is a von Neumann algebra and $\Gamma : \mathfrak{M} \longrightarrow \mathfrak{M} \otimes \mathfrak{M}$ is a normal, unital $*$-homomorphism satisfying $(\iota \otimes \Gamma)\Gamma = (\Gamma \otimes \iota)\Gamma$.

Let $(\mathfrak{M}, \Gamma)$ be a Hopf-von Neumann algebra. Then, the unique predual $\mathfrak{M}_*$ of $\mathfrak{M}$ turns into a Banach algebra under the product $*$ given by:

$$(\omega * \omega')(x) = (\omega \otimes \omega')(\Gamma(x)) \quad (\omega, \omega' \in \mathfrak{M}_*, x \in \mathfrak{M}).$$

Let G be a locally compact group. For the so-called Hopf-von Neumanna algebras $(\mathrm{L}^\infty(G), \Gamma_a)$ and $(\mathrm{VN}(G), \Gamma_s)$, we use the notations $\mathbb{H}_a$ and $\mathbb{H}_s$, respectively, where,

$$\Gamma_a(f)(s,t) = f(st) \qquad (f \in \mathrm{L}^\infty(G),\ s, t \in G),$$
$$\Gamma_s : \mathrm{VN}(G) \longrightarrow \mathrm{VN}(G) \otimes \mathrm{VN}(G), \qquad \lambda(t) \longrightarrow \lambda(t) \otimes \lambda(t) \quad (t \in G),$$

in which, λ is the left regular representation of G on $\mathrm{L}^2(G)$. It is worthwhile mentioning that for $\mathbb{H}_a$ the product $*$ imposed on $\mathrm{L}^1(G)$ is just the usual convolution on $\mathrm{L}^1(G)$, whereas for $\mathbb{H}_s$ it yields the usual pointwise product on $A(\mathrm{G})$.

For the rest of this section we fix a Hopf-von Neumann algebra $\mathbb{H} = (\mathfrak{M}, \Gamma)$. A state M of $\mathfrak{M}$ is called a left invariant mean for $\mathbb{H}$ if

$$M((\omega \otimes \iota)(\Gamma(x))) = \omega(1)M(x) \qquad (\omega \in \mathfrak{M}_*,\ x \in \mathfrak{M}).$$

If there is a left invariant mean on $\mathbb{H}$, then we call $\mathbb{H}$ left amenable. We shall require the notion of left amenability for a more general case, as presented in the next definition.

Definition 2.2. Let $\mathbb{H}$ be a Hopf-von Neumann algebra, and let $\mathfrak{N}$ be a von Neumann algebra. A left action of $\mathbb{H}$ on $\mathfrak{N}$ is a normal, injective $*$-homomorphism $\alpha : \mathfrak{N} \longrightarrow \mathfrak{M} \otimes \mathfrak{N}$ such that $(\iota \otimes \alpha)\alpha = (\Gamma \otimes \iota)\alpha$. A state N of $\mathfrak{N}$ is called an α-left invariant mean if

$$N((\omega \otimes \iota)(\alpha(y))) = \omega(1)N(y) \qquad (\omega \in \mathfrak{M}_*,\ y \in \mathfrak{N}).$$

If there is an α-left invariant mean on $\mathfrak{N}$, then we call $(\mathfrak{N}, \alpha)$ left amenable.

Definition 2.3. Let $\mathbb{H} = (\mathfrak{M}, \Gamma)$ be a Hopf-von Neumann algebra. We say that U is a unitary left corepresentation of $\mathbb{H}$ and write $U \in \mathcal{CR}(\mathbb{H})$ if there is a Hilbert space H_U such that U is a unitary element of $\mathfrak{M} \otimes \mathcal{B}(H_U)$ with $(\Gamma \otimes \iota)(U) = U_{13}U_{23}$.

Example 2.4.
(i) *The comultiplication* Γ *is a left action of* $\mathbb{H}$ *on* $\mathfrak{M}$ *and* $\mathbb{H}$ *is amenable if and only if the pair* $(\mathfrak{M}, \Gamma)$ *is left amenable.*
(ii) *If* $U \in \mathcal{CR}(\mathbb{H})$*, then* $\alpha_U : \mathcal{B}(H_U) \longrightarrow \mathfrak{M} \otimes \mathcal{B}(H_U)$*, where* $\alpha_U(x) = U^*(1 \otimes x)U$ $(x \in \mathcal{B}(H_U))$ *is a left action of* $\mathbb{H}$ *on* $\mathcal{B}(H_U)$ *and* $(\mathcal{B}(H_U), \alpha_U)$ *is left amenable if and only if* U *is left amenable in the sense of* [2, Definition 4.1]. *Moreover,* $(\Gamma \otimes \iota)(U^*) = U_{23}^* U_{13}^*$ *and* α_{U^*} *is also a left action of* $\mathbb{H}$ *on* $\mathcal{B}(H_U)$. $(\mathcal{B}(H_U), \alpha_{U^*})$ *is left amenable if and only if* U *is right amenable in the sense of* [2, Definition 4.1].
(iii) *If* G *is a locally compact group and* π *is a unitary representation of* G *on a Hilbert space* H_π*, then* $\alpha_\pi : \mathcal{B}(H_\pi) \longrightarrow L^\infty(G) \otimes \mathcal{B}(H_\pi) \simeq L^\infty(G, \mathcal{B}(H_\pi))$*, where* $\alpha_\pi(T)(s) = \pi(s^{-1})T\pi(s)$ $(T \in \mathcal{B}(H_\pi), s \in G)$ *is a left action of* $\mathbb{H}_a$ *on the von Neumann algebra* $\mathcal{B}(H_\pi)$. $(\mathcal{B}(H_\pi), \alpha_\pi)$ *is left amenable if and only if* π *is amenable in the sense of* [3, Definition 1.1].
(iv) *If* G *is a locally compact group and* $s \longrightarrow \beta_s$ *is an action of* G *on the von Neumann algebra* $\mathfrak{N}$*, then* $\alpha_G : \mathfrak{N} \longrightarrow L^\infty(G) \otimes \mathfrak{N} \simeq L^\infty(G, \mathfrak{N})$*, where* $\alpha_G(y)(s) = \beta_{s^{-1}}(y)$ $(y \in \mathfrak{N}, s \in G)$ *is a left action of* $\mathbb{H}_a$ *on* $\mathfrak{N}$. *Moreover, by* [6, Proposition I.3] *there is a bijective correspondence between actions of* G *and left action of* $\mathbb{H}_a$ *on the von Neumann algebra* $\mathfrak{N}$. $(\mathfrak{N}, \alpha_G)$ *is left amenable if and only if* $\mathfrak{N}$ *is* G*-amenable in the sense of* [11, Definition 3.1] *or* G *acts amenably on* $\mathfrak{N}_*$ *in the sense of* [18, Definition 1.4].

We commence with the following proposition.

Proposition 2.5. *For a Hopf-von Neumann algebra* $\mathbb{H}$*, the following assertions are equivalent:*
(i) $\mathbb{H}$ *is left amenable.*
(ii) *For every von Neumann algebra* $\mathfrak{N}$ *and every left action* α *of* $\mathbb{H}$ *on it,* $(\mathfrak{N}, \alpha)$ *is left amenable.*

Proof. (ii)$\Rightarrow$(i) is clear. For the converse, let M be a left invariant mean for $\mathbb{H}$, and fix any state $\nu \in \mathfrak{N}_*$ and define $N \in \mathfrak{N}^*$, by $N(y) = M((\iota \otimes \nu)(\alpha(y)))$ $(y \in \mathfrak{N})$. For $\omega \in \mathfrak{M}_*$ and $y \in \mathfrak{N}$, we have

$$\begin{aligned} N((\omega \otimes \iota)\alpha(y)) &= M((\omega \otimes \iota \otimes \nu)(\iota \otimes \alpha)\alpha(y)) \\ &= M((\omega \otimes \iota \otimes \nu)(\Gamma \otimes \iota)\alpha(y)) \\ &= M((\omega \otimes \iota)\Gamma((\iota \otimes \nu)\alpha(y))) \\ &= \omega(1)N(y), \end{aligned}$$

that is, N is an α-left invariant mean. □

For the left action α of $\mathbb{H}$ on $\mathfrak{N}$, let α_* denote the restriction of the adjoint of α to $(\mathfrak{M} \otimes \mathfrak{N})_*$. Thus, we have, $\alpha_* : \mathfrak{M}_* \hat{\otimes} \mathfrak{N}_* \longrightarrow \mathfrak{N}_*$ [5, Corollary 4.1.9]. The following result needs a standard argument which is skipped here.

Proposition 2.6. *Let α be a left action of $\mathbb{H}$ on a von Neumann algebra $\mathfrak{N}$ Then, the following assertions are equivalent:*

- (i) *For any $\epsilon > 0$ and any finite subset $\{\omega_1, \omega_2, ..., \omega_n\}$ of $\mathfrak{M}_*$, there exists a state $\nu \in \mathfrak{N}_*$ such that $\|\alpha_*(\omega_k \otimes \nu) - \omega_k(1)\nu\| < \epsilon$, for $k = 1, 2, ..., n$.*
- (ii) *There is a net $\{\nu_i\}$ of states in $\mathfrak{N}_*$ such that $\lim_i \|\alpha_*(\omega \otimes \nu_i) - \omega(1)\nu_i\| = 0$, for all $\omega \in \mathfrak{M}_*$.*
- (iii) *There is a net $\{\nu_i\}$ of states in $\mathfrak{N}_*$ such that $\alpha_*(\omega \otimes \nu_i) - \omega(1)\nu_i \xrightarrow{w} 0$ in $\mathfrak{N}_*$, for all $\omega \in \mathfrak{M}_*$.*

We say that a pair $(\mathfrak{N}, \alpha)$ has the Reiter's property FP_1 if it satisfies one of the equivalent conditions of Proposition 2.6. In the cases presented in parts (i), (iii) and (iv) of Example 2.4, we respectively have: $(\mathfrak{M}, \Gamma)$ has Reiter's property FP_1 if and only if the condition (xi) of [8, Theorem 2.4] holds. $(\mathcal{B}(H_\pi), \alpha_\pi)$ has Reiter's property FP_1 if and only if the representation π satisfies condition (iii) of [3, Theorem 3.6], and $(\mathfrak{N}, \alpha_G)$ has Reiter's property FP_1 if and only if $s \to \beta_s$ satisfies condition (3) of [18, Corollary 1.12].

For an action α of $\mathbb{H}$ on a von Neumann algebra $\mathfrak{N}$, we define the closed linear subspace $\mathrm{LUC}(\mathfrak{N}, \alpha)$ of $\mathfrak{N}$ by

$$\mathrm{LUC}(\mathfrak{N}, \alpha) = \overline{\mathrm{span}}\{(\omega \otimes \iota)\alpha(y); \quad y \in \mathfrak{N}, \omega \in \mathfrak{M}_*\}.$$

Since $(\omega \otimes \iota)\alpha(1) = \omega(1)1$ and $((\omega \otimes \iota)\alpha(y))^* = (\bar{\omega} \otimes \iota)\alpha(y^*)$, where $\bar{\omega}(x) = \overline{\omega(x^*)}$, $\mathrm{LUC}(\mathfrak{N}, \alpha)$ contains the identity and is also self adjoint.

In the cases (i), (iii) and (iv) of Example 2.4, we respectively have: $\mathrm{LUC}(\mathfrak{M}, \Gamma) = \mathrm{LUC}(\mathbb{G})$ as defined in [16, Definition 2.2], $\mathrm{LUC}(\mathcal{B}(H_\pi), \alpha_\pi) = X(H_\pi)$ as defined in [3, Definition 3.1] and $\mathrm{LUC}(\mathfrak{N}, \alpha_G) = \mathrm{UC}(\mathfrak{N})$ as defined in [18, Definition 1.5].

The next result contains some fairly familiar characterizations of amenability of an action which covers [8, Theorem 2.4], [1, Proposition 6.4], [3, Theorem 3.5, 3.6] and [18, Proposition 1.10, Corollary 1.12] for the special cases presented in Example 2.4, respectively. See also [16, Theorem 3.4].

Lemma 2.7. *Let α be a left action of $\mathbb{H}$ on a von Neumann algebra $\mathfrak{N}$. Consider the following assertions:*
(i) $(\mathfrak{N}, \alpha)$ *has Reiter's property FP_1.*
(ii) $(\mathfrak{N}, \alpha)$ *is left amenable.*
(iii) *There is a left invariant mean on* $\mathrm{LUC}(\mathfrak{N}, \alpha)$.

Then, (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) and these are equivalent in the case where $\mathfrak{M}_*$ has a bounded approximate identity consisting of states.

Proof. (i)$\Rightarrow$(ii): A direct verification reveals that any w*-cluster point of the net $\{\nu_i\}$, presented in Proposition 2.6, is an α-left invariant mean.
(ii)$\Rightarrow$ (i): Let $N \in \mathfrak{N}^*$ be an α-left invariant mean. There exists a net $\{\nu_i\}$ of states in $\mathfrak{N}_*$ such that $\nu_i \xrightarrow{w^*} N$ in $\mathfrak{N}^*$, and thus $\alpha_*(\omega \otimes \nu_i) - \omega(1)\nu_i \xrightarrow{w} 0$ in $\mathfrak{N}_*$, for all $\omega \in \mathfrak{M}_*$.
(ii)$\Rightarrow$(iii) is trivial. Just restrict the α-left invariant mean to $\mathrm{LUC}(\mathfrak{N}, \alpha)$.
(iii)$\Rightarrow$(ii): Let $\{\omega_i\}$ be a bounded approximate identity for $\mathfrak{M}_*$ consisting of states and let M_0 be a left invariant mean on $\mathrm{LUC}(\mathfrak{N}, \alpha)$. Define $N : \mathfrak{N} \longrightarrow \mathbb{C}$ by

$$N(y) = \lim_{i \in I} M_0((\omega_i \otimes \iota)\alpha(y)) \qquad (y \in \mathfrak{N}).$$

It is immediate that N is a state on $\mathfrak{N}$. Since for each $\omega \in \mathfrak{M}_*$, $\lim_i(\omega \otimes \omega_i)\Gamma = \lim_i(\omega_i \otimes \omega)\Gamma$, we have

$$\begin{aligned} N((\omega\otimes\iota)\alpha(y)) &= \lim_{i\in I} M_0((\omega\otimes\omega_i\otimes\iota)(\iota\otimes\alpha)\alpha(y)) \\ &= \lim_{i\in I} M_0((\omega\otimes\omega_i\otimes\iota)(\Gamma\otimes\iota)\alpha(y)) \\ &= \lim_{i\in I} M_0((\omega_i\otimes\omega\otimes\iota)(\Gamma\otimes\iota)\alpha(y)) \\ &= \lim_{i\in I} M_0((\omega\otimes\iota)\alpha((\omega_i\otimes\iota)\alpha(y))) \\ &= \omega(1)N(y). \end{aligned}$$

Therefore, N is an α-left invariant mean on $\mathfrak{N}$. □

3. Reiter's property P_1 for $(\mathfrak{N},\alpha)$

We start this section with recalling some basic definitions and properties of the von Neumann algebraic locally compact quantum groups developed by Kustermans and Vaes in [9, 10]. We begin with recalling some notions about normal semi-finite faithful (n.s.f.) weights on von Neumann algebras [19].

Let $\mathfrak{M}$ be a von Neumann algebra and let $\mathfrak{M}^+$ denote its positive elements. For a weight φ on $\mathfrak{M}$, let

$$\mathcal{M}_\varphi^+ := \{x\in\mathfrak{M}^+ : \varphi(x)<\infty\} \quad\text{and}\quad \mathcal{N}_\varphi := \{x\in\mathfrak{M} : x^*x\in\mathcal{M}_\varphi^+\}.$$

Definition 3.1. A locally compact quantum group is a Hopf-von Neumann algebra $\mathbb{H}=(\mathfrak{M},\Gamma)$ if

- there is an n.s.f. weight φ on $\mathfrak{M}$ which is left invariant; i.e.,
$$\varphi((\omega\otimes\iota)(\Gamma(x)))=\omega(1)\varphi(x) \qquad (\omega\in\mathfrak{M}_*, x\in\mathcal{M}_\varphi^+),$$
- there is an n.s.f. weight ψ on $\mathfrak{M}$ which is right invariant; i.e.,
$$\psi((\iota\otimes\omega)(\Gamma(x)))=\omega(1)\psi(x) \qquad (\omega\in\mathfrak{M}_*, x\in\mathcal{M}_\psi^+).$$

For a locally compact group G, we have two locally compact quantum groups $(\mathrm{L}^\infty(G),\Gamma_a,\varphi_a,\psi_a)$ and $(\mathrm{VN}(G),\Gamma_s,\varphi_s,\psi_s)$, in which φ_a and ψ_a are the left and right Haar integrals, respectively, and $\varphi_s=\psi_s$ is the Plancherel weight on $\mathrm{VN}(G)$ [19, Definition VII.3.2].

Let $\mathfrak{M}$ be in its standard form related to the GNS-construction (H,ι,Λ) for the left invariant n.s.f. weight φ. Then, there exists a unique unitary—the multiplicative unitary—$W \in B(H \otimes H)$ such that

$$W^*(\Lambda(x) \otimes \Lambda(y)) = (\Lambda \otimes \Lambda)((\Gamma(y))(x \otimes 1)) \qquad (x, y \in \mathcal{N}_\varphi),$$

Satisfying $(\Gamma \otimes i)(W) = W_{13}W_{23}$ and $\Gamma(x) = W^*(1 \otimes x)W$ $(x \in \mathfrak{M})$; see [10, Theorem 1.2] and [9, p. 913].

Equivalently, the von Neumann algebraic quantum group $(\mathfrak{M}, \Gamma, \varphi)$ has an underlying C*-algebraic quantum group $(\mathcal{A}, \Gamma_c, \varphi_c)$ as discussed in [9], where,

$$\mathcal{A} = \{(i \otimes \omega)(W);\ \omega \in \mathcal{B}(H)_*\}^{-\|\cdot\|},$$

and Γ_c and φ_c are the restriction of Γ and φ to $\mathcal{A}$ and $\mathcal{A}^+$, respectively [10, Proposition 1.7] and [21, Proposition A.2, A.5]. The dual space of $\mathcal{A}$ becomes a Banach algebra with the product $*_c$ given by $(f *_c g)(x) = (f \otimes g)\Gamma_c(x)$ $(f, g \in \mathcal{A}^*,\ x \in \mathcal{A})$. It canonically contains $\mathfrak{M}_*$ as a closed ideal [9, p. 193].

Similar to the group setting we shall use the following notation: the locally compact quantum group $(\mathfrak{M}, \Gamma, \varphi)$ is denoted by $\mathbb{G}$, and we write $\mathrm{L}^\infty(\mathbb{G})$ for $\mathfrak{M}$, $\mathrm{L}^1(\mathbb{G})$ for $\mathfrak{M}_*$, $\mathrm{L}^2(\mathbb{G})$ for H, $\mathrm{C}_0(\mathbb{G})$ for $\mathcal{A}$ and $\mathrm{M}(\mathbb{G})$ for $\mathcal{A}^*$.

We shall apply notions such as left amenability and unitary left corepresentation to locally compact quantum groups whenever they make sense for the underlying Hopf-von Neumann algebras and we write $\mathcal{CR}(\mathbb{G})$ instead of $\mathcal{CR}(\mathbb{H})$. It should be mentioned that $U \in \mathcal{CR}(\mathbb{G})$ on a Hilbert space H_U is, in a sense, automatically continuous; i.e., $U \in M(\mathrm{C}_0(\mathbb{G}) \otimes \mathcal{K}(H_U))$ [22, Theorem 1.6].

Before we proceed unit the definitions, let us describe our main aim with more details. Let G be a locally compact group, $p \in \{1, 2\}$ and $h \in \mathrm{L}^p(G)$. The mapping $F[h] : G \longrightarrow \mathrm{L}^p(G)$, given by $F[h](s) = L_{s^{-1}}h \quad (s \in G)$, is bounded and continuous, where $(L_s h)(t) = h(st) \quad (t \in G)$. G is said to have Reiter's property P_p if there is a net $\{h_i\}$ of non-negative, norm one functions in $\mathrm{L}^p(G)$ such that for each compact subset K of G,

$$\lim_i \sup_{x \in K} \|F[h_i] - h_i\|_p = 0.$$

Moreover, these properties are equivalent to the amenability of G [13, Proposition 6.12]. Let $a \in \mathrm{C}_0(G)$ and define $F_a[h] : G \longrightarrow \mathrm{L}^p(G)$ by $F_a[h](s) = a(s)L_{s^{-1}}h$ $(s \in G)$. Since $a \in \mathrm{C}_0(G)$, we have $F_a[h] \in \mathrm{C}_0(G, \mathrm{L}^p(G)) \cong \mathrm{C}_0(G) \otimes^\lambda \mathrm{L}^p(G)$, where $\otimes^\lambda$ denotes the injective tensor product of Banach spaces. It is straightforward to verify that G has

Reiter's property P_p if and only if there is a net $\{h_i\}$ of non-negative, norm one functions in $\mathrm{L}^p(G)$ such that for each $a \in \mathrm{C}_0(G)$,

$$\lim_i \|F_a[h_i] - a \otimes h_i\|_{\otimes^\lambda} = 0.$$

Since $\mathrm{C}_0(G)$ is a minimal operator space, $\mathrm{C}_0(G) \otimes^\lambda \mathrm{L}^1(G) \cong \mathrm{C}_0(G)\check{\otimes}\mathrm{L}^1(G)$ and $\mathrm{C}_0(G) \otimes^\lambda \mathrm{L}^2(G) \cong \mathrm{C}_0(G)\check{\otimes}(\mathrm{L}^2(G))_c$ [5, §8.2]. Daws and Runde [17], by the canonical embedding $\mathrm{C}_0(G)\check{\otimes}\mathrm{L}^1(G)$ and $\mathrm{C}_0(G)\check{\otimes}(\mathrm{L}^2(G))_c$ into $\mathcal{CB}(\mathrm{L}^\infty(G), \mathrm{C}_0(G))$ and $\mathcal{CB}((\mathrm{L}^2(G))^*_c, \mathrm{C}_0(G))$, respectively, could extend Reiter's properties P_1 and P_2 from locally compact groups to locally compact quantum groups. Our motivation comes from the fact that $\mathrm{C}_0(G)\check{\otimes}\mathrm{L}^1(G)$ and $\mathrm{C}_0(G)\check{\otimes}(\mathrm{L}^2(G))_c$ can canonically embed into $\mathcal{CB}(\mathrm{M}(G), \mathrm{L}^1(G))$ and $\mathcal{CB}(\mathrm{M}(G), (\mathrm{L}^2(G))_c)$, respectively.

Let $\alpha : \mathfrak{N} \longrightarrow \mathrm{L}^\infty(\mathbb{G}) \otimes \mathfrak{N}$ be a left action of a locally compact quantum group $\mathbb{G}$ on a von Neumann algebra $\mathfrak{N}$, and let $a, b \in \mathrm{C}_0(\mathbb{G})$ and $\nu \in \mathfrak{N}_*$. Define,

$$F^\alpha_{a,b}[\nu] : \mathrm{M}(\mathbb{G}) \longrightarrow \mathfrak{N}^*, \quad F^\alpha_{a,b}[\nu](f) = \alpha^*(bfa \otimes \nu); \quad (f \in \mathrm{M}(\mathbb{G})).$$

It is immediate that $F^\alpha_{a,b}[\nu]$ is bounded.

Proposition 3.2. *Let α be a left action of $\mathbb{G}$ on $\mathfrak{N}$, $\nu \in \mathfrak{N}_*$, and let $a, b \in \mathrm{C}_0(\mathbb{G})$. Then, $F^\alpha_{a,b}[\nu]$ lies in $\mathcal{CB}(\mathrm{M}(\mathbb{G}), \mathfrak{N}^*)$ and can be identified with an element of $\mathrm{C}_0(\mathbb{G})\check{\otimes}\mathfrak{N}^*$.*

Proof. Let θ be an n.s.f. weight on $\mathfrak{N}$, and let $(\Lambda_\theta, \iota, H_\theta)$ be the GNS-construction for it. Since $\mathfrak{N}$ is in standard form on H_θ, there are $L, S \in \mathcal{K}(H_\theta)$ such that $\nu(y) = \nu(LyS)$ $(y \in \mathfrak{N})$. For $f \in \mathrm{M}(\mathbb{G})$ and $y \in \mathfrak{N}$, we have

$$(F^\alpha_{a,b}[\nu](f))(y) = (bfa \otimes \nu)\alpha(y) = (f \otimes \nu)((a \otimes S)\alpha(y)(b \otimes L)).$$

Let $U_\alpha \in \mathrm{L}^\infty(\mathbb{G}) \otimes \mathcal{B}(H_\theta)$ be the unitary corepresentation such that U^*_α is the unitary implementation, as defined in [20, Definition 3.6], of the left action α. Since $\alpha = \alpha_{U_\alpha}$, the mapping

$$y \rightarrow (i \otimes \nu)((a \otimes S)\alpha(y)(b \otimes L)) : \mathfrak{N} \rightarrow \mathrm{C}_0(\mathbb{G})$$

is cb-norm limit of a net of finite rank operators in $\mathcal{CB}(\mathfrak{N}, \mathrm{C}_0(\mathbb{G}))$. By taking adjoint, $F^\alpha_{a,b}[\nu]$ is also cb-norm limit of a net of finite rank operators in $\mathcal{CB}(\mathrm{M}(\mathbb{G}), \mathfrak{N}^*)$. Thus, $F^\alpha_{a,b}[\nu] \in \mathcal{CB}(\mathrm{M}(\mathbb{G}), \mathfrak{N}^*)$ and can be identified with an element of $\mathrm{C}_0(\mathbb{G})\check{\otimes}\mathfrak{N}^*$. □

Definition 3.3. Let α be a left action of $\mathbb{G}$ on $\mathfrak{N}$. We say that $(\mathfrak{N},\alpha)$ has Reiter's property P_1 if there is a net $\{\nu_i\}$ of states in $\mathfrak{N}_*$ such that

$$\lim_i \|F^{\alpha}_{a,b}[\nu_i] - ab\otimes\nu_i\|_{\check{\otimes}} = 0$$

in $\mathrm{C}_0(\mathbb{G})\check{\otimes}\mathfrak{N}^*$, for all $a,b\in \mathrm{C}_0(\mathbb{G})$.

In the cases (i), (iii) and (iv) presented in Example 2.4, we respectively have: $(\mathfrak{M},\Gamma)$ has Reiter's property P_1 if and only if $\mathbb{G}$ has Reiter's property P_1 in the sense of [17, Definition 3.3]; in particular, G has Reiter's property P_1 if and only if $(\mathrm{L}^\infty(G),\Gamma_a)$ has Reiter's property P_1. $(\mathcal{B}(H_\pi),\alpha_\pi)$ has Reiter's property P_1 if and only if the representation π has Reiter's property $(P_1)_\pi$ in the sense of [3, Definition 4.1]; and $(\mathfrak{N},\alpha_G)$ has Reiter's property P_1 if and only if the action $s\longrightarrow\beta_s$ satisfies the condition (iii) of [11, Proposition 3.2] or satisfies the condition (2) of [18, Proposition 1.13].

For the proof of the main theorem of this section that covers [17, Theorem 4.5], [11, Proposition 3.2], [3, Theorem 4.3] and [18, Proposition 1.13], we quote the following technical lemma from [17].

Lemma 3.4. ([17, Lemma 4.4]) *Let E_0, E and F be operator spaces, and let $S\in\mathcal{CB}(E,E_0)$ lies in the cb-norm closure of the finite rank operators. Then, for every norm bounded net $\{T_i\}$ in $\mathcal{CB}(E_0,F)$ that converges to $T\in\mathcal{CB}(E_0,F)$ pointwise on E_0, we have*

$$\lim_i \|T_i\circ S\|_{\check{\otimes}} = \lim_i \sup_{n\in\mathbb{N}} \sup_{\boldsymbol{f}\in\mathbb{M}_n(E)_1} \|(T_i\circ S)^{(n)}(\boldsymbol{f})\|_n = 0,$$

where $\mathbb{M}_n(E)_1 = \{\boldsymbol{f}\in\mathbb{M}_n(E) \;\; ; \;\|\boldsymbol{f}\|_n\le 1\}$.

Theorem 3.5. *Let α be a left action of $\mathbb{G}$ on $\mathfrak{N}$. Then, the following assertions are equivalent:*

(i) *$(\mathfrak{N},\alpha)$ is left amenable.*

(ii) *$(\mathfrak{N},\alpha)$ has Reiter's property P_1.*

Proof. (i)⇒(ii): Let $a,b\in \mathrm{C}_0(\mathbb{G})$ and $\omega_0\in \mathrm{L}^1(\mathbb{G})$ be an arbitrary state. By Lemma 2.7, there exists a net $\{\nu'_i\}_{i\in I}$ of states in $\mathfrak{N}_*$ such that $\lim_i\|\alpha_*(\omega\otimes\nu'_i)-\omega(1)\nu'_i\|=0$, for all $\omega\in \mathrm{L}^1(\mathbb{G})$. For $i\in I$, set $\nu_i=\alpha_*(\omega_0\otimes\nu'_i)$ and define $T_i:\mathrm{L}^1(\mathbb{G})\to\mathfrak{N}_*$ by

$$T_i(\omega)=\alpha_*(\omega\otimes\nu'_i)-\omega(1)\nu'_i.$$

Thus, $\lim_i \|\nu_i - \nu_i'\| = 0$ and the net $\{T_i\}$, which lies in $\mathcal{CB}(\mathrm{L}^1(\mathbb{G}), \mathfrak{N}_*)$, is norm bounded and also converges to 0 pointwise on $\mathrm{L}^1(\mathbb{G})$.

By Proposition 3.2, $S := F^{\Gamma}_{a,b}[\omega_0] \in \mathcal{CB}(\mathrm{M}(\mathbb{G}), \mathrm{L}^1(\mathbb{G}))$ belongs to the cb-norm closure of the finite rank operators. For $f \in \mathrm{M}(\mathbb{G})$, we have

$$\begin{aligned} F^{\alpha}_{a,b}[\nu_i](f) &= \alpha_*(bfa \otimes \alpha_*(\omega_0 \otimes \nu_i')) \\ &= (bfa \otimes \omega_0 \otimes \nu_i')(i \otimes \alpha)\alpha \\ &= (bfa \otimes \omega_0 \otimes \nu_i')(\Gamma \otimes i)\alpha \\ &= (bfa *_c \omega_0 \otimes \nu_i')\alpha \\ &= \alpha_*(S(f) \otimes \nu_i'). \end{aligned}$$

So, by Lemma 3.4, we have,

$$\lim_i \|F^{\alpha}_{a,b}[\nu_i] - ab \otimes \nu_i\|_{\hat{\otimes}} \le \lim_i \left(\|T_i \circ S\|_{\hat{\otimes}} + \|ab \otimes \nu_i - ab \otimes \nu_i'\|_{\hat{\otimes}} \right) = 0.$$

This proves that (ii) holds.
(ii) ⇒ (i): Let $\{\nu_i\}$ be a net satisfying Definition 3.3, and let $\omega \in \mathrm{L}^1(\mathbb{G})$. By the Cohen's factorization theorem, [4, Corollary 2.9.26], there are $a, b \in \mathrm{C}_0(\mathbb{G})$ and $\omega' \in \mathrm{L}^1(\mathbb{G})$ such that $\omega = b\omega' a$. We then have:

$$\begin{aligned} \lim_i \|\alpha_*(\omega \otimes \nu_i) - \omega(1)\nu_i\| &= \lim_i \|F^{\alpha}_{a,b}[\nu_i](\omega') - \omega'(ab)\nu_i\| \\ &\le \|\omega'\| \lim_i \|F^{\alpha}_{a,b}[\nu_i] - ab \otimes \nu_i\|_{\hat{\otimes}} = 0. \end{aligned}$$

This together with Lemma 2.7 imply that (i) holds. □

Applying Theorem 3.5 together with Lemma 2.7 for the action presented in Example 2.4 (ii), we have the next result, which provides some equivalences for the left amenability of $U \in \mathcal{CR}(\mathbb{G})$, as in [2, Defintion 4.1]. Recall that a locally compact quantum group $\mathbb{G}$ is called co-amenable if the Banach algebra $\mathrm{L}^1(\mathbb{G})$ has a bounded approximate identity.

Corollary 3.6. *For a locally compact quantum group $\mathbb{G}$, consider the following assertions:*

(i) *$U \in \mathcal{CR}(\mathbb{G})$ is left amenable.*
(ii) *$(\mathcal{B}(H_U), \alpha_U)$ has the Reiter's property FP_1.*
(iii) *$(\mathcal{B}(H_U), \alpha_U)$ has the Reiter's property P_1.*
(iv) *There is a left invariant mean on $\mathrm{LUC}(\mathcal{B}(H_U), \alpha_U)$.*

Then, (i) ⇔ (ii) ⇔ (iii) ⇒ (iv) and these are equivalent in the case where $\mathbb{G}$ is co-amenable.

4. **Reiter's Property** P_2 **for** $(\mathfrak{N}, \alpha)$

Let $\alpha : \mathfrak{N} \longrightarrow \mathrm{L}^\infty(\mathbb{G})\bar{\otimes}\mathfrak{N}$ be a left action of a locally compact quantum group $\mathbb{G}$ on $\mathfrak{N}$. Fix an n.s.f. weight θ on $\mathfrak{N}$ with the corresponding GNS-construction $(\Lambda_\theta, \iota, H_\theta)$. Let $U_\alpha \in \mathcal{CR}(\mathbb{G})$, acting on H_θ, be such that U_α^* is the unitary implementation, as defined in [20, Definition 3.6], of the left action α; in other words, $\alpha = \alpha_{U_\alpha}$ on $\mathfrak{N}$. Let $\xi \in H_\theta$ and $a, b \in \mathrm{C}_0(\mathbb{G})$ and define

$$F^\alpha_{a,b}[\xi] : \mathrm{M}(\mathbb{G}) \to H_\theta \quad \text{by} \quad F^\alpha_{a,b}[\xi](f) = (bfa \otimes i)(U_\alpha)\xi.$$

Then, we have the following result.

Lemma 4.1. *Let α be a left action of $\mathbb{G}$ on $\mathfrak{N}$, $\xi \in H_\theta$, and let $a, b \in \mathrm{C}_0(\mathbb{G})$. Then, $F^\alpha_{a,b}[\xi]$ lies in $\mathcal{CB}(\mathrm{M}(\mathbb{G}), (H_\theta)_c)$ and can be identified with an element of $\mathrm{C}_0(\mathbb{G})\check{\otimes}(H_\theta)_c$.*

Proof. Let $L \in \mathcal{K}(H_\theta)$ be such that $L\xi = \xi$. Since U_α is automatically continuous, i.e., $U_\alpha \in M(\mathrm{C}_0(\mathbb{G})\check{\otimes}\mathcal{K}(H_\theta))$, we have

$$(i \otimes M_\xi)((a \otimes 1)U_\alpha(b \otimes L)) \in \mathrm{C}_0(\mathbb{G})\check{\otimes}(H_\theta)_c,$$

where $M_\xi : \mathcal{K}(H_\theta) \to (H_\theta)_c$ is the completely bounded map given by $M_\xi(S) = S\xi$ $(S \in \mathcal{K}(H_\theta))$. From the canonical embedding $\mathrm{C}_0(\mathbb{G})\check{\otimes}(H_\theta)_c$ into $\mathcal{CB}(\mathrm{M}(\mathbb{G}), (H_\theta)_c)$, we have

$$(f \otimes M_\xi)((a \otimes 1)U_\alpha(b \otimes L)) = F^\alpha_{a,b}[\xi](f).$$

In other words, $F^\alpha_{a,b}[\xi] \in \mathcal{CB}(\mathrm{M}(\mathbb{G}), (H_\theta)_c)$ and it can be identified with an element of $\mathrm{C}_0(\mathbb{G})\check{\otimes}(H_\theta)_c$. □

Now, we define the property P_2 for a left action of a locally compact quantum group on a von Neumann algebra; see also [17, Definition 5.2].

Definition 4.2. Let α be a left action of $\mathbb{G}$ on $\mathfrak{N}$. We say that $(\mathfrak{N}, \alpha)$ has Reiter's property P_2 if there is a net $\{\xi_i\}$ of unit vectors in H_θ such that for every $a, b \in \mathrm{C}_0(\mathbb{G})$,

$$\lim_i \|F^\alpha_{a,b}[\xi_i] - ab \otimes \xi_i\|_{\check{\otimes}} = 0$$

in $\mathrm{C}_0(\mathbb{G})\check{\otimes}(H_\theta)_c$.

It is obvious that $(\mathrm{L}^\infty(\mathbb{G}), \Gamma)$ has Reiter's property P_2 if and only if $\mathbb{G}$ has Reiter's property P_2 in the sense of [17, Definition 5.2].

Recall that $U \in \mathcal{CR}(\mathbb{G})$ has the weak containment property (WCP) if there exists a net $\{\xi_i\}_{i\in I}$ of unit vectors in H_U such that

$$\lim_i \|U(\eta \otimes \xi_i') - \eta \otimes \xi_i'\| = 0 \qquad (\eta \in \mathrm{L}^2(\mathbb{G}));$$

see [2, §5] for details.

Now, we can prove the main result of this section that covers [17, Theorem 5.4].

Theorem 4.3. *Let α be a left action of $\mathbb{G}$ on $\mathfrak{N}$. Then, the following assertions are equivalent:*

(i) *$U_\alpha \in \mathcal{CR}(\mathbb{G})$ has the WCP.*

(ii) *$(\mathfrak{N}, \alpha)$ has Reiter's property P_2.*

Proof. (i) $\Rightarrow$ (ii): Let $a, b \in \mathrm{C}_0(\mathbb{G})$ and let $\omega_0 \in \mathrm{L}^1(\mathbb{G})$ be an arbitrary state. Let $\{\xi_i'\}_{i\in I}$, be a net of unit vectors in H_θ such that

$$\lim_i \|U_\alpha(\eta \otimes \xi_i') - \eta \otimes \xi_i'\| = 0 \qquad (\eta \in \mathrm{L}^2(\mathbb{G})),$$

or equivalently, $\lim_i \|(\omega \otimes i)(U_\alpha)\xi_i' - \omega(1)\xi_i'\| = 0$, for all $\omega \in \mathrm{L}^1(\mathbb{G})$. For $i \in I$, set $\xi_i = (\omega_0 \otimes i)(U_\alpha)\xi_i'$ and define $T_i : \mathrm{L}^1(\mathbb{G}) \to H_\theta$ by

$$T_i(\omega) = (\omega \otimes i)(U_\alpha)\xi_i' - \omega(1)\xi_i'.$$

Then, $\lim_i \|\xi_i - \xi_i'\| = 0$ and the net $\{T_i\}$, which lies in $\mathcal{CB}(\mathrm{L}^1(\mathbb{G}), (H_\theta)_c)$, is norm bounded and also converges to 0 pointwise on $\mathrm{L}^1(\mathbb{G})$. Let S be defined as in the proof of Theorem 3.5. Then, for $f \in \mathrm{M}(\mathbb{G})$,

$$\begin{aligned} F^\alpha_{a,b}[\xi_i](f) &= (bfa \otimes i)(\omega_0 \otimes i)(U_\alpha)\xi_i' \\ &= (bfa \otimes \omega_0 \otimes i)((U_\alpha)_{13}(U_\alpha)_{23})\xi_i' \\ &= (bfa \otimes \omega_0 \otimes i)((\Gamma \otimes i)(U_\alpha))\xi_i' \\ &= (bfa *_c \omega_0 \otimes i)(U_\alpha)\xi_i' \\ &= (S(f) \otimes i)(U_\alpha)\xi_i'. \end{aligned}$$

Thus, by Lemma 3.4, we have

$$\lim_i \|F^\alpha_{a,b}[\xi_i] - ab \otimes \xi_i\|_{\check{\otimes}} \le \lim_i \Big(\|T_i \circ S\|_{\check{\otimes}} + \|ab \otimes \xi_i' - ab \otimes \xi_i\|_{\check{\otimes}} \Big) = 0.$$

(ii) $\Rightarrow$ (i): Let $\{\xi_i\}$ be a net satisfying Definition (4.2), and let $\omega \in \mathrm{L}^1(\mathbb{G})$. By the Cohen's factorization theorem, [4, Corollary 2.9.26], there are

$a, b \in \mathrm{C}_0(\mathbb{G})$ and $\omega' \in \mathrm{L}^1(\mathbb{G})$ such that $\omega = b\omega' a$. Thus,

$$\begin{aligned}\lim_i \|(\omega \otimes i)(U_\alpha)\xi_i - \omega(1)\xi_i\| &= \lim_i \|F^\alpha_{a,b}[\xi_i](\omega') - \omega'(ab)\xi_i\| \\ &\leq \|\omega'\| \lim_i \|F^\alpha_{a,b}[\xi_i] - ab \otimes \xi_i\|_{\check{\otimes}} = 0,\end{aligned}$$

and this completes the proof. □

Applying Theorem 4.3 for the action presented in Example 2.4 (ii), we obtain the next result which provides an equivalence for the WCP of $U \in \mathcal{CR}(\mathbb{G})$; see [2, Theorem 5.1].

Corollary 4.4. *For a locally compact quantum group $\mathbb{G}$, the following assertions are equivalent:*

(i) *$U \in \mathcal{CR}(\mathbb{G})$ has the WCP.*
(ii) *$(\mathcal{B}(H_U), \alpha_U)$ has Reiter's property P_2.*

At this stage, we remark that it would be desirable if one could present the quantum group version of the property $(P_2)_\pi$ for $U \in \mathcal{CR}(\mathbb{G})$, in the sense of [3, Definition 4.1].

5. **The right actions**

Since Definition 2.2 and all of its consequences are given in terms of left actions, a natural question that arises is what happens if we reset our definition and results with the right actions. Here, we explain further.

Let $\mathbb{H}$ and $\mathfrak{N}$ be a Hopf-von Neumann algebra and a von Neumann algebra, respectively. A right action of $\mathbb{H}$ on $\mathfrak{N}$ is a normal, injective $*$-homomorphism $\delta : \mathfrak{N} \longrightarrow \mathfrak{N} \otimes \mathfrak{M}$ such that $(\delta \otimes \iota)\delta = (\iota \otimes \Gamma)\delta$. A state N of $\mathfrak{N}$ is called a δ-right invariant mean if

$$N((\iota \otimes \omega)(\delta(y))) = \omega(1)N(y) \qquad (\omega \in \mathfrak{M}_*,\ y \in \mathfrak{N}).$$

If there is a δ-right invariant mean on $\mathfrak{N}$, Then we call $(\delta, \mathfrak{N})$ right amenable. Two important example of such actions are as follows:

(i) The comultiplication Γ is a right action of $\mathbb{H}$ on $\mathfrak{M}$ and $\mathbb{H}$ is amenable if and only if the pair $(\Gamma, \mathfrak{M})$ is right amenable.
(ii) If $V \in \mathcal{CR}'(\mathbb{H})$, i.e., there is a Hilbert space H_V such that V is a unitary element of $\mathcal{B}(H_V) \otimes \mathfrak{M}$ with $(\iota \otimes \Gamma)(V) = V_{13}V_{12}$, then $\delta_V : \mathcal{B}(H_V) \longrightarrow \mathcal{B}(H_V) \otimes \mathfrak{M}$, where $\delta_V(x) = V^*(x \otimes 1)V$ $(x \in \mathcal{B}(H_V))$ is a right action of $\mathbb{H}$ on $\mathcal{B}(H_V)$ and $(\delta_V, \mathcal{B}(H_V))$ is right

amenable if and only if V is right amenable in the sense of [2, Defiition 4.1]. Moreover, $(\iota \otimes \Gamma)(V^*) = V^*_{12}V^*_{13}$ and δ_{V^*} is also a right action of $\mathbb{H}$ on $\mathcal{B}(H_V)$. $(\delta_{V^*}, \mathcal{B}(H_V))$ is right amenable if and only if V is left amenable in the sense of [2, Defiition 4.1].

If δ is a right action, then $\sigma\delta$ will be a left action of $\mathbb{H}^{\mathrm{op}} = (\mathfrak{M}, \Gamma^{\mathrm{op}})$ on $\mathfrak{N}$, where σ denotes the flip map and $\Gamma^{\mathrm{op}} = \sigma\Gamma$ denotes the opposite comultiplication. So, we can easily prove the right version of results presented in the left case. It should be mentioned that we must place $\mathrm{RUC}(\delta, \mathfrak{N})$ instead of $\mathrm{LUC}(\mathfrak{N}, \alpha)$ in the right version of Lemma 2.7, where $\mathrm{RUC}(\delta, \mathfrak{N})$ is defined by

$$\mathrm{RUC}(\delta, \mathfrak{N}) = \overline{\mathrm{span}}\{(\iota \otimes \omega)\delta(y); \quad y \in \mathfrak{N}, \omega \in \mathfrak{M}_*\}.$$

Since $\mathrm{RUC}(\Gamma, \mathfrak{M}) = \mathrm{RUC}(\mathbb{G})$, as defined in [16, Definition 2.2], the right version of Lemma 2.7 covers [16, Theorem 3.4] and [12, Proposition 4.7].

Let $V \in \mathcal{CR}'(\mathbb{G})$ act on the Hilbert space H_V. Set $\bar{V} = (j \otimes R)(V)$. Then, $\bar{V} \in \mathcal{CR}'(\mathbb{G})$ and $H_{\bar{V}} = \bar{H}_V$, where $j : \mathcal{B}(H_V) \longrightarrow \mathcal{B}(\bar{H}_V)$ is the canonical anti-isomorphism given by $j(x)(\bar{\xi}) = \overline{x^*(\xi)}$ $(\bar{\xi} \in \bar{H}_V)$ and $\bar{H}_V$ is the conjugate Hilbert space of H_V. Let $\mathcal{HS}(H_V)$ denote the Hilbert space of the Hilbert-Schmidt operators on H_V. Let $\Theta : H_V \otimes \bar{H}_V \longrightarrow \mathcal{HS}(H_V)$ be the canonical isometric isomorphism which is given by $\Theta(\xi \otimes \bar{\eta})(\zeta) = \langle \zeta, \eta \rangle \xi$ $(\xi, \zeta \in H_V, \bar{\eta} \in \bar{H}_V)$. Define a normal unital $*$-isomorphism,

$$\widetilde{\Theta} : \mathcal{B}(H_V \otimes \bar{H}_V) \to \mathcal{B}(\mathcal{HS}(H_V) \text{ by } \widetilde{\Theta}(T) = \Theta T \Theta^* \quad (T \in \mathcal{B}(H_V \otimes \bar{H}_V)),$$

and also define the Hilbert-Schmidt operator V_{HS} by $V_{\mathrm{HS}} = (\widetilde{\Theta} \otimes \iota)(V \times \bar{V})$, where $V \times \bar{V} \in \mathcal{CR}'(\mathbb{G})$ is given by $V \times \bar{V} = V_{13}\bar{V}_{23}$. Then, $V_{\mathrm{HS}} \in \mathcal{CR}'(\mathbb{G})$ and $H_{V_{\mathrm{HS}}} = \mathcal{HS}(H_V)$.

Now, apply the right case version of Theorem 4.3 for $V \in \mathcal{CR}'(\mathbb{G})$ to obtain the next result which provides some equivalences for the property $(\check{P}_2)$ of V^*, as in [12, Definition 4.2(a)]; see [12, Proposition 4.5].

Corollary 5.1. *Let $V \in \mathcal{CR}'(\mathbb{G})$ act on the Hilbert space H_V. Then, the following assertions are equivalent:*

(i) *V_{HS} has the WCP.*
(ii) *V^* has property $(\check{P}_2)$.*
(iii) *$(\delta_{V_{\mathrm{HS}}}, \mathcal{B}(\mathcal{HS}(H_V)))$ has Reiter's property P_2.*
(iv) *$(\delta_{(V \times \bar{V})}, \mathcal{B}(H_V \otimes \bar{H}_V))$ has Reiter's property P_2.*

Proof. Since for every $\xi \in \mathcal{HS}(H_V)$ we have $F_{a,b}^{\delta_{V_{\mathrm{HS}}}}[\xi] = F_{a,b}^{\delta_{V\times\bar{V}}}[\Theta^*(\xi)]$, the proof is straightforward. □

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M. Ramezanpour
Department of pure Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran.
Email: md_ramezanpour@yahoo.com

H. R. Ebrahimi Vishki
Department of pure Mathematics and Centre of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, Iran.
Email: vishki@um.ac.ir, vishki@ams.org