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FIXED POINT THEOREMS FOR α-ψ-CONTRACTIVE MAPPINGS IN PARTIALLY ORDERED SETS AND APPLICATION TO ORDINARY DIFFERENTIAL EQUATIONS

M. S. ASGARI AND Z. BADEHIAN*

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Abstract. In this paper, we introduce α-ψ-contractive mapping in partially ordered sets and construct fixed point theorems to solve a first-order ordinary differential equation by existence of its lower solution.
Keywords: Fixed point, α-ψ-contractive mappings, partially ordered sets, lower and upper solutions.
MSC(2010): Primary: 47H10; Secondary: 34A12.

1. Introduction

Fixed point theory has been an active research topic in mathematics. During the last four decades this theory has been extended to various areas of research. Recently, the existence of a fixed point in partially ordered sets has been considered in [1–3, 5–7, 9–11, 14, 15, 18]. Furthermore, some applications to periodic boundary value problems and matrix equations are given in [12, 13, 16]. Samet et al. [17] introduced α-ψ-contractive type mappings in complete metric space and established some fixed point theorems with applications to a second-order ordinary differential equation.

In this paper, we consider α-ψ-contractive type mappings with an additional condition for partially ordered sets and solve a first-order ordinary differential equation with respect to its lower solution.

We consider the following periodic boundary value problem

$$\left\{\begin{array}{l} u'(t) = h(t, u(t)), \quad t \in I = [0, T], \\ u(0) = u(T), \end{array}\right. \tag{1.1}$$

where $T > 0$, and $h : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

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Definition 1.1. A solution to (1.1) is a function $u \in C^1(I, \mathbb{R})$ satisfying conditions in (1.1).

Definition 1.2. A lower solution for (1.1) is a function $u \in C^1(I, \mathbb{R})$ such that

$$\begin{cases} u'(t) \leq h(t, u(t)), & t \in I = [0, T], \\ u(0) \leq u(T). \end{cases}$$

An upper solution for (1.1) satisfies the reversed inequalities.

It is well-known [8] that the existence of a lower solution u and an upper solution v with $u \leq v$ implies the existence of a solution of (1.1) between u and v. In this paper, the existence of a unique solution for problem (1.1) is obtained under suitable conditions.

2. Preliminaries

Definition 2.1. Let $(X, \leq)$ be a partially ordered set. We say that $f : X \to X$ is monotone non-decreasing if

$$x \leq y \Longrightarrow f(x) \leq f(y), \qquad (x, y \in X).$$

Definition 2.2. Let $\alpha : X \times X \to [0, \infty)$ be a function. We say that $f : X \to X$ is an ordered α-admissible function if for all $x, y \in X$,

$$x \geq y, \qquad \alpha(x, y) \geq 1 \Longrightarrow \alpha(f(x), f(y)) \geq 1. \tag{2.1}$$

Definition 2.3. ([17]) Let $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing function such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the n^{th} iterate of ψ.

Lemma 2.4. *([17]) Let $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing function, If for each $t > 0$, $\lim_{n\to\infty} \psi^n(t) = 0$ then $\psi(t) < t$.*

Let Ψ denote the class of functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

(i) ψ is nondecreasing,
(ii) for each $t > 0$, $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$,
(iii) for each $t > 0$, $\psi(t) < t$.

Definition 2.5. Let $(X, \leq)$ be a partially ordered space with a complete metric d. We say that $f : X \to X$ is an α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in X$ with $x \geq y$,

$$\alpha(x, y) d(f(x), f(y)) \leq \psi(d(x, y)). \tag{2.2}$$

3. **Fixed point theorems**

Theorem 3.1. *Let $(X, \leq)$ be a partially ordered space with a complete metric d. Let $f : X \to X$ be a non-decreasing, continuous, α-ψ-contractive and ordered α-admissible mapping. If there exists some $x_0 \in X$ such that $x_0 \leq f(x_0)$ and*

$$\alpha(f(x_0), x_0) \geq 1, \tag{3.1}$$

then, f has a fixed point.

Proof. Since $x_0 \leq f(x_0)$ and f is nondecreasing, then

$$x_0 \leq f(x_0) \leq f^2(x_0) \leq f^3(x_0) \leq \ldots \leq f^n(x_0) \leq f^{n+1}(x_0). \tag{3.2}$$

Since f is ordered α-admissible by (3.1), we get

$$\alpha(f(x_0), x_0) \geq 1 \to \alpha(f^2(x_0), f(x_0)) \geq 1 \to \ldots \to \alpha(f^{n+1}(x_0), f^n(x_0)) \geq 1. \tag{3.3}$$

Now by (2.2) and (3.3), we obtain

$$d(f^2(x_0), f(x_0)) \leq \alpha(f(x_0), x_0)d(f^2(x_0), f(x_0)) \leq \psi(d(f(x_0), x_0)).$$

Continuing this process, we get

$$d(f^{n+1}(x_0), f^n(x_0)) \leq \psi^n(d(f(x_0), x_0)).$$

Now, as $n \to \infty$ then $d(f^{n+1}(x_0), f^n(x_0)) \to 0$. We prove that $\{f^n(x_0)\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Fix $\epsilon > 0$ and let $n(\epsilon) \in \mathbb{N}$ such that

$$\sum_{n \geq n(\epsilon)} \psi^n(d(f(x_0), x_0)) < \epsilon.$$

Let $m, n \in \mathbb{N}$ with $m > n > n(\epsilon)$. By triangular inequality,

$$\begin{aligned} d(f^n(x_0), f^m(x_0)) &\leq d(f^n(x_0), f^{n+1}(x_0)) + \ldots + d(f^{m-1}(x_0), f^m(x_0)) \\ &\leq \psi^n(d(f(x_0), x_0)) + \ldots + \psi^{m-1}(d(f(x_0), x_0)) \\ &= \sum_{k=n}^{m-1} \psi^k(d(f(x_0), x_0)) \\ &\leq \sum_{n \geq n(\epsilon)} \psi^n(d(f(x_0), x_0)) < \epsilon. \end{aligned}$$

Since (X, d) is a complete metric space, there exists $x \in X$ such that $\lim_{n\to\infty} f^n(x_0) = x$. Now, we show that x is a fixed point for f. Suppose $\epsilon > 0$ is given. Since f is a continuous function, there exists $\delta > 0$ such that, for each $z \in X$, $d(z, x) < \delta$ implies that $d(f(z), f(x)) < \frac{\epsilon}{2}$. Given $\eta = min\{\frac{\epsilon}{2}, \delta\}$, by convergence of $\{f^n(x_0)\}_{n=1}^{\infty}$ to x, there exists $n_0 \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $n \geq n_0$, $d(f^n(x_0), x) < \eta$.

Taking $n \in \mathbb{N}$, $n \geq n_0$, we get

$$\begin{aligned} d(f(x), x) &\leq d(f(x), f(f^n(x_0)) + d(f^{n+1}(x_0), x) \\ &< \frac{\epsilon}{2} + \eta \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

therefore, $d(f(x), x) = 0$. Consequently, $f(x) = x$. □

In the next theorem, the continuity hypothesis of f has been removed.

Theorem 3.2. *Let $(X, \leq)$ be a partially ordered space with a complete metric d. Let $f : X \to X$ be a nondecreasing, α-ψ-contractive and ordered α-admissible mapping satisfying the following conditions:*

(i) there exists $x_0 \in X$ such that $x_0 \leq f(x_0)$ and $\alpha(f(x_0), x_0) \geq 1$;
(ii) if $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$, then $\alpha(x_n, x) \geq 1$;
(iii) if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$ then $x_n \leq x$, for all $n \in \mathbb{N}$.

Then, f has a fixed point.

Proof. Following the proof of Theorem 3.1, there exists $x \in X$ such that $\lim_{n\to\infty} f^n(x_0) = x$. We show that x is a fixed point of $f(x)$. Given $\epsilon > 0$, since $\{f^n(x_0)\}_{n=1}^{\infty}$ converges to x, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$d(f^n(x_0), x) < \frac{\epsilon}{2}.$$

Then, from (3.2) and the hypothesis (iii),

$$f^n(x_0) \leq x. \tag{3.4}$$

Now, from (2.2), (3.3), (3.4), Lemma 2.4 and (ii) we get

$$\begin{aligned} d(x, f(x)) &\leq d(f(f^n(x_0), f(x))) + d(f^{n+1}(x_0), x) \\ &\leq \alpha(f^n(x_0), x) d(f(f^n(x_0), f(x))) + d(f^{n+1}(x_0), x) \\ &\leq \psi(d(f^n(x_0), x)) + d(f^{n+1}(x_0), x) \\ &< d(f^n(x_0), x) + d(f^{n+1}(x_0), x) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore, $f(x) = x$. □

Example 3.3. Let $(\mathbb{R}, \leq)$ and $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$, then $(\mathbb{R}, d)$ is a complete metric space. Define $f : \mathbb{R} \to \mathbb{R}$ and $\alpha : X \times X \to [0, +\infty)$, by

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x; \\ 0 & \text{if } x < 0, \end{cases}$$

and

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \geq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Let $\psi(t) = \frac{t}{2}$ for each $t > 0$. Clearly, f is an α-ψ-contractive mapping. Moreover, f is nondecreasing and continuous. Now, we show that f is ordered α-admissible. For all $x, y \in [0, +\infty)$ with $x \geq y$, we obtain

$$\alpha(x,y) \geq 1 \implies \alpha(f(x), f(y)) = \alpha(\frac{x}{2}, \frac{y}{2}) \geq 1.$$

In addition, there exists $x_0 \in \mathbb{R}$ such that $\alpha(f(x_0), x_0) \geq 1$. Let $x_0 = 0$, then

$$\alpha(f(x_0), x_0) = \alpha(f(0), 0) = \alpha(0,0) = 1 \geq 1.$$

Further, since $0 \leq f(0) = 0$, we have $x_0 \leq f(x_0)$. Now, all the hypotheses of Theorem 3.1 are satisfied. Consequently, f has a fixed point. Here, 0 is a fixed point of f.

In the following example, the continuity of f has been removed.

Example 3.4. Let $(\mathbb{R}, \leq)$ and $d(x,y) = |x-y|$ for all $x, y \in \mathbb{R}$, then $(\mathbb{R}, d)$ is a complete metric space. Define $f : \mathbb{R} \to \mathbb{R}$ and $\alpha : X \times X \to [0, +\infty)$, by

$$f(x) = \begin{cases} 2x - \frac{1}{2} & \text{if } x \geq \frac{1}{2}; \\ \frac{x}{2} & \text{if } 0 \leq x < \frac{1}{2}; \\ 0 & \text{if } x < 0, \end{cases}$$

and

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in [0, \frac{1}{2}]; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, f is nondecreasing and discontinuous. Let $\psi(t) = \frac{t}{2}$ for each $t > 0$ then f is an α-ψ-contractive mapping. For all $x, y \in \mathbb{R}$ and $x \geq y$, if $\alpha(x,y) \geq 1$, then we get

$$d(f(x), f(y)) = |f(x) - f(y)| = |\frac{x}{2} - \frac{y}{2}| = \frac{|x-y|}{2}$$

and

$$\psi(d(x,y)) = \frac{d(x,y)}{2} = \frac{|x-y|}{2},$$

therefore,

$$1 \times \frac{|x-y|}{2} \leq \frac{|x-y|}{2}.$$

In other words

$$\alpha(x,y) d(f(x), f(y)) \leq \psi(d(x,y)).$$

Moreover, there exists $x_0 \in \mathbb{R}$ such that $\alpha(f(x_0), x_0) \geq 1$. Let $x_0 = 0$ then

$$\alpha(f(x_0), x_0) = \alpha(f(0), 0) = \alpha(0,0) = 1 \geq 1.$$

Also, since $0 = x_0 \leq 0 = f(x_0)$ we have $x_0 \leq f(x_0)$. Now, let $x, y \in \mathbb{R}$ with $x \geq y$, such that $\alpha(x, y) \geq 1$. This implies that $x, y \in [0, \frac{1}{2}]$. Thus

$$\alpha(x, y) = 1 \geq 1 \implies \alpha(f(x), f(y)) = \alpha(\frac{x}{2}, \frac{y}{2}) = 1 \geq 1.$$

Finally, if $\{x_n\}$ is a nondecreasing sequence in $\mathbb{R}$ such that $\alpha(x_{n+1}, x_n) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ then, by definition of α, $x_n \in [0, \frac{1}{2}]$, consequently, $x \in [0, \frac{1}{2}]$. In addition, $\{x_n\}$ is nondecreasing hence $x_n \leq x$. Therefore, all required hypotheses of Theorem 3.2 are satisfied, thus f has a fixed point. Here, 0 and $\frac{1}{2}$ are two fixed points of f.

Regarding the example 3.4, it is seen that f may have more than one fixed points. In the following, additional condition is imposed to the hypotheses of Theorems 3.1 and 3.2 to obtain uniqueness of the fixed point.

Theorem 3.5. *Suppose that all the hypotheses of Theorems 3.1 and 3.2 are satisfied. If there exists $z \in X$ such that for all $x, y \in X$,*

$$\alpha(x, z) \geq 1 \quad \text{and} \quad \alpha(y, z) \geq 1, \quad x \geq z, \quad y \geq z. \tag{3.5}$$

Then, f has a unique fixed point.

Proof. Suppose that $x^\star$ and $y^\star$ are two fixed points of f, then $f(x^\star) = x^\star$ and $f(y^\star) = y^\star$. By (3.5), there exists $z \in X$ such that

$$\alpha(x^\star, z) \geq 1 \quad \text{and} \quad \alpha(y^\star, z) \geq 1, \quad x^\star \geq z, \quad y^\star \geq z. \tag{3.6}$$

Since f is ordered α-admissible, from (2.1) and (3.6), we get

$$\alpha(f(x^\star), f(z)) \geq 1 \quad \text{and} \quad \alpha(f(y^\star), f(z)) \geq 1, \quad f(x^\star) \geq f(z), \quad f(y^\star) \geq f(z).$$

Therefore,

$$\alpha(x^\star, f(z)) \geq 1 \quad \text{and} \quad \alpha(y^\star, f(z)) \geq 1, \quad x^\star \geq f(z), \quad y^\star \geq f(z).$$

Continuing this process, we get

$$\alpha(x^\star, f^n(z)) \geq 1 \quad \text{and} \quad \alpha(y^\star, f^n(z)) \geq 1, \quad x^\star \geq f^n(z), \quad y^\star \geq f^n(z), \tag{3.7}$$

for all $n \in \mathbb{N}$. Using (2.2) and, first part of (3.7), we have

$$\begin{aligned} d(x^\star, f^n(z)) &= d(f(x^\star), f(f^{n-1}(z))) \\ &\leq \alpha(x^\star, f^{n-1}(z)) d(f(x^\star), f(f^{n-1}(z))) \\ &\leq \psi(d(x^\star, f^{n-1}(z))) \\ &\leq \psi(\psi(d(x^\star, f^{n-2}(z)))) \\ &\vdots \\ &\leq \psi^n(d(x^\star, z)), \end{aligned}$$

which implies that

$$d(x^\star, f^n(z)) \leq \psi^n(d(x^\star, z)),$$

for all $n \in \mathbb{N}$. Now, if $n \to \infty$ then $\lim_{n\to\infty} f^n(z) = x^\star$. Similarly, for the second part of (3.7), $\lim_{n\to\infty} f^n(z) = y^\star$. Therefore, $x^\star = y^\star$. That means f has a unique fixed point. $\square$

Theorem 3.6. *Let $(X, \leq)$ be a partially ordered space with a complete metric d. Let $f : X \to X$ be a nondecreasing, continuous , α-ψ-contractive and ordered α-admissible mapping. If there exists $x_0 \in X$ such that $x_0 \geq f(x_0)$ and*

$$\alpha(x_0, f(x_0)) \geq 1, \tag{3.8}$$

then f has a fixed point.

Theorem 3.7. *Let $(X, \leq)$ be a partially ordered space with complete metric d. Let $f : X \to X$ be a nondecreasing, α-ψ-contractive and ordered α-admissible mapping satisfying the following conditions:*

(i) there exists $x_0 \in X$ such that $x_0 \geq f(x_0)$ and $\alpha(x_0, f(x_0)) \geq 1$;
(ii) if $\{x_n\}_{n=1}^{\infty}$ is a sequence in X such that $\alpha(x_{n+1}, x_n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$, then $\alpha(x, x_n) \geq 1$;
(iii) if $\{x_n\}$ is a nonincreasing sequence in X such that $x_n \to x$ then $x \leq x_n$, for all $n \in \mathbb{N}$,

then f has a fixed point.

Theorem 3.8. *Suppose all the hypotheses of Theorems 3.6 and 3.7 are satisfied. If there exists $z \in X$ such that for all $x, y \in X$,*

$$\alpha(z, x) \geq 1 \quad \text{and} \quad \alpha(z, y) \geq 1, \quad z \geq x, \quad z \geq y, \tag{3.9}$$

then f has a unique fixed point.

4. Application to ordinary differential equations

Recently, Nieto and Lopez [12] have solved problem (1.1) in presence of a lower solution. This theorem is as follows.

Theorem 4.1. *Consider problem (1.1) with $h : I \times \mathbb{R} \to \mathbb{R}$ continuous. Suppose that there exist $\lambda > 0$ and $\mu > 0$ with $\mu < \lambda$ such that for all $x, y \in \mathbb{R}$, with $y \geq x$,*

$$0 \leq h(t, y) + \lambda y - h(t, x) - \lambda x \leq \mu(y - x),$$

then, the existence of a lower solution for (1.1) implies the existence of a unique solution of (1.1).

Also, Harjani and Sadarangani [6] have established the following theorem.

Theorem 4.2. *Consider problem (1.1) with $h : I \times \mathbb{R} \to \mathbb{R}$ continuous. Suppose that there exists $\lambda > 0$ such that for all $x, y \in \mathbb{R}$, with $y \geq x$,*

$$0 \leq h(t, y) + \lambda y - h(t, x) - \lambda x \leq \lambda\psi(y - x),$$

where $\psi : [0, \infty) \to [0, \infty)$ can be written by $\psi(x) = x - \phi(x)$ with $\phi : [0, \infty) \to [0, \infty)$ continuous, increasing, positive in $(0, \infty)$, $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) =$

∞. *Then the existence of a lower solution of (1.1) provides the existence of a unique solution of (1.1).*

Now, we prove the existence of a solution of problem (1.1) in presence of a lower solution with α-ψ-contractive mappings. Recall that, Ψ is the class of functions ψ defined in Section 2.

Theorem 4.3. *Consider the differential equation (1.1) with continuous $h : I \times \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:*

(i) there exists $\lambda > 0$ such that for all $x, y \in \mathbb{R}$, with $y \geq x$, and $\psi \in \Psi$

$$0 \leq h(t, y) + \lambda y - h(t, x) - \lambda x \leq \lambda \psi(y - x);$$

(ii) there exists a function $\xi : \mathbb{R}^2 \to \mathbb{R}$ such that for all $t \in I$ and for all $a, b \in \mathbb{R}$ with $\xi(a, b) \geq 0$,

$$\xi\left(\int_0^T G(t,s)[h(s,u(s)) + \lambda u(s)]ds, \gamma(t)\right) \geq 0;$$

where $\gamma \in C(I, \mathbb{R})$ is a lower solution of (1.1),

(iii) for all $t \in I$ and all $x, y \in C(I, \mathbb{R})$, $\xi(x(t), y(t)) \geq 0$ implies,

$$\xi\left(\int_0^T G(t,s)[h(s,x(s)) + \lambda u(s)]ds, \int_0^T G(t,s)[h(s,y(s)) + \lambda u(s)]ds\right) \geq 0;$$

(iv) if $x_n \to x \in C(I, \mathbb{R})$ and $\xi(x_n, x_{n+1}) \geq 0$ then $\xi(x_n, x) \geq 0$ for all $n \in \mathbb{N}$.

Then, the existence of a lower solution for (1.1) provides a unique solution of (1.1).

Proof. Problem (1.1) is written as

$$\begin{cases} u'(t) + \lambda u(t) = h(t, u(t)) + \lambda u(t), & t \in I = [0, T]; \\ u(0) = u(T). \end{cases}$$

This differential equation is equivalent to the integral equation

$$u(t) = \int_0^T G(t,s)[h(s,u(s)) + \lambda u(s)]ds,$$

where

$$G(t,s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & 0 \leq s < t \leq T; \\ \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & 0 \leq t < s \leq T. \end{cases}$$

Define $\mathcal{A} : C(I, \mathbb{R}) \to C(I, \mathbb{R})$ by

$$[\mathcal{A}u](t) = \int_0^T G(t,s)[h(s,u(s)) + \lambda u(s)]ds, \quad t \in I.$$

Note that if $u \in C(I, \mathbb{R})$ is a fixed point of $\mathcal{A}$, then $u \in C^1(I, \mathbb{R})$ is a solution of (1.1). Let $X = C(I, \mathbb{R})$. By the following order relation, X is a partially ordered set,

$$x, y \in X, \quad x \leq y \Longleftrightarrow x(t) \leq y(t), \qquad t \in I.$$

If we choose

$$d(x,y) = \sup_{t \in I} |x(t) - y(t)|, \qquad x, y \in X$$

then (X, d) is a complete metric space. Consider a monotone nondecreasing sequence $\{x_n\} \subseteq C(I, \mathbb{R})$ converging to $x \in C(I, \mathbb{R})$. Then for each $t \in I$,

$$x_1(t) \leq x_2(t) \leq x_3(t) \leq \cdots \leq x_n(t) \leq \cdots .$$

The convergence of this sequence to $x(t)$ implies that $x_n(t) \leq x(t)$, for all $t \in I$ and all $n \in \mathbb{N}$. Therefore, $x_n \leq x$ for all $n \in \mathbb{N}$. Moreover, $\mathcal{A}$ is a non-decreasing mapping, since for all $u, v \in X$ with $u \geq v$,

$$h(t,u) + \lambda u \geq h(t,v) + \lambda v$$

and also $G(t,s) > 0$ for all $(t,s) \in I \times I$,

$$\begin{aligned} [\mathcal{A}u](t) &= \int_0^T G(t,s)[h(s,u(s)) + \lambda u(s)]ds \\ &\geq \int_0^T G(t,s)[h(s,v(s)) + \lambda v(s)]ds = [\mathcal{A}v](t). \end{aligned}$$

In addition, for $u \geq v$ by (i) and definition of $G(t,s)$, we obtain

$$\begin{aligned} d(\mathcal{A}u, \mathcal{A}v) &= \sup_{t \in I} |\mathcal{A}u(t) - \mathcal{A}v(t)| \\ &\leq \sup_{t \in I} \int_0^T G(t,s)|h(s,u(s)) + \lambda u(s) - h(s,v(s)) - \lambda v(s)|ds \\ &\leq \sup_{t \in I} \int_0^T G(t,s)|\lambda \psi(u(s) - v(s))|ds \\ &\leq \sup_{t \in I} \int_0^T G(t,s)\lambda \psi(|u(s) - v(s)|)ds \\ &\leq \lambda \psi(d(u,v)) \sup_{t \in I} \int_0^T G(t,s)ds \\ &= \lambda \psi(d(u,v)) \sup_{t \in I} \frac{1}{e^{\lambda T} - 1} \left(\frac{1}{\lambda} \, e^{\lambda(T+s-t)} \Big|_0^t + \frac{1}{\lambda} \, e^{\lambda(s-t)} \Big|_t^T \right) \\ &= \lambda \psi(d(u,v)) \times \frac{1}{\lambda} = \psi(d(u,v)), \end{aligned}$$

then

$$d(\mathcal{A}u, \mathcal{A}v) \leq \psi(d(u,v)).$$

Define the function $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(u,v) = \begin{cases} 1 & \text{if } \xi(u(t), v(t)) \geq 0,\ t \in I; \\ 0 & \text{otherwise,} \end{cases}$$

for all $u, v \in X$ with $u \geq v$. Then,

$$\alpha(u,v) d(\mathcal{A}u, \mathcal{A}v) \leq \psi(d(u,v)),$$

which implies that $\mathcal{A}$ is an α-ψ-contractive mapping. Now, by (iii), for all $u, v \in X$ with $u \geq v$, we get

$$\alpha(u,v) \geq 1 \Longrightarrow \xi(u(t), v(t)) \geq 0 \Longrightarrow \xi(\mathcal{A}u(t), \mathcal{A}v(t)) \geq 0 \Longrightarrow \alpha(\mathcal{A}u, \mathcal{A}v) \geq 1.$$

Therefore, $\mathcal{A}$ is ordered α-admissible. Let β be a lower solution of (1.1), then from (ii),

$$\xi((\mathcal{A}\beta)(t), \beta(t)) \geq 0 \Longrightarrow \alpha(\mathcal{A}\beta, \beta) \geq 1.$$

Now, we show that $\mathcal{A}\beta \geq \beta$. Since β is a lower solution of (1.1), we have

$$\begin{cases} \beta'(t) \leq h(t, \beta(t)), \quad t \in I = [0, T]; \\ \beta(0) \leq \beta(T). \end{cases}$$

For all $t \in I$ and $\lambda > 0$ we have

$$\beta'(t) + \lambda\beta(t) \leq h(t, \beta(t)) + \lambda\beta(t).$$

Multiplying this by $e^{\lambda t}$, we get

$$(\beta(t)e^{\lambda t})' \leq (h(t, \beta(t)) + \lambda\beta(t))e^{\lambda t}.$$

By integration, we obtain

$$\beta(t)e^{\lambda t} \leq \beta(0) + \int_0^t [h(s, \beta(s)) + \lambda\beta(s)]e^{\lambda s} ds, \tag{4.1}$$

which implies that

$$\beta(0)e^{\lambda T} \leq \beta(T)e^{\lambda T} \leq \beta(0) + \int_0^T [h(s, \beta(s)) + \lambda\beta(s)]e^{\lambda s} ds,$$

and so

$$\beta(0) \leq \int_0^T \frac{e^{\lambda s}}{e^{\lambda T} - 1}[h(s, \beta(s)) + \lambda\beta(s)] ds. \tag{4.2}$$

From (4.2) and (4.3),

$$\begin{aligned} \beta(t)e^{\lambda t} &\leq \int_0^T \frac{e^{\lambda s}}{e^{\lambda T} - 1}[h(s, \beta(s)) + \lambda\beta(s)] ds + \int_0^t [h(s, \beta(s)) + \lambda\beta(s)]e^{\lambda s} ds \\ &\leq \int_0^t \frac{e^{\lambda(T+s)}}{e^{\lambda T} - 1}[h(s, \beta(s)) + \lambda\beta(s)] ds + \int_t^T \frac{e^{\lambda s}}{e^{\lambda T} - 1}[h(s, \beta(s)) + \lambda\beta(s)] ds. \end{aligned}$$

Dividing by $e^{\lambda t}$, we obtain

$$\beta(t) \leq \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}[h(s,\beta(s))+\lambda\beta(s)]ds + \int_t^T \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}[h(s,\beta(s))+\lambda\beta(s)]ds.$$

Then, by the definition of $G(t,s)$, we have

$$\beta(t) \leq \int_0^T G(t,s)[h(s,\beta(s))+\lambda\beta(s)]ds = [\mathcal{A}\beta](t),$$

for all $t \in I$. Thus, $\mathcal{A}\beta \geq \beta$. Finally, from (iv) if $x_n \to x \in X$, for all n,

$$\xi(x_n, x_{n+1}) \geq 0 \Longrightarrow \xi(x_n, x) \geq 0.$$

Therefore,

$$\alpha(x_n, x_{n+1}) \geq 1 \Longrightarrow \alpha(x_n, x) \geq 1.$$

Then, all the hypotheses of Theorem 3.2 are satisfied. Consequently, $\mathcal{A}$ has a fixed point and so equation (1.1) has a solution. The uniqueness of the solution follows from (3.5). □

Theorem 4.4. *If we replace the existence of lower solution to (1.1) by upper solution, Theorem 4.3 still holds.*

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