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EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS FOR SINGULAR MONGE-AMPÈRE SYSTEM

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Abstract. Using the fixed point theorem in a cone, the existence and multiplicity of radial convex solutions of singular system of Monge-Ampère equations are established.

Keywords: Singular Monge-Ampère system; radial convex solution; Fixed point theorem.
MSC(2010): Primary: 34B16; Secondary: 47E05.

1. Introduction

The Monge-Ampère equation

$$\begin{cases} \det(D^2u) = \lambda f(u), & in\ B, \\ u(x) = 0, & on\ \partial B, \end{cases}$$

arises from the differential geometry, optimization or mass-transfer problems. So there are many authors who pay more attention to the study of the Monge-Ampère equations, see [4–6, 11, 16] and references therein. Recently, using the fixed point theorem, S. Hu, H. Wang consider the existence, multiplicity and nonexistence of radial convex solutions of the Monge-Ampère equations, see [12,13]. In this paper, we are concerned with the existence and multiplicity of convex solutions of the singular boundary value problem

$$(1.1) \qquad \begin{cases} ((u_1'(r))^N)' = \lambda N r^{N-1} f_1(-u_1, \cdots, -u_n), & 0 < r < 1, \\ \cdots \\ ((u_n'(r))^N)' = \lambda N r^{N-1} f_n(-u_1, \cdots, -u_n), & 0 < r < 1, \\ u_i'(0) = u_i(1) = 0, \ \ i = 1, \cdots, n, \end{cases}$$

where $N > 1$, $\lambda > 0$ is a positive parameter, and the function f_i is singular at $(0, \cdots, 0)$. Such a problem arises in the study of the existence of radial convex

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solutions to the Dirichlet problem of system of Monge-Ampère equations

$$(1.2)\qquad \begin{cases} \det(D^2u_1) = \lambda f_1(-u_1, \cdots, -u_n), \ \ in\ B, \\ \cdots \\ \det(D^2u_n) = \lambda f_n(-u_1, \cdots, -u_n), \ \ in\ B, \\ u_i(x) = 0, \ \ on\ \partial B, \end{cases}$$

where $D^2u_i(x) = (\frac{\partial^2 u_i}{\partial x_j \partial x_k})$, $j, k = 1, 2\cdots n$, is the Hessian matrix of $u_i(x)$, $B = \{x \in R^N : |x| < 1\}$. For radial solution $u_i(r)$ with $r = \sqrt{\sum_1^N x_i^2}$, the Monge-Ampère operator simply becomes

$$\det(D^2u_i) = \frac{(u_i')^{N-1}u_i''}{r^{N-1}} = \frac{1}{Nr^{N-1}}((u_i')^N)'.$$

To the authors' knowledge, there is a considerable interest in the existence of positive solutions of the singular differential equations, see [1,2,7–10,15] and references therein. The common techniques are the Krasnoselskii fixed point theorem, the method of lower and upper solutions and the Schauder's fixed point theorem. In particular, H. Wang [14] demonstrates that the Krasnoselskii fixed point theorem on compression and expansion of cones can be effectively used to deal with singular problems. Inspired by the above references, our aim in this present paper is to investigate the existence and multiplicity of radial convex solutions of singular systems of Monge-Ampère equations using the Krasnoselskii fixed point theorem on compression and expansion of cones.

This paper is organized as follows: In Section 2, some preliminaries are given; in Section 3, we give the main results.

2. **Preliminaries**

For convenience, with a simple transformation $v_i(t) = -u_i(r)$, the (1.1) can be brought to the following system

$$(2.1)\qquad \begin{cases} ((-v_1'(t))^N)' = \lambda N t^{N-1} f_1(v_1, \cdots, v_n), \ \ 0 < t < 1, \\ \cdots \\ ((-v_n'(t))^N)' = \lambda N t^{N-1} f_n(v_1, \cdots, v_n), \ \ 0 < t < 1, \\ v_i'(0) = v_i(1) = 0, \ \ i = 1, \cdots, n. \end{cases}$$

Now we mainly treat the positive concave solutions of (2.1). The following lemma is a standard result due to the concavity of $v(t)$:

Lemma 2.1. *([12]) Let $v(t) \in C^1[0,1]$.If $v(t) \geq 0$, $v'(t)$ is nonincreasing on $[0,1]$, then*

$$v(t) \geq \min\{t, 1-t\}\|v\|, \ \ t \in [0,1],$$

where $\|v\| = \max_{t\in[0,1]} |v(t)|$. *In particular,*

$$\min_{\frac{1}{4}\le t\le \frac{3}{4}} v(t) \ge \frac{1}{4}\|v\|.$$

Let E denote the Banach space $\underbrace{C[0,1]\times\cdots\times C[0,1]}_{n}$ with the norm defined as

$$\|v\| = \sum_{i=1}^{n} \sup_{t\in[0,1]} |v_i(t)|,\ for\ v(t) = (v_1(t),\cdots,v_n(t)) \in E.$$

Let K be a cone in E defined as

$$K = \{v(t) \in E : v_i(t) \ge 0 \text{ is concave for } t \in [0,1],\ and\ v_i'(0) = v_i(1) = 0, i = 1,\cdots,n\}.$$

For $r > 0$, set

$$K_r = \{v \in K : \|v\| < r\},$$

and

$$\partial K_r = \{v \in K : \|v\| = r\}.$$

The following well-known result of the fixed point theorem is crucial in our arguments.

Lemma 2.2. *[3] Let E be a Banach space, and $K \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*
(i)$\|Tu\| \le \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Tu\| \ge \|u\|$, $u \in K \cap \partial\Omega_2$;or
(ii)$\|Tu\| \ge \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Tu\| \le \|u\|$, $u \in K \cap \partial\Omega_2$.
Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

For convenience, we now introduce some notation. Let $R_+ = [0,+\infty)$, $R_+^n = \underbrace{R_+\times\cdots\times R_+}_{n}$, and $|v| = \sum_{i=1}^{n} |v_i|$, for $v \in R_+^n$. The following conditions hold throughout this paper.
(H1) $f_i(v)$ is a scalar continuous function defined for $|v| > 0$, and $f_i(v) > 0$ for $|v| > 0$;
(H2) For each pair of positive numbers $r < R$, there exists a nonnegative function $F_i(t) \in C([0,1]\backslash\{0,1\})$ with

$$0 < \int_0^1 \varphi^{-1}(\int_0^s \tau^{N-1} F_i(\tau)d\tau)ds < +\infty$$

such that

$$f_i(v(t)) \le F_i(t),$$

for $t \in [0,1]\backslash\{0,1\}$, $v(t) \in K$ with $\min\{t, 1-t\}r \le |v(t)| = \sum_{i=1}^{n} v_i(t) \le R$, where $\varphi(t) = t^N$, $\varphi^{-1}(t) = t^{\frac{1}{N}}$.

Define an operator T as $Tv(t) = (T_1v(t), \cdots, T_nv(t))$, for $v(t) \in K$, where T_i is

$$T_iv(t) = \int_t^1 \varphi^{-1}(\lambda \int_0^s [N\tau^{N-1} f_i(v_1(\tau), \cdots, v_n(\tau))]d\tau)ds,\ 0 \le t \le 1,$$

$i = 1, \cdots n$.

Remark 2.3. *We point out that* (2.1) *is equivalent to the fixed point equation*

$$Tv(t) = v(t), \ \ in\ K.$$

In fact, if $v(t) \in K$ is a positive fixed point of T, then it is a concave solution of (2.1). *It is further clear that $v'' < 0$ for $t \in (0,1)$ and hence $-v(t)$ must be a strictly convex solution of* (1.1). *Conversely, if $u(r) < 0$ is a strictly convex solution of* (1.1)*, then $-u(r)$ is a positive fixed point of T in K.*

Lemma 2.4. *Assume that (H1) and (H2) hold. Let $0 < r < R < +\infty$, then $T : \overline{K_R}\backslash K_r \to K$ is completely continuous.*

Proof. Let $v(t) \in \overline{K_R}\backslash K_r$. From Lemma 2.1, it is easy to obtain that

$$\min\{t, 1-t\}r \le \sum_{i=1}^{n} v_i(t) \le R, \ \ 0 \le t \le 1.$$

By (H2), there exists a $F_i(t) \in C([0,1]\backslash\{0,1\})$ such that

$$f_i(v) \le F_i(t), \ \ t \in [0,1]\backslash\{0,1\},\ \min\{t, 1-t\}r \le \sum_{i=1}^{n} v_i(t) \le R.$$

Hence, we have

$$\begin{aligned}
&\max_{0\le t\le 1} \int_t^1 \varphi^{-1}(\lambda \int_0^s [N\tau^{N-1} f_i(v_1, \cdots, v_n)]d\tau)ds \\
&\le \int_0^1 \varphi^{-1}(\lambda \int_0^s [N\tau^{N-1} F_i(\tau)]d\tau)ds \\
&< +\infty.
\end{aligned}$$

Consequently, $T_iv(t) \in C[0,1]$.

For any $v(t) \in \overline{K_R}\backslash K_r$, it is clear to see that $T_iv(1) = 0$. Since

$$(T_iv)'(t) = -\varphi^{-1}(\lambda \int_0^t [N\tau^{N-1} f_i(v_1(\tau), \cdots, v_n(\tau))]d\tau)ds,$$

we also obtain that $(T_iv)'(0) = 0$ and $(T_iv)'(t)$ is nonincreasing. Furthermore, we have that $T_iv(t)$ is concave. So $T : \overline{K_R}\backslash K_r \to K$.

Let $F_i^k(t) = \min\{F_i(t), k\}$, where $k > 0$. Then $F_i^k(t) \in C([0,1]\backslash\{0,1\})$ and $F_i^k(t) \le k$, for $t \in [0,1]\backslash\{0,1\}$. By (H2), we have

$$\varphi^{-1}(\lambda) \int_0^1 \{\varphi^{-1}(\int_0^s N\tau^{N-1} F_i(\tau)d\tau) - \varphi^{-1}(\int_0^s N\tau^{N-1} F_i^k(\tau)d\tau)\}ds \to 0\ (k \to +\infty).$$

In addition, define the function $f_i^k(v(t))$ as follows:

$$f_i^k(v(t)) = \begin{cases} f_i(v(t)), & f_i(v(t)) \leq F_i^k(t), \\ F_i^k(t), & f_i(v(t)) \geq F_i^k(t). \end{cases}$$

Then $f_i^k : [0,+\infty)\times\cdots\times[0,+\infty) \to [0,\infty)$ is continuous and $f_i^k(v)$ is bounded.

Define the operator T_i^k as follows:

$$T_i^k v(t) = \int_t^1 \varphi^{-1}(\lambda \int_0^s [N\tau^{N-1} f_i^k(v_1(\tau),\cdots,v_n(\tau))]d\tau)ds.$$

Since the cone K is closed, let $v^n(t), v^0(t) \in K$ and $\|v^n(t) - v^0(t)\| \to 0$ $(n \to +\infty)$. Then $v^n(t) \to v^0(t),\ 0 \leq t \leq 1$ and

$$f_i^k(v^n(t)) \to f_i^k(v^0(t)),\ \ t \in [0,1]\backslash\{0,1\}.$$

By the Lebesgue dominated convergence theorem, we get that

$$\begin{aligned} & \lim_{n\to+\infty} \|T_i^k v^n - T_i^k v^0\| \\ = \ & \lim_{n\to+\infty} \max_{0\leq t\leq 1} |\int_t^1 \varphi^{-1}(\lambda \int_0^s N\tau^{N-1} f_i^k(v^n(\tau))d\tau)ds \\ & -\int_t^1 \varphi^{-1}(\lambda \int_0^s N\tau^{N-1} f_i^k(v^0(\tau))d\tau)ds| \\ \leq \ & \lim_{n\to+\infty} \int_0^1 |\{\varphi^{-1}(\lambda \int_0^s N\tau^{N-1} f_i^k(v^n(\tau))d\tau) - \varphi^{-1}(\lambda \int_0^s N\tau^{N-1} f_i^k(v^0(\tau))d\tau)\}|ds \\ \to \ & 0(n \to +\infty) \end{aligned}$$

Consequently, $T_i^k : K \to C[0,1]$ is continuous. By the Arzela-Ascoli theorem, it is easy to prove that $T_i^k : K \to C[0,1]$ is compact.

For any $v(t) \in \overline{K_R}\backslash K_r$, by the definition of $f_i^k(v(t))$, we have

$$0 \leq f_i(v(t)) - f_i^k(v(t)) \leq F_i(t) - F_i^k(t),\ \ for\ t \in [0,1]\backslash\{0,1\}.$$

Furthermore, we can get

$$
\begin{aligned}
& \sup_{v\in\overline{K_R}\setminus K_r} \|T_i v - T_i^k v\| \\
= \quad & \sup_{v\in\overline{K_R}\setminus K_r} \max_{0\le t\le 1} \int_t^1 \{\varphi^{-1}(\lambda\int_0^s N\tau^{N-1} f_i(v(\tau))d\tau) \\
& -\varphi^{-1}(\lambda\int_0^s N\tau^{N-1} f_i^k(v(\tau))d\tau)\}ds \\
= \quad & \sup_{v\in\overline{K_R}\setminus K_r} \int_0^1 \{\varphi^{-1}(\lambda\int_0^s N\tau^{N-1} f_i(v(\tau))d\tau) \\
& -\varphi^{-1}(\lambda\int_0^s N\tau^{N-1} f_i^k(v(\tau))d\tau)\}ds \\
\le \quad & \sup_{\{v\in\overline{K_R}\setminus K_r : f_i(v(t))\le F_i^k(t)\}} \int_0^1 \{\varphi^{-1}(\lambda\int_0^s N\tau^{N-1} f_i(v(\tau))d\tau) \\
& -\varphi^{-1}(\lambda\int_0^s N\tau^{N-1} f_i^k(v(\tau))d\tau)\}ds \\
& + \sup_{\{v\in\overline{K_R}\setminus K_r : f_i(v(t))\ge F_i^k(t)\}} \int_0^1 \{\varphi^{-1}(\lambda\int_0^s N\tau^{N-1} f_i(v(\tau))d\tau) \\
& -\varphi^{-1}(\lambda\int_0^s N\tau^{N-1} f_i^k(v(\tau))d\tau)\}ds \\
= \quad & \sup_{\{v\in\overline{K_R}\setminus K_r : f_i(v(t))\le F_i^k(t)\}} \int_0^1 \{\varphi^{-1}(\lambda\int_0^s N\tau^{N-1} f_i(v(\tau))d\tau) \\
& -\varphi^{-1}(\lambda\int_0^s N\tau^{N-1} f_i(v(\tau))d\tau)\}ds \\
& + \sup_{\{v\in\overline{K_R}\setminus K_r : f_i(v(t))\ge F_i^k(t)\}} \int_0^1 \{\varphi^{-1}(\lambda\int_0^s N\tau^{N-1} f_i(v(\tau))d\tau) \\
& -\varphi^{-1}(\lambda\int_0^s N\tau^{N-1} f_i^k(v(\tau))d\tau)\}ds \\
= \quad & \sup_{\{v\in\overline{K_R}\setminus K_r : f_i(v(t))\ge F_i^k(t)\}} \int_0^1 \{\varphi^{-1}(\lambda\int_0^s N\tau^{N-1} f_i(v(\tau))d\tau) \\
& -\varphi^{-1}(\lambda\int_0^s N\tau^{N-1} F_i^k(\tau)d\tau)\}ds \\
\le \quad & \sup_{v\in\overline{K_R}\setminus K_r} \int_0^1 \{\varphi^{-1}(\lambda\int_0^s N\tau^{N-1} F_i(\tau)d\tau) - \varphi^{-1}(\lambda\int_0^s N\tau^{N-1} F_i^k(\tau)d\tau)\}ds \\
\to \quad & 0 (k\to+\infty).
\end{aligned}
$$

Therefore, the completely continuous operators T_i^k uniformly converge to the operator T_i on the bounded closed set $\overline{K_R}\setminus K_r$. Therefore $T_i : K \to K$ is completely continuous. Furthermore, $T : \overline{K_R}\backslash K_r \to K$ is completely continuous. $\square$

3. Main results

For any $\rho > 0$, let

$$M_i(t,\rho) = \max\{f_i(v(t)) : v(t) \in K,\ \min\{t, 1-t\}\rho \le |v(t)| \le \rho,\ i = 1, \cdots, n\} > 0,$$

$$m_i(t,\rho) = \min\{f_i(v(t)) : v(t) \in K,\ \min\{t, 1-t\}\rho \le |v(t)| \le \rho,\ i = 1, \cdots, n\} > 0.$$

If (H1) and (H2) hold, then $M_i(\cdot,\rho)$,$m_i(\cdot,\rho) \in C([0,1] \setminus \{0,1\})$, and satisfy

$$0 < \int_0^1 \varphi^{-1}(\int_0^s \tau^{N-1} M_i(\tau,\rho)d\tau)ds < +\infty,$$

$$0 < \int_0^1 \varphi^{-1}(\int_0^s \tau^{N-1} m_i(\tau,\rho)d\tau)ds < +\infty.$$

Theorem 3.1. Assume (H1) and (H2) hold. If $\lim_{|v|\to 0} f_i(v) = \infty$ for some $i = 1, 2, \cdots, n$, then there exists a $\lambda_0 > 0$ such that (2.1) has a positive concave solution for $0 < \lambda < \lambda_0$.

Proof. Fix a $R > 0$. Then for any $v(t) \in \partial K_R$, we have $\min\{t, 1-t\}R \le |v(t)| \le R$. Furthermore, we can obtain that

$$\begin{aligned} \|Tv\| &= \Sigma_{i=1}^n \|T_i v\| \\ &= \Sigma_{i=1}^n \max_{t\in[0,1]} \int_t^1 \varphi^{-1}(\lambda \int_0^s [N\tau^{N-1} a_i(\tau) f_i(v_1(\tau), \cdots, v_n(\tau))]d\tau)ds \\ &\le \Sigma_{i=1}^n \int_0^1 \varphi^{-1}(\lambda \int_0^1 [N\tau^{N-1} f_i(v_1(\tau), \cdots, v_n(\tau))]d\tau)ds \\ &\le \Sigma_{i=1}^n \int_0^1 \varphi^{-1}(\lambda \int_0^1 [N\tau^{N-1} M_i(\tau,R)d\tau)ds \\ &= \varphi^{-1}(\lambda)\Sigma_{i=1}^n \int_0^1 \varphi^{-1}(\int_0^1 [N\tau^{N-1} M_i(\tau,R)d\tau)ds. \end{aligned}$$

Therefore, there exists a $\lambda_0 > 0$ such that

$$\|Tv\| < \|v\|,\ for\ v(t) \in \partial K_R,\ 0 < \lambda < \lambda_0.$$

From the assumption, there exists a $i_0 \in \{1, 2, \cdots, n\}$ such that $\lim_{|v|\to 0} f_{i_0}(v) = \infty$. Then by the definition of $\lim_{|v|\to 0} f_{i_0}(v) = \infty$, there exists a $r > 0$ with $r < R$ such that

$$f_{i_0}(v) \ge \varphi(\mu)\varphi(|v|),\ for\ v \in R_+^n\ with\ 0 < |v| \le r,$$

where μ satisfies $\frac{\mu}{4}\varphi^{-1}(\lambda)\int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^{s} N\tau^{N-1}d\tau)^{\frac{1}{N}}ds > 1$. For any $v(t) = (v_1(t), \cdots, v_n(t)) \in \partial K_r$, it is clear to see that

$$|v(t)| = \sum_{i=1}^{n} v_i(t) \geq \min\{t, 1-t\}\|v(t)\| = \min\{t, 1-t\}r, \ for\ t \in [0,1].$$

Especially, $|v(t)| = \sum_{i=1}^{n} v_i(t) \geq \min\{t, 1-t\}\|v(t)\| = \frac{1}{4}r$, for $t \in [\frac{1}{4}, \frac{3}{4}]$.

Furthermore, we have

$$\begin{aligned}
\|Tv\| &\geq \max_{t\in[0,1]} T_{i_0}(v(t)) \\
&= \max_{t\in[0,1]} \int_t^1 \varphi^{-1}(\lambda\int_0^s [N\tau^{N-1} f_{i_0}(v_1(\tau), \cdots, v_n(\tau))]4\tau)ds \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1}(\lambda\int_0^s [N\tau^{N-1} f_{i_0}(v_1(\tau), \cdots, v_n(\tau))]d\tau)ds \\
&\geq \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1}(\lambda\int_{\frac{1}{4}}^s [N\tau^{N-1}\varphi(\mu)\varphi(\frac{1}{4}\|v\|)]d\tau)ds \\
&\geq \frac{\mu}{4}\varphi^{-1}(\lambda)\int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^{s} N\tau^{N-1}d\tau)^{\frac{1}{N}}ds\|v\| \\
&> \|v\|,
\end{aligned}$$

namely,

$$\|Tv\| > \|v\|, \ for\ \forall\ v(t) \in \partial K_r.$$

Therefore, by Lemma 2.2, T has a fixed point in $\overline{K_R}\backslash K_r$, which is a positive concave solution of (2.1). □

Example 3.1. *Consider the following system:*

$$(3.1)\quad \begin{cases} ((u_1'(r))^N)' = \lambda N r^{N-1}[(\sum_{i=1}^{n}(-u_i))^{-\alpha_1} + (\sqrt{\sum_{i=1}^{n} u_i^2})^{\beta_1}], \\ \cdots \\ ((u_n'(r))^N)' = \lambda N r^{N-1}[(\sum_{i=1}^{n}(-u_i))^{-\alpha_n} + (\sqrt{\sum_{i=1}^{n} u_i^2})^{\beta_n}], \\ u_i'(0) = u_i(1) = 0, \ \ i = 1, \cdots, n \end{cases}$$

where $0 < \alpha_i < N$, $\beta_i > 0$. *Let* $v_i(t) = -u_i(r)$, *then* (3.1) *can be brought to the following system*

$$(3.2)\quad \begin{cases} ((-v_1'(t))^N)' = \lambda N t^{N-1}[(\sum_{i=1}^{n} v_i)^{-\alpha_1} + (\sqrt{\sum_{i=1}^{n} v_i^2})^{\beta_i}], \\ \cdots \\ ((-v_n'(t))^N)' = \lambda N t^{N-1}[(\sum_{i=1}^{n} v_i)^{-\alpha_n} + (\sqrt{\sum_{i=1}^{n} v_i^2})^{\beta_i}], \\ v_i'(0) = v_i(1) = 0, \ \ i = 1, \cdots, n. \end{cases}$$

It is clear to see that (H1) holds. Now we show that this example satisfies the condition (H2). For each pair of positive numbers $0 < r < R$*, and* $v(t) \in K$ *with* $\min\{t, 1-t\}r \le |v(t)| = \sum_{i=1}^{n} v_i(t) \le R$*, we have*

$$\begin{aligned} f_i(v(t)) &= \lambda N t^{N-1}[(\sum_{i=1}^{n} v_i)^{-\alpha_i} + (\sqrt{\sum_{i=1}^{n} v_i^2})^{\beta_i}] \\ &\le \lambda N t^{N-1}[(\min\{t, 1-t\}r)^{-\alpha_i} + R^{\beta_i}] \\ &= \lambda N t^{N-1}(\min\{t, 1-t\})^{-\alpha_i} r^{-\alpha_i} + \lambda N t^{N-1} R^{\beta_i} \\ &= F_i(t). \end{aligned}$$

Finally, we only need to verify that

$$0 \le \int_0^1 \varphi^{-1}(\int_0^s \tau^{N-1}(\min\{t, 1-t\})^{-\alpha_i} d\tau) ds < +\infty.$$

Since

$$\begin{aligned} & \int_0^1 \varphi^{-1}(\int_0^s \tau^{N-1}(\min\{\tau, 1-\tau\})^{-\alpha_i} d\tau) ds \\ = & \int_0^{\frac{1}{2}} \varphi^{-1}(\int_0^s \tau^{N-1}(\min\{\tau, 1-\tau\})^{-\alpha_i} d\tau) ds \\ & + \int_{\frac{1}{2}}^1 \varphi^{-1}(\int_0^s \tau^{N-1}(\min\{\tau, 1-\tau\})^{-\alpha_i} d\tau) ds \\ = & \int_0^{\frac{1}{2}} \varphi^{-1}(\int_0^s \tau^{N-1}\tau^{-\alpha_i} d\tau) ds \\ & + \int_{\frac{1}{2}}^1 \varphi^{-1}(\int_0^s \tau^{N-1}(1-\tau)^{-\alpha_i} d\tau) ds, \end{aligned}$$

then we have

$$\begin{aligned} \int_0^{\frac{1}{2}} \varphi^{-1}(\int_0^s \tau^{N-1}\tau^{-\alpha_i} d\tau) ds &= \int_0^{\frac{1}{2}} \varphi^{-1}(\frac{1}{N-\alpha_i} s^{N-\alpha_i}) ds \\ &= \varphi^{-1}(\frac{1}{N-\alpha_i}) \int_0^{\frac{1}{2}} s^{\frac{N-\alpha_i}{N}} ds \\ &= \varphi^{-1}(\frac{1}{N-\alpha_i}) \frac{N}{2N-\alpha_i} (\frac{1}{2})^{\frac{2N-\alpha_i}{N}} < +\infty, \end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}\varphi^{-1}(\int_0^s \tau^{N-1}(1-\tau)^{-\alpha_i}d\tau)ds\\
&\le \int_{\frac{1}{2}}^{1}\varphi^{-1}(\int_0^s (1-\tau)^{-\alpha_i}d\tau)ds\\
&= \begin{cases} \int_{\frac{1}{2}}^{1}\varphi^{-1}(\frac{1}{1-\alpha_i}-\frac{1}{1-\alpha_i}(1-s)^{1-\alpha_i})ds, \ if\ 0<\alpha_i<N,\ \alpha_i\neq 1\\ \int_{\frac{1}{2}}^{1}\varphi^{-1}(-\ln(1-s))ds, \ \ if\ \ \alpha_i=1\end{cases}\\
&= \begin{cases} \int_0^{\frac{1}{2}}\varphi^{-1}(\frac{1}{1-\alpha_i}-\frac{1}{1-\alpha_i}s^{1-\alpha_i})ds, \ if\ 0<\alpha_i<N,\ \alpha_i\neq 1\\ \int_{\ln 2}^{+\infty}\frac{s^{\frac{1}{N}}}{e^s}ds, \ \ if\ \ \alpha_i=1\end{cases}\\
&< +\infty.
\end{aligned}$$

So $F_i(t)$ satisfies the condition (H2). Therefore, by Theorem 3.1, (3.1) *has a convex solution.*

Theorem 3.2. *Assume (H1) and (H2) hold. In addition, suppose that* $\lim_{|v|\to 0} f_i(v)=\infty$ *for some* $i=1,2,\cdots,n$. *If* $\lim_{\|v\|\to\infty}\frac{f_i(v)}{\varphi(\|v\|)}=0$, $i=1,\cdots,n$, *then for any* $\lambda>0$, (2.1) *has a positive concave solution.*

Proof. On one hand, as the proof of Theorem 3.1, for any $\lambda>0$, there exists a $r>0$ such that

$$\|Tv\|>\|v\|,\ for\ \forall\ v(t)\in\partial K_r.$$

On the other hand, since $\lim_{\|v\|\to\infty}\frac{f_i(v)}{\varphi(\|v\|)}=0$, then there exists $\widehat{R}>2r$ such that

$$f_i(v)\le\varphi(\epsilon)\varphi(\|v\|),\ \ \|v\|\ge\widehat{R},$$

where ϵ satisfies $\varphi^{-1}(\lambda)\epsilon n<1$. For any $v(t)=(v_1(t),\cdots,v_n(t))\in\partial K_{\widehat{R}}$, we have

$$\begin{aligned}
\|Tv\| &= \Sigma_{i=1}^n\|T_iv\|\\
&= \Sigma_{i=1}^n\max_{t\in[0,1]}\int_t^1\varphi^{-1}(\lambda\int_0^s[N\tau^{N-1}a_i(\tau)f_i(v_1(\tau),\cdots,v_n(\tau))]d\tau)ds\\
&\le \Sigma_{i=1}^n\int_0^1\varphi^{-1}(\lambda\int_0^1[N\tau^{N-1}f_i(v_1(\tau),\cdots,v_n(\tau))]d\tau)ds\\
&\le \Sigma_{i=1}^n\int_0^1\varphi^{-1}(\lambda\int_0^1[N\tau^{N-1}\varphi(\epsilon)\varphi(\|v\|)d\tau)ds\\
&= \varphi^{-1}(\lambda)\epsilon n\|v\|\\
&< \|v\|,
\end{aligned}$$

namely,

$$\|Tv\|<\|v\|,\ \ for\ v(t)\in\partial K_{\widehat{R}}$$

Therefore, by Lemma 2.2, T has a fixed point in $\overline{K_{\widehat{R}}}\backslash K_r$, which is a positive concave solution of (2.1). □

Example 2. *Consider the following system:*

$$(3.3)\begin{cases} ((u_1'(r))^N)' = \lambda N r^{N-1}[(\sum_{i=1}^n(-u_i))^{-\alpha_1} + (\sum_{i=1}^n(-u_i))^{\frac{N}{2}}], \\ \cdots \\ ((u_n'(r))^N)' = \lambda N r^{N-1}[(\sum_{i=1}^n(-u_i))^{-\alpha_n} + (\sum_{i=1}^n(-u_i))^{\frac{N}{n+1}}], \\ u_i'(0) = u_i(1) = 0, \ \ i \ = \ 1, \ \cdots, \ n \end{cases}$$

where $0 < \alpha_i < N$. It is clear to see that Example 2 satisfies the conditions of Theorem 3.2. Therefore, (3.3) *has a convex solution.*

Theorem 3.2. *Assume (H1) and (H2) hold. In addition, suppose that $\lim_{|v|\to 0} f_i(v) = \infty$ for some $i = 1, 2, \cdots, n$. If $\lim_{|v|\to\infty} \frac{f_i(v)}{\varphi(|v|)} = \infty$, then* (2.1) *has two positive concave solutions for all sufficient small $\lambda > 0$.*

Proof. On one hand, it follows, from Theorem 3.1, that there exists a $\lambda_0 > 0$ such that T has a fixed point in $\overline{K_R}\backslash K_r$ for $0 < \lambda < \lambda_0$, which is a positive concave solution of (2.1).

On the other hand, since $\lim_{|v|\to\infty} \frac{f_i(v)}{\varphi(|v|)} = \infty$, there is a $R' > 2R$ such that

$$f_i(v) \geq \varphi(\eta)\varphi(|v|), \ \ for \ \ v \in R_+^n \ and \ |v| \geq R',$$

where η satisfies $\frac{\eta n}{4}\varphi^{-1}(\lambda)\int_{\frac{1}{4}}^{\frac{3}{4}}(\int_{\frac{1}{4}}^{s} N\tau^{N-1}d\tau)^{\frac{1}{N}}ds > 1$

Let $\overline{R} = 4R'$. Then for any $v(t) = (v_1(t), \cdots, v_n(t)) \in \partial K_{\overline{R}}$, we have

$$\min_{t\in[\frac{1}{4},\frac{3}{4}]}\sum_{i=1}^n v_i(t) \geq \frac{1}{4}\|v\| \geq \frac{1}{4}\overline{R} \geq R',$$

which implies that

$$f_i(v(t)) \geq \varphi(\eta)\varphi(|v(t)|), \ \ for \ t \in [\frac{1}{4}, \frac{3}{4}].$$

Furthermore, we have

$$
\begin{aligned}
\|Tv\| &\geq \sum_{i=1}^{n} \max_{t\in[0,1]} T_i(v(t)) \\
&= \sum_{i=1}^{n} \max_{t\in[0,1]} \int_t^1 \varphi^{-1}(\lambda \int_0^s [N\tau^{N-1} f_i(v_1(\tau),\cdots,v_n(\tau))]d\tau)ds \\
&= \sum_{i=1}^{n} \int_0^1 \varphi^{-1}(\lambda \int_0^s [N\tau^{N-1} f_i(v_1(\tau),\cdots,v_n(\tau))]d\tau)ds \\
&\geq \sum_{i=1}^{n} \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1}(\lambda \int_0^s [N\tau^{N-1} f_i(v_1(\tau),\cdots,v_n(\tau))]d\tau)ds \\
&\geq \sum_{i=1}^{n} \int_{\frac{1}{4}}^{\frac{3}{4}} \varphi^{-1}(\lambda \int_{\frac{1}{4}}^s [N\tau^{N-1} \varphi(\eta)\varphi(\frac{1}{4}\|v\|)]d\tau)ds \\
&\geq \frac{\eta n}{4}\varphi^{-1}(\lambda)\int_{\frac{1}{4}}^{\frac{3}{4}} (\int_{\frac{1}{4}}^s N\tau^{N-1}d\tau)^{\frac{1}{N}} ds\|v\| \\
&> \|v\|,
\end{aligned}
$$

namely,

$$\|Tv\| > \|v\|,\ for\ \forall\ v(t) \in \partial K_{\overline{R}}.$$

Therefore, by Lemma 2.2, T has a fixed point in $\overline{K_{\overline{R}}}\backslash K_R$, which is another positive concave solution of (2.1). □

Example 3.3. *Consider the following system:*

$$
(3.4)\begin{cases}
((u_1'(r))^N)' = \lambda N r^{N-1}[(\sum_{i=1}^n(-u_i))^{-\alpha_1} + (\sum_{i=1}^n(-u_i))^{2N}], \\
\cdots \\
((u_n'(r))^N)' = \lambda N r^{N-1}[(\sum_{i=1}^n(-u_i))^{-\alpha_n} + (\sum_{i=1}^n(-u_i))^{(n+1)N}], \\
u_i'(0) = u_i(1) = 0,\ \ i\ =\ 1,\ \cdots,\ n
\end{cases}
$$

where $0 < \alpha_i < N$. *It is easy to verify that all conditions of Theorem 3.3 hold. Therefore,* (3.4) *has at least two convex solutions.*

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