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SOME PROPERTIES FOR A CLASS OF MEROMORPHICALLY MULTIVALENT FUNCTIONS WITH LINEAR OPERATOR

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ABSTRACT. Making use of a linear operator, which is defined here by means of the Hadamard product (or convolution), we define a subclass $\overline{\mathcal{T}}_p(a, c, \gamma, \lambda; h)$ of meromorphically multivalent functions. The main object of this paper is to investigate some important properties for the class. We also derive many results for the Hadamard products of functions belonging to the class.

Keywords: Meromorphically multivalent functions, convex univalent functions, Hadamard product, subordination.

MSC(2010): Primary: 30C45; Secondary: 26D15.

1. Introduction

Let Σ_p denote the class of meromorphically multivalent function $f(z)$ of the form

$$(1.1) \quad f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^{n-p} \quad (n \in \mathbb{N} = \{1, 2, 3, \dots\}),$$

which are analytic in the punctured open unit disc $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U \setminus \{0\}$. For functions $f(z) \in \Sigma_p$ given by (1.1) and $g(z) \in \Sigma_p$ given by

$$g(z) = z^{-p} + \sum_{n=1}^{\infty} b_n z^{n-p} \quad (n \in \mathbb{N}),$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$(f * g)(z) = z^{-p} + \sum_{n=1}^{\infty} a_n b_n z^{n-p} \quad (n \in \mathbb{N}).$$

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Let the function $\varphi_p(a, c; z)$ be defined by

$$(1.2) \quad \varphi_p(a, c; z) = z^{-p} + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n-p},$$

where $(a)_n = a(a + 1) \cdots (a + n - 1), n \in \mathbb{N}$. Corresponding to the function $\varphi_p(a, c; z)$, Liu [6] and Liu and Srivastava [7] have introduced a linear operator $\mathcal{L}_p(a, c)$ which is defined by means of the following Hadamard product (or convolution)

$$(1.3) \quad \mathcal{L}_p(a, c)f(z) = \varphi_p(a, c; z) * f(z) \quad (f(z) \in \Sigma_p).$$

Just as in [6] and [7], it is easily verified from the definitions (1.2) and (1.3) that

$$z(\mathcal{L}_p(a, c)f(z))' = a\mathcal{L}_p(a + 1, c)f(z) - (a + p)\mathcal{L}_p(a, c)f(z).$$

The operator $\mathcal{L}_p(a, c)$ is also studied by many authors(see examples in [11,12], [14]. We note, for any integer $n > -p$ and for $f(z) \in \Sigma_p$, that

$$\mathcal{L}_p(n + p, 1)f(z) = D^{n+p-1}f(z) = \frac{1}{z^{p(1-z)^{n+p}}} * f(z),$$

where D^{n+p-1} is the differential operator studied by (for detail, see [2, 4, 5]) Uralegaddi and Somanatha [13] and Aouf [1].

Let

$$(1.4) \quad \mathcal{F}_{p,a,c,\gamma}f(z) = (1 - \gamma)\mathcal{L}_p(a, c)f(z) + \frac{\gamma}{p}z(\mathcal{L}_p(a, c)f(z))' \quad (f \in \Sigma_p; 0 \leq \gamma < \frac{1}{2}).$$

The operator $\mathcal{F}_{p,a,c,\gamma}$ is introduced by Aouf [3] . We easily obtain that

$$(1.5) \quad \mathcal{F}_{p,a,c,\gamma}f(z) = (1 - 2\gamma)z^{-p} + \sum_{n=1}^{\infty} \left(1 - \gamma + \gamma \frac{n - p}{p}\right) \frac{(a)_n}{(c)_n} a_n z^{n-p}$$

since $f(z) \in \Sigma_p$ is given by (1.1). From (1.5), it is easily verified that

$$(1.6) \quad \mathcal{F}_{p,a,c,0}f(z) = \mathcal{L}_p(a, c)f(z),$$

and

$$(1.7) \quad z(\mathcal{F}_{p,a,c,\gamma}f(z))' = a\mathcal{F}_{p,a+1,c,\gamma}f(z) - (a + p)\mathcal{F}_{p,a,c,\gamma}f(z) = \mathcal{F}_{p,a,c,\gamma}(zf'(z)).$$

Let Ω be the class of functions $h(z)$ with $h(0) = 1$, which are analytic and convex univalent in the open unit disk U . For functions $f(z)$ and $g(z)$ analytic in U , we say that $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$, if $g(z)$ is univalent in $U, f(0) = g(0)$ and $f(U) \subset g(U)$.

Let \mathcal{A} be the class of functions of the form

$$(1.8) \quad h(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in U . A function $h(z)$ is said to be in the class $S^*(\alpha)$, if

$$\operatorname{Re} \left\{ \frac{zh'(z)}{h(z)} \right\} > \alpha \quad (z \in U)$$

for some α ($\alpha < 1$). When $0 \leq \alpha < 1$, $S^*(\alpha)$ is the class of starlike functions of order α in U . A function $h(z) \in \mathcal{A}$ is said to be prestarlike of order α in U , if

$$\frac{z}{(1-z)^{2(1-\alpha)}} * h(z) \in S^*(\alpha) \quad (\alpha < 1).$$

We denote this class by $R(\alpha)$ (see [9]). A function $h(z) \in \mathcal{A}$ is in the class $R(0)$ if and only if $h(z)$ is convex univalent in U and

$$R\left(\frac{1}{2}\right) = S^*\left(\frac{1}{2}\right).$$

In this paper, we introduce and investigate the following subclass of Σ_p .

Definition 1.1. A function $f(z) \in \Sigma_p$ is said to be in the class $\mathcal{T}_p(a, c, \gamma, \lambda; h)$ if it satisfies the subordination condition

$$(1.9) \quad (1 + \lambda) \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} f(z) + \frac{\lambda}{p} \frac{1}{1 - 2\gamma} z^{p+1} (\mathcal{F}_{p,a,c,\gamma} f(z))' \prec h(z),$$

where $0 \leq \gamma < \frac{1}{2}$, $\lambda \in \mathbb{C}$, $h(z) \in \Omega$.

The special class $\mathcal{T}_p(a, c, 0, \lambda; h) = \mathcal{M}_p(a, c, \lambda; h)$ was investigated by Yang and Liu [15]. In order to prove our main results, we need the following lemmas.

Lemma 1.2. (see [4]) Let $g(z)$ be analytic in U and $h(z)$ be analytic and convex univalent in U with $h(0) = g(0)$. If

$$(1.10) \quad g(z) + \frac{1}{\mu} z g'(z) \prec h(z),$$

where $\operatorname{Re} \mu \geq 0$ and $\mu \neq 0$, then

$$g(z) \prec \tilde{h}(z) = \mu z^{-\mu} \int_0^z t^{\mu-1} h(t) dt \prec h(z),$$

and $\tilde{h}(z)$ is the best dominant of (1.10).

Lemma 1.3. (see [9]) Let $\alpha < 1$, $f(z) \in S^*(\alpha)$ and $g(z) \in R(\alpha)$. Then, for any analytic function $F(z)$ in U ,

$$\frac{g*(fF)}{g*f}(U) \subset \overline{co}(F(U)),$$

where $\overline{co}(F(U))$ denotes the convex hull of $F(U)$.

2. Main results

In this section, we obtain some results of $\mathcal{T}_p(a, c, \gamma, \lambda; h)$. The first set of inclusion relationships are given below.

2.1. Inclusion relations.

Theorem 2.1. *Let $0 \leq \lambda_1 < \lambda_2$, then*

$$\mathcal{T}_p(a, c, \gamma, \lambda_2; h) \subset \mathcal{T}_p(a, c, \gamma, \lambda_1; h).$$

Proof. Let $0 \leq \lambda_1 < \lambda_2$, and suppose that

$$(2.1) \quad H(z) = \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} f(z)$$

for $f(z) \in \mathcal{T}_p(a, c, \gamma, \lambda_2; h)$. Then the function $H(z)$ is analytic in U with $H(0) = 1$. Differentiating both sides of (2.1) with respect to z and using (1.7), we have

$$(2.2) \quad H(z) + \frac{\lambda_2}{p} z H'(z) = (1 + \lambda_2) \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} f(z) + \frac{\lambda_2}{p} \frac{1}{1 - 2\gamma} z^{p+1} (\mathcal{F}_{p,a,c,\gamma} f(z))' \prec h(z).$$

From Lemma 1.2 with $\mu = \frac{p}{\lambda_2} > 0$, we get

$$(2.3) \quad H(z) \prec h(z).$$

Noting that $0 \leq \frac{\lambda_1}{\lambda_2} < 1$ and that $H(z)$ is convex univalent in U , it follows from (2.1)-(2.3) that

$$\begin{aligned} & (1 + \lambda_1) \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} f(z) + \frac{\lambda_1}{p} \frac{1}{1 - 2\gamma} z^{p+1} (\mathcal{F}_{p,a,c,\gamma} f(z))' \\ &= \frac{\lambda_1}{\lambda_2} \left[(1 + \lambda_2) \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} f(z) + \frac{\lambda_2}{p} \frac{1}{1 - 2\gamma} z^{p+1} (\mathcal{F}_{p,a,c,\gamma} f(z))' \right] \\ &+ (1 - \frac{\lambda_1}{\lambda_2}) H(z) \prec h(z). \end{aligned}$$

Thus, $f(z) \in \mathcal{T}_p(a, c, \gamma, \lambda_1; h)$. □

Theorem 2.2. *Let*

$$(2.4) \quad \operatorname{Re} \{ z^p \varphi_p(a_1, a_2; z) \} > \frac{1}{2} \quad (z \in U; a_2 \neq \{0, -1, -2, \dots\}),$$

where $\varphi_p(a_1, a_2; z)$ is defined as in (1.2). Then,

$$\mathcal{T}_p(a_2, c, \gamma, \lambda; h) \subset \mathcal{T}_p(a_1, c, \gamma, \lambda; h).$$

Proof. For $f(z) \in \Sigma_p$, we can easily verify that

$$(2.5) \quad \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a_1,c,\gamma} f(z) = (z^p \varphi_p(a_1, a_2; z)) * \left(\frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a_2,c,\gamma} f(z) \right),$$

and

$$(2.6) \quad \frac{1}{1 - 2\gamma} z^{p+1} (\mathcal{F}_{p,a_1,c,\gamma} f(z))' = (z^p \varphi_p(a_1, a_2; z)) * \left(\frac{1}{1 - 2\gamma} z^{p+1} (\mathcal{F}_{p,a_2,c,\gamma} f(z))' \right).$$

We suppose $f(z) \in \mathcal{T}_p(a_2, c, \gamma, \lambda; h)$, then from (2.5) and (2.6), we deduce that

$$(2.7) \quad (1+\lambda) \frac{1}{1-2\gamma} z^p \mathcal{F}_{p,a_1,c,\gamma} f(z) + \frac{\lambda}{p} \frac{1}{1-2\gamma} z^{p+1} (\mathcal{F}_{p,a_1,c,\gamma} f(z))' = (z^p \varphi_p(a_1, a_2; z)) * \Psi(z),$$

where

$$(2.8) \quad \Psi(z) = (1+\lambda) \frac{1}{1-2\gamma} z^p \mathcal{F}_{p,a_2,c,\gamma} f(z) + \frac{\lambda}{p} \frac{1}{1-2\gamma} z^{p+1} (\mathcal{F}_{p,a_2,c,\gamma} f(z))' \prec h(z).$$

In view of (2.4), the function $z^p \varphi_p(a_1, a_2; z)$ has the Herglotz representation

$$(2.9) \quad z^p \varphi_p(a_1, a_2; z) = \int_{|x|=1} \frac{d\mu(x)}{1-xz} \quad (z \in U),$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x| = 1$ and

$$\int_{|x|=1} d\mu(x) = 1.$$

Since $h(z)$ is convex univalent in U , it follows from (2.7)-(2.9) that

$$(1+\lambda) \frac{1}{1-2\gamma} z^p \mathcal{F}_{p,a_1,c,\gamma} f(z) + \frac{\lambda}{p} \frac{1}{1-2\gamma} z^{p+1} (\mathcal{F}_{p,a_1,c,\gamma} f(z))' = \int_{|x|=1} \Psi(xz) d\mu(x) \prec h(z).$$

This shows that $f(z) \in \mathcal{T}_p(a_1, c, \gamma; h)$, and the proof of Theorem 2.2 is completed. □

Theorem 2.3. *Let $0 < a_1 < a_2$. Then,*

$$\mathcal{T}_p(a_2, c, \gamma, \lambda; h) \subset \mathcal{T}_p(a_1, c, \gamma, \lambda; h).$$

Proof. Define

$$w(z) = z + \sum_{n=1}^{\infty} \frac{(a_1)_n}{(a_2)_n} z^{n+1} \quad (z \in U; 0 < a_1 < a_2).$$

Then,

$$(2.10) \quad z^p \varphi_p(a_1, a_2; z) = w(z) \in \mathcal{A},$$

where $\varphi_p(a_1, a_2; z)$ is defined as in (1.2), and

$$(2.11) \quad \frac{z}{(1-z)^{a_2}} * w(z) = \frac{z}{(1-z)^{a_1}}.$$

By (2.11), we have

$$\frac{z}{(1-z)^{a_2}} * w(z) \in S^*(1 - \frac{a_1}{2}) \subset S^*(1 - \frac{a_2}{2})$$

for $0 < a_1 < a_2$, which implies that

$$(2.12) \quad w(z) \in R(1 - \frac{a_2}{2}).$$

Let $f(z) \in \mathcal{T}_p(a_2, c, \gamma, \lambda; h)$. Then we deduce from (2.7) and (2.8) (used in the proof of Theorem 2.2) and (2.10) that

$$(2.13) \quad (1 + \lambda) \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a_1,c,\gamma} f(z) + \frac{\lambda}{p} \frac{1}{1 - 2\gamma} z^{p+1} (\mathcal{F}_{p,a_1,c,\gamma} f(z))' \\ = \frac{w(z)}{z} * \Psi(z) = \frac{w(z) * (z\Psi(z))}{w(z) * z},$$

where

$$(2.14) \quad \Psi(z) = (1 + \lambda) \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a_2,c,\gamma} f(z) \\ + \frac{\lambda}{p} \frac{1}{1 - 2\gamma} z^{p+1} (\mathcal{F}_{p,a_2,c,\gamma} f(z))' \prec h(z).$$

Since the function z belongs to $S^*(1 - \frac{a_2}{2})$ and $h(z)$ is convex univalent in U , it follows from (2.12) and (2.13) and Lemma 1.3 that

$$(1 + \lambda) \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a_1,c,\gamma} f(z) + \frac{\lambda}{p} \frac{1}{1 - 2\gamma} z^{p+1} (\mathcal{F}_{p,a_1,c,\gamma} f(z))' \prec h(z).$$

Thus, $f(z) \in \mathcal{T}_p(a_1, c, \gamma, \lambda; h)$ and the proof is completed. □

As a special case of theorem 2.3, we have $\mathcal{T}_p(a+1, c, \gamma, \lambda; h) \subset \mathcal{T}_p(a, c, \gamma, \lambda; h)$ for $a > 0$.

Theorem 2.4. *Let $Re a \geq 0$ and $a \neq 0$. Then*

$$\mathcal{T}_p(a + 1, c, \gamma, \lambda; h) \subset \mathcal{T}_p(a, c, \gamma, \lambda; \tilde{h})$$

where

$$\tilde{h}(z) = az^{-a} \int_0^z t^{a-1} h(t) dt \prec h(z).$$

Proof. Define

$$(2.15) \quad g(z) = (1 + \lambda) \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} f(z) + \frac{\lambda}{p} \frac{1}{1 - 2\gamma} z^{p+1} (\mathcal{F}_{p,a,c,\gamma} f(z))'$$

for $f(z) \in \Sigma_p$. From (1.7) and (2.15), we obtain

$$(2.16) \quad \frac{1}{1 - 2\gamma} \mathcal{F}_{p,a+1,c,\gamma} f(z) + (p - a\lambda) \frac{1}{1 - 2\gamma} \mathcal{F}_{p,a,c,\gamma} f(z).$$

Differentiating both sides of (2.16) and using (2.13), we arrive at

$$(2.17) \quad \frac{1}{1 - 2\gamma} pz^{-p} [zg'(z) - pg(z)] = a\lambda \frac{1}{1 - 2\gamma} z (\mathcal{F}_{p,a+1,c,\gamma} f(z))' + \\ (p - a\lambda) \frac{1}{1 - 2\gamma} [a\mathcal{F}_{p,a+1,c,\gamma} f(z) - (a + p)\mathcal{F}_{p,a,c,\gamma} f(z)].$$

By (2.16) and (2.17), we have

$$pz^{-p}(zg'(z) + ag(z)) = a\lambda z \frac{1}{1 - 2\gamma} (\mathcal{F}_{p,a+1,c,\gamma} f(z))' + ap(1 + \lambda) \frac{1}{1 - 2\gamma} \mathcal{F}_{p,a+1,c,\gamma} f(z),$$

that is,

$$(2.18) \quad g(z) + \frac{z}{a}g'(z) = (1 + \lambda) \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a+1,c,\gamma} f(z) + \frac{\lambda}{p} \frac{1}{1 - 2\gamma} z^{p+1} (\mathcal{F}_{p,a+1,c,\gamma} f(z))'.$$

If $f(z) \in \mathcal{T}_p(a + 1, c, \gamma, \lambda; h)$, then it follows from (2.18) that

$$g(z) + \frac{z}{a}g'(z) \prec h(z) \quad (Re\ a \geq 0, a \neq 0).$$

Hence an application of Lemma 1.2 yields

$$g(z) \prec \tilde{h}(z) = az^{-a} \int_0^z t^{a-1} h(t) dt \prec h(z),$$

which shows that $f(z) \in \mathcal{T}_p(a, c, \gamma, \lambda; \tilde{h}) \subset \mathcal{T}_p(a, c, \gamma, \lambda; h)$. □

Theorem 2.5. *Let $\lambda > 0, \beta > 0$ and $f(z) \in \mathcal{T}_p(a, c, \gamma, \lambda; \beta h + 1 - \beta)$. If $\beta \leq \beta_0$, where*

$$(2.19) \quad \beta_0 = \frac{1}{2} \left(1 - \frac{p}{\lambda} \int_0^1 \frac{u^{\frac{p}{\lambda}-1}}{1+u} du \right)^{-1},$$

then $f(z) \in \mathcal{T}_p(a, c, \gamma, 0; h)$. The bound β_0 is sharp when $h(z) = \frac{1}{1-z}$.

Proof. Define

$$(2.20) \quad g(z) = \frac{1}{1 - 2\lambda} z^p \mathcal{F}_{p,a,c,\gamma} f(z),$$

for $f(z) \in \mathcal{T}_p(a, c, \gamma, \lambda; \beta h + 1 - \beta)$ with $\lambda > 0, \beta > 0$. Then we have

$$g(z) + \frac{\lambda}{p} z g'(z) = (1 + \lambda) \frac{1}{1 - 2\lambda} z^p \mathcal{F}_{p,a,c,\gamma} f(z) + \frac{\lambda}{p} \frac{1}{1 - 2\lambda} z^{p+1} (\mathcal{F}_{p,a,c,\gamma} f(z))' \prec \beta h(z) + 1 - \beta.$$

Hence an application of Lemma 1.2 yields

$$(2.21) \quad g(z) \prec \tilde{h}(z) = \frac{p}{\lambda} z^{-\frac{p}{\lambda}} \int_0^z t^{\frac{p}{\lambda}-1} (\beta h(t) + 1 - \beta) dt = \frac{p\beta}{\lambda} z^{-\frac{p}{\lambda}} \int_0^z t^{\frac{p}{\lambda}-1} h(t) dt + 1 - \beta = (h * \Psi)(z),$$

where

$$(2.22) \quad \Psi(z) = \frac{p\beta}{\lambda} z^{-\frac{p}{\lambda}} \int_0^z \frac{t^{\frac{p}{\lambda}-1}}{1-t} dt + 1 - \beta.$$

If $0 < \beta \leq \beta_0$, where $\beta_0 > 1$ is given by (2.19), then it follows from (2.22) that

$$Re \Psi(z) = \frac{p\beta}{\lambda} \int_0^1 u^{\frac{p}{\lambda}-1} Re \left(\frac{1}{1-uz} \right) du + 1 - \beta > \frac{p\beta}{\lambda} \int_0^1 \frac{u^{\frac{p}{\lambda}-1}}{1+u} du + 1 - \beta \geq \frac{1}{2}.$$

By using the Herglotz representation for $\Psi(z)$, from (2.20) and (2.21), we arrive at

$$g(z) = \frac{1}{1-2\lambda} z^p \mathcal{F}_{p,a,c,\gamma} f(z) \prec (h * \Psi)(z) \prec h(z),$$

because $h(z)$ is convex univalent in U . This shows that $f(z) \in \mathcal{T}_p(a, c, \gamma, 0; h)$. For $h(z) = \frac{1}{1-z}$ and $f(z) \in \Sigma_p$ defined by

$$\frac{1}{1-2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} f(z) = \frac{p\beta}{\lambda} z^{-\frac{p}{\lambda}} \int_0^z \frac{t^{\frac{p}{\lambda}-1}}{1-t} dt + 1 - \beta,$$

it is easy to verify that

$$(1 + \lambda) \frac{1}{1-2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} f(z) + \frac{\lambda}{p} \frac{1}{1-2\gamma} z^{p+1} (\mathcal{F}_{p,a,c,\gamma} f(z))' = \beta h(z) + 1 - \beta.$$

Thus, $f(z) \in \mathcal{T}_p(a, c, \gamma, 0; \beta h + 1 - \beta)$. Also, for $\beta > \beta_0$, we have

$$Re \left\{ \frac{1}{1-2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} f(z) \right\} \rightarrow \frac{p\beta}{\lambda} \int_0^1 \frac{u^{\frac{p}{\lambda}-1}}{1+u} du + 1 - \beta < \frac{1}{2} \quad (z \rightarrow -1),$$

which implies that $f(z) \notin \mathcal{T}_p(a, c, \gamma, 0; h)$. Hence the bound β_0 cannot be increased when $h(z) = \frac{1}{1-z}$. \square

2.2. Convolution properties.

Theorem 2.6. *Let $f(z) \in \mathcal{T}_p(a, c, \gamma, \lambda; h)$, $g(z) \in \Sigma_p$ and $Re\{z^p g(z)\} > \frac{1}{2}$ ($z \in U$). Then*

$$(f * g)(z) \in \mathcal{T}_p(a, c, \gamma, \lambda; h).$$

Proof. For $f(z) \in \mathcal{T}_p(a, c, \gamma, \lambda; h)$ and $g(z) \in \Sigma_p$, we have

$$\begin{aligned} (2.23) \quad & (1 + \lambda) \frac{1}{1-2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} (f * g)(z) + \frac{\lambda}{p} \frac{1}{1-2\gamma} z^{p+1} (\mathcal{F}_{p,a,c,\gamma} (f * g)(z))' \\ &= (1 + \lambda) (z^p g(z)) * \left(\frac{1}{1-2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} f(z) \right) + \frac{\lambda}{p} (z^p g(z)) * \left(\frac{1}{1-2\gamma} z^{p+1} (\mathcal{F}_{p,a,c,\gamma} f(z))' \right) \\ &= (z^p g(z)) * (\Psi(z)), \end{aligned}$$

where

$$(2.24) \quad \Psi(z) = (1 + \lambda) \frac{1}{1-2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} f(z) + \frac{\lambda}{p} \frac{1}{1-2\gamma} z^{p+1} (\mathcal{F}_{p,a,c,\gamma} f(z))' \prec h(z).$$

The remaining part of the proof of Theorem 2.6 is similar to that of Theorem 2.2. \square

Corollary 2.7. *Let $f(z) \in \mathcal{T}_p(a, c, \gamma, \lambda; h)$ be given by (1.1) and let*

$$s_m(z) = z^{-p} + \sum_{n=1}^{m-1} a_n z^{n-p} \quad (m \in \mathbb{N} \setminus \{0\}).$$

Then the function

$$\sigma_m(z) = \int_0^1 t^p s_m(tz) dt$$

is also in the class $\mathcal{T}_p(a, c, \gamma, \lambda; h)$.

Proof. we easily obtain that

$$(2.25) \quad \sigma_m(z) = z^{-p} + \sum_{n=1}^{m-1} \frac{a_n}{n+1} z^{n-p} = (f * g_m)(z),$$

where

$$f(z) = z^{-p} + \sum_{n=1}^{m-1} a_n z^{n-p} \in \mathcal{T}_p(a, c, \gamma, \lambda; h)$$

and

$$g_m(z) = z^{-p} + \sum_{n=1}^{m-1} \frac{1}{n+1} z^{n-p} \in \Sigma_p.$$

It is known from [10] that

$$(2.26) \quad \operatorname{Re}\{z^p g_m(z)\} = \operatorname{Re}\left\{1 + \sum_{n=1}^{m-1} \frac{1}{n+1} z^{n-p}\right\} > \frac{1}{2}.$$

In view of (2.25) and (2.26), an application of Theorem 2.6 leads to $\sigma_m(z) \in \mathcal{T}_p(a, c, \gamma, \lambda; h)$. □

Theorem 2.8. *Let $f(z) \in \mathcal{T}_p(a, c, \gamma, \lambda; h)$, $g(z) \in \Sigma_p$ and $z^{p+1}g(z) \in R(\alpha)$ ($\alpha < 1$). Then*

$$(f * g)(z) \in \mathcal{T}_p(a, c, \gamma, \lambda; h).$$

Proof. For $f(z) \in \mathcal{T}_p(a, c, \gamma, \lambda; h)$, and $g(z) \in \Sigma_p$, from (2.23) we have

$$(2.27) \quad (1 + \lambda) \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a,c,\gamma}(f * g)(z) + \frac{\lambda}{p} \frac{1}{1 - 2\gamma} z^{p+1} (\mathcal{F}_{p,a,c,\gamma}(f * g)(z))' \\ = \frac{(z^{p+1}g(z)) * (z\Psi(z))}{(z^{p+1}g(z)) * z},$$

where $\Psi(z)$ is defined as in (2.24). Since $h(z)$ is convex univalent in U ,

$$\Psi(z) \prec h(z), z^{p+1}g(z) \in R(\alpha), z \in S^*(\alpha) \ (\alpha < 1).$$

From (2.28) and Lemma 1.3 the desired result follows. □

Taking $\alpha = 0$ and $\alpha = \frac{1}{2}$, Theorem 2.8 reduces to the following.

Corollary 2.9. *Let $f(z) \in \mathcal{T}_p(a, c, \gamma, \lambda; h)$, and let $g(z) \in \Sigma_p$ satisfy either of the following conditions:*

- (1) $z^{p+1}g(z)$ is convex univalent in U or
- (2) $z^{p+1}g(z) \in R(\frac{1}{2})$.

Then $(f * g)(z) \in \mathcal{T}_p(a, c, \gamma, \lambda; h)$.

2.3. Integral operators.

Theorem 2.10. *Let $f(z) \in \mathcal{T}_p(a, c, \gamma, \lambda; h)$. Then the function $F(z)$ defined by*

$$(2.28) \quad F(z) = \frac{\mu - p}{z^\mu} \int_0^z t^{\mu-1} f(t) dt \quad (Re \mu > p)$$

is in the class $\mathcal{T}_p(a, c, \gamma, \lambda; \tilde{h})$, where

$$\tilde{h}(z) = (\mu - p)z^{-(\mu-p)} \int_0^z t^{\mu-p-1} h(t) dt \prec h(z).$$

Proof. For $f(z) \in \Sigma_p$ and $Re \mu > p$, we find from (2.28) that $F(z) \in \Sigma_p$ and

$$(2.29) \quad (\mu - p)f(z) = \mu F(z) + zF'(z).$$

Define $G(z)$ by

$$(2.30) \quad z^{-p}G(z) = (1 + \lambda) \frac{1}{1 - 2\gamma} \mathcal{F}_{p,a,c,\gamma} F(z) + \frac{\lambda}{p} \frac{1}{1 - 2\gamma} z(\mathcal{F}_{p,a,c,\gamma} F(z))'.$$

Differentiating both sides of (2.30) with respect to z , implies

$$(2.31) \quad zG'(z) - pG(z) = (1 + \lambda) \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} (zF'(z)) + \frac{\lambda}{p} \frac{1}{1 - 2\gamma} z^{p+1} (\mathcal{F}_{p,a,c,\gamma} (zF'(z)))'.$$

Furthermore, it follows from (2.29)-(2.31) that

$$(2.32) \quad \begin{aligned} & (1 + \lambda) \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} f(z) + \frac{\lambda}{p} \frac{1}{1 - 2\gamma} z^{p+1} (\mathcal{F}_{p,a,c,\gamma} f(z))' = \\ & (1 + \lambda) \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} \left(\frac{\mu F(z) + zF'(z)}{\mu - p} \right) \\ & \quad + \frac{\lambda}{p} \frac{1}{1 - 2\gamma} z^{p+1} \left(\mathcal{F}_{p,a,c,\gamma} \left(\frac{\mu F(z) + zF'(z)}{\mu - p} \right) \right)' \\ & = \frac{\mu}{\mu - p} G(z) + \frac{1}{\mu - p} (zG'(z) - G(z)) = G(z) + \frac{zG'(z)}{\mu - p}. \end{aligned}$$

Let $f(z) \in \mathcal{T}_p(a, c, \gamma, \lambda; h)$. Then, by (2.32),

$$G(z) + \frac{zG'(z)}{\mu - p} \prec h(z) \quad (Re \mu > p),$$

and so it follows from Lemma 1.2 that

$$G(z) \prec \tilde{h}(z) = (\mu - p)z^{-(\mu-p)} \int_0^z t^{\mu-p-1} h(t) dt \prec h(z).$$

Therefore, we conclude that

$$F(z) \in \mathcal{T}_p(a, c, \gamma, \lambda; \tilde{h}) \subset \mathcal{T}_p(a, c, \gamma, \lambda; h).$$

□

Theorem 2.11. Let $f(z) \in \Sigma_p$ and $F(z)$ defined as in Theorem 2.10. If

$$(2.33) \quad (1 - \alpha) \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} F(z) + \alpha \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} f(z) \prec h(z) \quad (\alpha > 0),$$

then $f(z) \in \mathcal{T}_p(a, c, \gamma, 0; \tilde{h})$, where $\operatorname{Re} \mu > p$, and

$$\tilde{h}(z) = \frac{\mu - p}{\alpha} z^{-\frac{\mu - p}{\alpha}} \int_0^z t^{\frac{\mu - p}{\alpha} - 1} h(t) dt \prec h(z).$$

Proof. Define

$$(2.34) \quad G(z) = \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} F(z).$$

Then $G(z)$ is analytic in U , with $G(0) = 1$, and

$$(2.35) \quad zG'(z) = pG(z) + \frac{1}{1 - 2\gamma} z^{p+1} (\mathcal{F}_{p,a,c,\gamma} F(z))'.$$

Making use of (2.29) and (2.33), (2.34), (2.35), implies that

$$\begin{aligned} & (1 - \alpha) \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} F(z) + \alpha \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} f(z) \\ &= (1 - \alpha) \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} F(z) + \alpha \frac{1}{1 - 2\gamma} z^p \mathcal{F}_{p,a,c,\gamma} \left(\frac{\mu F(z) + zF'(z)}{\mu - p} \right) \\ &= (1 - \alpha)G(z) + \alpha \frac{1}{1 - 2\gamma} z^p \left[\mathcal{F}_{p,a,c,\gamma} \frac{\mu}{\mu - p} F(z) + \mathcal{F}_{p,a,c,\gamma} \left(\frac{1}{\mu - p} zF'(z) \right) \right] \\ &= (1 - \alpha)G(z) + \frac{\mu\alpha}{\mu - p} G(z) + \frac{\alpha}{\mu - p} (zG'(z) - pG(z)) = G(z) + \frac{\alpha}{\mu - p} zG'(z) \prec h(z) \end{aligned}$$

for $\operatorname{Re} \mu > p$ and $\alpha > 0$. Therefore, an application of Lemma 1.2 yields the assertion of Theorem 2.11. \square

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