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SOME EXISTENCE RESULTS FOR GENERALIZED VECTOR QUASI-EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we introduce and study a class of generalized vector quasi-equilibrium problem, which includes many vector equilibrium problems, equilibrium problems, vector variational inequalities and variational inequalities as special cases. Using one person game theorems, the concept of escaping sequences and without convexity assumptions, we prove some existence results for generalized vector quasi-equilibrium problem.
Keywords: Vector quasi-equilibrium problem, escaping sequence, existence, continuity.
MSC(2010): Primary: 49J40, 47H19; Secondary: 47H10.

1. Introduction

Equilibrium problems include variational inequality problems as well as fixed point problems, optimization problems, saddle point problems and Nash equilibrium problems. Equilibrium problems provide us a systematic framework to study a wide class of problems arising in finance, economics, optimization and operations research. General equilibrium problems have been extended to the case of vector-valued bi-functions, known as vector equilibrium problem, see for example [1, 2, 8, 13, 15, 16, 22, 24, 26].

The generalized vector equilibrium problems have been studied by many researchers and include as special cases different types of vector variational inequalities, vector complementarity problems, see [3,4,6,14,20]. Quasi-equilibria constitute an extension of Nash equilibria, which is of fundamental importance in the theory of noncooperative games.

The aim of this paper is to establish some existence results for a generalized vector quasi-equilibrium problem under compact and noncompact settings

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by using one person game theorems and concept of escaping sequences. An existence result is proved in H-spaces.

2. Preliminaries

Let K be a subset of a topological space X. We denote by 2^K the family of all subsets of K and by $int_X K$ the interior of K in X, $cl_X K$ the closure of K in X. The convex hull of K is denoted by CoK. Let $P, Q : K \to 2^K$ be the multi-valued mappings. The the multi-valued mapping $P \cap Q : K \to 2^X$ is defined by $(P \cap Q)(x) = P(x) \cap Q(x), (C \circ T)(x) = C \circ T(x)$ and $(clT)(x) = cl_X T(x)$ for each $x \in K$, respectively.

Let X and Y be topological spaces and $P : X \to 2^Y$ be a multi-valued mapping. The mapping P is said to be open or have open graph if the graph of $P = \{(x, y) \in X \times Y : x \in X, y \in P(x)\}$ is open in $X \times Y$. The mapping P is said to be uppersemicontinuous if for each $x \in X$ and each open set V in Y with $P(x) \subset V$, there exists an open neighborhood U of x in X such that $P(y) \subset V$ for each $y \in U$. Let K be a non-empty convex subset of X and $C : K \to 2^Y$ be a multi-valued mapping such that for each $x \in K, C(x)$ is a closed convex cone in Y with int$C(x) \neq \emptyset$, where int$C(x)$ denotes the interior of $C(x)$.

Given $A : K \to 2^K$ as a continuous multi-valued mapping, $\eta : K \times K \to X$ and $f : K \to K$ continuous mappings. Suppose that $g : K \times X \to Y$ is a vector-valued mapping.We consider the following generalized vector quasi-equilibrium problem.

Find $x_0 \in K$ such that for all $z \in K, \lambda \in (0, 1], x_0 \in cl_K A(x_0)$ and

$$g(\lambda x_0 + (1-\lambda)z, \eta(x, f(x_0))) \notin -\mathrm{int_Y} C(x_0), \text{ for all } x \in A(x_0). \tag{2.1}$$

If f is an identity mapping and $\eta(x, x_0) = y$ and $\lambda = 1$, then problem (2.1) reduces to the problem of finding $x_0 \in K$ such that $x_0 \in cl_K A(x_0)$ and

$$g(x_0, y) \notin -\mathrm{int_Y} C(x_0), \text{ for all } y \in A(x_0). \tag{2.2}$$

This problem is studied by Khaliq and Krishan [18].

When $A(x) = K$ for each $x \in K$, problem (2.2) was considered by Ansari [5]. When $A(x) = K$ and $C(x) = P$ for each $x \in K$, where P is a convex cone in Y, then it was studied by Tan and Tinh [27]. The case $C(x) = R_+$ for each $x \in K$ and $Y = R$, problem (2.2) was studied by Lin and Park [23]. Also when $A(x) = K, C(x) = R_+$ for each $x \in K$ and $Y = R$, Blum and Oettli [9], Konnov and Schaible [21] and Kalmoun [17].

Clearly problem (2.1) is much more general than many problems studied in recent past.

We need the following two lemmas which are special cases of Theorem 2 of Ding et al. [12] and Theorem 2 of Ding et al. [11], respectively.

Lemma 2.1. *Let $\Gamma = (X, A, P)$ be a 1-person game such that*

(i) *X is a non-empty compact convex subset of a Hausdorff topological vector space,*
(ii) *$A : X \to 2^X$ is a correspondence such that for each $x \in X, A(x)$ is non-empty convex for each $y \in X, A^{-1}(y)$ is open in X,*
(ii) *the correspondence $clA : X \to 2^X$ is uppersemicontinuous,*
(iv) *the correspondence $P : X \to 2^X$ is such that $P^{-1}(y)$ is open in X for each $y \in X$,*
(v) *for each $x \in X, x \notin \mathrm{Co}P(x)$.*

Then Γ has an equilibrium choice $\hat{x} \in X$ such that $\hat{x} \in cl_X A(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$.

Lemma 2.2. *Let $\Gamma = (X, A, P)$ be a 1-person game such that*

(i) *X is a non-empty convex subset of a locally convex Hausdorff topological vector space and D is a non-empty compact subset of X,*
(1) *$A : X \to 2^D$ is a correspondence such that for each $x \in X, A(x)$ is non-empty convex and for each $y \in D, A^{-1}(y)$ is open in X,*
(ii) *the correspondence $clA : X \to 2^X$ is uppersemicontinuous,*
(iii) *the correspondence $P : X \to 2^D$ is such that $P^{-1}(y)$ is open in X for each $y \in D$,*
(iv) *for each $x \in X, x \notin \mathrm{Co}P(x)$.*

Then Γ has an equilibrium choice $\hat{x} \in D$ such that $\hat{x} \in cl_X A(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$.

3. Existence of solution for generalized vector quasi-equilibrium problem in compact and noncompact settings

We prove the following result in compact setting.

Theorem 3.1. *Let K be a non-empty compact convex subset of a Hausdorff topological vector space X and let Y be an ordered Hausdorff topological vector space. Let $C : K \to 2^Y$ be a multi-valued mapping such that, for all $x \in K, C(x)$ is a closed, convex and pointed cone in Y with $\mathrm{int}_Y C(x) \neq \emptyset$. Let $A : K \to 2^K$ be a multi-valued mapping such that $A(x)$ non-empty convex, $A^{-1}(y)$ is open in K for all $y \in K$ and $cl_K A : K \to 2^K$ is uppersemicontinuous for all $x, y \in K$.*

Let $g : K \times X \to Y$ be a vector-valued mapping which is continuous and affine in the second argument and let $\eta : K \times K \to X$ be a mapping which is affine in the first argument. Let $f : K \to K$ be a continuous mapping. Let $x_\alpha \to x_0$, for all $x_0 \in K$ whenever $\alpha \in \Lambda$. Let the multi-valued mapping $W(x) = Y \setminus (-\mathrm{int}_Y C(x))$ is uppersemicontinuous on K for all $x \in K$ and

$$g(\lambda x + (1-\lambda)z, \eta(x, f(x))) \notin -\mathrm{int}_Y C(x), \text{ for all } x \in K.$$

Then there exists a point $x_0 \in K$ such that for all $z \in K, \lambda \in (0,1]$, $x_0 \in cl_K A(x_0)$ and

$$(\lambda x_0 + (1-\lambda)z, \eta(x, f(x_0))) \notin -\mathrm{int}_Y C(x_0), \text{ for all } x \in A(x_0).$$

Proof. We define a multi-valued mapping $M : K \to 2^K$ by

$$M(x) = \{y \in K : g(\lambda x + (1-\lambda)z, \eta(x, f(x))) \in -\mathrm{int}_Y C(x)\}, \text{ for each } x \in K.$$

First we show that for $x \in K, x \notin \mathrm{Co}M(x)$. Suppose that $x_0 \in \mathrm{Co}M(x_0)$ for some $x_0 \in K$. Indeed, let $\{y_1, y_2, \cdots, y_n\}$ be a finite subset of K. Let $I \subset N$ be non-empty and $x_0 \in \mathrm{Co}\{y_i, i \in I\}$, then $x_0 = \sum_{i \in I} t_i y_i$ with $t_i \geq 0$ and $\sum_{i \in I} t_i = 1$. Then for each $i \in I$, we have

$$g(\lambda x_0 + (1-\lambda)z, \eta(y_i, f(x_0))) \in -\mathrm{int}_Y C(x_0).$$

Since $C(x_0)$ is a cone, $-\mathrm{int}_Y C(x_0)$ is convex and g and η are affine in the second and first argument, respectively, we have

$$\begin{aligned} g(\lambda x_0 + (1-\lambda)z, \eta(x_0, f(x_0))) &= g(\lambda x_0 + (1-\lambda)z, \eta(\sum_{i=1}^n t_i y_i, f(x_0))) \\ &= \sum_{i=1}^n t_i g(\lambda x_0 + (1-\lambda)z, \eta(y_i, f(x_0))) \in -\mathrm{int}_Y C(x_0), \end{aligned}$$

which contradicts the assumption. This shows that condition (v) of Lemma 2.2 is satisfied. It remains to show that $M^{-1}(y)$ is open in K, which is equivalent to show that $[M^{-1}(y)]^C = K \setminus M^{-1}(y)$ is closed. Clearly $M^{-1}(y) = \{x \in K : y \in M(x), g(\lambda x + (1-\lambda)z, \eta(y, f(x))) \in -\mathrm{int}_Y C(x)\}$. Let $\{x_\alpha\}_{\alpha \in \Lambda}$ be a net in $[M^{-1}(y)]^C$ converging to $x_0 \in K$. Since $x_\alpha \in [M^{-1}(y)]^C$, by the continuity of g and f and by uppersemicontinuity of W, we have

$$g(\lambda x_0 + (1-\lambda)z, \eta(y, f(x_0))) \in W(x_0) = Y \setminus -\mathrm{int}_Y C(x_0)$$

i.e., $g(\lambda x_0 + (1-\lambda)z, \eta(y, f(x_0))) \notin -\mathrm{int}_Y C(x_0)$, which implies that $x_0 \in [M^{-1}(y)]^C$, so that $[M^{-1}(y)]^C$ is closed. Thus $M^{-1}(y)$ is open in K. This shows that condition (iv) of Lemma 2.2 is also satisfied.

By the hypothesis, rest of the conditions of Lemma 2.2 are also satisfied. Thus, by Lemma 2.2, there exists $x_0 \in K$ such that $x_0 \in cl_K A(x_0)$ and $A(x_0) \cap M(x_0) = \emptyset$. This implies that $x_0 \in K$ such that for all $z \in K, \lambda \in (0, 1]$, there exists $x_0 \in cl_K A(x_0)$ and

$$g(\lambda x_0 + (1-\lambda)z, \eta(x, f(x_0))) \notin -\mathrm{int}_Y C(x_0), \text{ for all } x \in A(x_0).$$

□

The following example ensures the existence of solution of problem (2.1) and satisfies all the conditions of Theorem 3.1.

Example 3.2. Let $X = Y = \mathbb{R}$ and $K = [0, 2]$. We consider the following mappings
$A : K \to 2^K, f : K \to K, C : K \to 2^Y, \eta : K \times K \to X$ and $g : K \times X \to Y$

defined by

$$A(x) = \begin{cases} \{0\}, & \text{if } x = 0, \\ (0,x), & \text{if } x \in (0,2]. \end{cases}$$

$f(x) = 2x, C(x) = \mathbb{R}_+, \eta(x,y) = \frac{x+y}{2}$ and $g(x,y) = 2x + \frac{3}{2}y$, for all $x, y \in K$, satisfy all the conditions of the Theorem 3.1. Then for every $x_0 \in K = [0,2]$, $\lambda \in (0,1]$, $x_0 \in cl_K A(x_0)$, the generalized vector quasi-equilibrium problem $g(\lambda x_0 + (1-\lambda)z, \eta(x, f(x_0))) \notin -\text{int}_Y C(x_0)$, for all $x_0 \in A(x_0)$, is satisfied.

If $g(\lambda x + (1-\lambda)z, \eta(y, f(x))) = \langle T(\lambda x + (1-\lambda)z, \eta(y, f(x)))\rangle$, where $T : K \to L(X,Y)$ be a mapping and $L(X,Y)$ be the set of all continuous linear operators from X to Y. For $l \in L(X,Y)$, the value of the linear operator l at x is denoted by $\langle l, x\rangle$.

For the following vector quasi-variational-like inequality problem: find $x \in K$ such that for all $z \in K, \lambda \in (0,1], x \in cl_K A(x_0)$ and $\langle T(\lambda x + (1-\lambda)z, \eta(y, f(x)))\rangle \notin -\text{int}_Y C(x)$, for all $y \in K$, we can obtain the following existence result from Theorem 3.1.

Theorem 3.3. *Let K be a non-empty compact convex subset of a Hausdorff topological vector space X and let Y be an ordered Hausdorff topological vector space. Assume that $C : K \to 2^Y$ a multi-valued mapping such that for all $x \in K, C(x)$ is closed, convex and pointed cone in Y with $\text{int}_Y C(x) \neq \emptyset$. Let $A : K \to 2^K$ be a multi-valued mapping such that for all $x \in K, A(x)$ is non-empty convex and, for all $y \in K, A^{-1}(y)$ is open in K and, for all $x, y \in K, cl_K A : K \to 2^K$ is uppersemicontinuous. Let $T : K \to L(X,Y)$ be a mapping such that $\langle T(\lambda x_\alpha + (1-\lambda)z, \eta(y, f(x_\alpha)))\rangle \to \langle T(\lambda x_0 + (1-\lambda)z, \eta(y, f(x_0)))\rangle$, whenever $x_\alpha \to x_0$, for all $x_0 \in K$ and $\alpha \in \Lambda$; where $\eta : K \times K \to X$ be a mapping which is affine in the first argument and $f : K \to K$ be a continuous mapping. Furthermore, suppose that $W(x) = Y \setminus (-\text{int}_Y C(x))$ is uppersemicontinuous on K and*

$$\langle T(\lambda x + (1-\lambda)z, \eta(x, f(x)))\rangle \notin -\text{int}_Y C(x), \text{ for all } x \in K.$$

Then there exists a point $x_0 \in K$ such that for all $z \in K, \lambda \in (0,1], x_0 \in cl_K A(x_0)$ and

$$\langle T(\lambda x_0 + (1-\lambda)z, \eta(x, f(x_0)))\rangle \notin -\text{int}_Y C(x_0), \text{ for all } x \in A(x_0).$$

If $\lambda = 1, \eta(x, f(x_0)) = x - f(x_0)$, then from Theorem 3.1, we can easily obtain Theorem 1 of Kim and Tan [19].

The following theorems are proved in non-compact setting.

Theorem 3.4. *Let K be a non-empty convex subset of a locally convex Hausdorff topological vector space X, D a non-empty compact subset of K and Y an ordered Hausdorff topological vector space. Let $C : K \to 2^Y$ be a multi-valued mapping such that $C(x)$ is a closed, convex and pointed cone in Y*

with $\mathrm{int}_Y C(x) \neq \emptyset$*. Let* $A : K \to 2^D$ *is a multi-valued mapping such that for each* $x \in K, A(x)$ *is non-empty convex, for all* $y \in K, A^{-1}(y)$ *is open in* K *and* $cl_K A : K \to 2^D$ *is uppersemicontinuous for each* $x, y \in K$*. Let* $g : K \times X \to Y, \eta : K \times D \to X$ *and* $f : K \to D$ *be the mappings such that these mappings preserve the conditions considered in Theorem 3.1 and* W *is same as in Theorem 3.1. Then problem (2.1) is solvable.*

Proof. Define a multi-valued mapping $M : K \to 2^D$ by

$$M(x) = \{y \in D : g(\lambda x + (1-\lambda)z, \eta(y, f(x))) \in -\mathrm{int}_Y C(x)\}, \text{ for each } x \in K.$$

Using the same arguments Theorem 3.1, we have $x \notin \mathrm{Co}M(x)$ and $M^{-1}(y)$ is open for each $y \in D$. Thus all hypothesis of Lemma 2.2 are satisfied. Hence there exists a solution of problem (2.1). □

Definition 3.5. [10]. Let X be a topological space and K be a subset of X such that $K = \bigcup_{n=1}^{\infty} K_n$, where $\{K_n\}_{n=1}^{\infty}$ is an increasing sequence of non-empty compact sets in the sense that $K_n \subseteq K_{n+1}$ for all $n \in N$. A sequence $\{x_n\}_{n=1}^{\infty}$ in K is said to be escaping sequence from K(relative to $\{K_n\}_{n=1}^{\infty}$) if for each n there is an $m > 0$ such that $k \geq m, x_k \notin K_n$.

Theorem 3.6. *Let* K *be a non-empty compact convex subset of a Hausdorff topological vector space* X *and let* Y *be an ordered topological vector space. Let* $C : K \to 2^Y$ *be a multi-valued mapping such that for all* $x \in K, C(x)$ *is a closed, convex and pointed cone in* Y *with* $\mathrm{int}_Y C(x) \neq \emptyset$*. Let* $A : K \to 2^K$ *be a multi-valued mapping such that* $A^{-1}(y)$ *is open in* K *for all* $y \in K$*, for each* $x \in X, A(x)$ *is non-empty convex and* $cl_K A : K \to 2^K$ *is uppersemicontinuous for all* $x, y \in K$*. Let* $g : K \times X \to Y$ *be a multi-valued mapping which is continuous and affine in the second argument,* $\eta : K \times K \to X$ *be a mapping which is affine in the first argument and let* $f : K \to K$ *be a continuous mapping. Let* $x_\alpha \to x_0$*, for all* $x_0 \in K$ *whenever* $\alpha \in \Lambda$*. Let the multi-valued mapping* $W(x) = Y \setminus (-\mathrm{int}_Y C(x))$ *is uppersemicontinuous on* K *for all* $x \in K$*.*

Suppose that for each sequence $\{x_n\}_{n=1}^{\infty}$ *in* K *with* $x_n \in K_n, n \in N$ *which is escaping from* K *relative to* $\{K_n\}_{n=1}^{\infty}$*, there exists* $m \in N$ *and* $y_m \in K_m \cap A(x_m)$ *such that for all* $z \in K_m$ *and* $\lambda \in (0,1], x_m \in cl_K A(x_m)$ *and* $g(\lambda x_m + (1-\lambda)z, \eta(y_m, f(x_m))) \in -\mathrm{int}_Y C(x_m)$*.*

Then there exists $x^* \in K$ *such that for all* $z \in K, \lambda \in (0,1]$*, and* $x^* \in cl_K A(x^*)$*, we have*

$$g(\lambda x^* + (1-\lambda)z, \eta(y, f(x^*))) \notin -\mathrm{int}_Y C(x^*).$$

Proof. Since for each $n \in N, K_n$ is compact and convex set in X, Theorem 3.1 implies that for all $n \in N$, there exists $x_n \in K_n$ such that for all $z \in K_n, \lambda \in (0,1], x_n \in cl_K A(x_n)$ and

$$(3.1) \qquad g(\lambda x_n + (1-\lambda)z, \eta(y, f(x_n))) \notin -\mathrm{int}_Y C(x_n), \text{ for all } y \in A(x_n).$$

Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ is escaping from K relative to $\{K_n\}_{n=1}^{\infty}$. By assumption of this theorem, there exists $m \in N$ and $y_m \in K_m \cap A(x_m)$ such that for all $z \in K_m, \lambda \in (0,1], x_m \in cl_K A(x_m)$ and $g(\lambda x_m + (1-\lambda)z, \eta(y_m, f(x_m))) \in -\text{int}_Y C(x_m)$, which contradicts (3.1). Hence $\{x_n\}_{n=1}^{\infty}$ is not an escaping sequence from K relative to $\{K_n\}_{n=1}^{\infty}$. Using the same arguments used by Qun [25] in proving [26, Theorem 2], there exists $r \in N$ and there is a subsequence $\{x_{j_n}\}$ of $\{x_n\}_{n=1}^{\infty}$, which must lie entirely in K_r. Since K_r is compact, there exists a subsequence $\{x_{i_n}\}_{i_n \in \Lambda}$ of $\{x_{j_n}\}$ in K_r and there exists $x^* \in K_r$ such that $x_{i_n} \to x^*$, when $i_n \to \infty$. Since $\{K_n\}_{n=1}^{\infty}$ is an increasing sequence for all $y \in K$, there exists $i_0 \in \Lambda$ with $i_0 > r$ such that $y \in K_{i_0}$, for all $i_n \in \Lambda$ and $i_n > i_0$, we have $y \in K_{i_0} \subseteq K_{i_n}$ such that for all $z \in K_r$ and $\lambda \in (0,1]$, we have

$$g(\lambda x_{i_n} + (1-\lambda)z, \eta(y, f(x_{i_n}))) \notin -\text{int}_Y C(x_{i_n}). \tag{3.2}$$

Since g and f are continuous mappings and $x_{i_n} \to x^*$ when $i_n \to \infty$, we have

$$g(\lambda x_{i_n} + (1-\lambda)z, \eta(y, f(x_{i_n}))) \to g(\lambda x^* + (1-\lambda)z, \eta(y, f(x^*))).$$

By uppersemicontinuity of W, we have

$$g(\lambda x^* + (1-\lambda)z, \eta(y, f(x^*))) \in W(x^*).$$

Since $cl_K A : K \to 2^K$ is uppersemicontinuous with compact values, there exists $x^* \in K$ such that for all $z \in K, \lambda \in (0,1], x^* \in cl_K A(x^*)$ and

$$g(\lambda x^* + (1-\lambda)z, \eta(y, f(x^*))) \notin -\text{int}_Y C(x^*), \text{ for all } y \in A(x^*).$$

□

4. **Existence theory without convexity**

In this section, we prove an existence theorem for generalized vector quasi-equilibrium problem, by replacing convexity assumptions with merely topological properties.

Definition 4.1. An H-space is a pair $(X, \{\Gamma_A\})$, where X is a topological space and $\{\Gamma_A\}$ is a given family of non-empty contractible subsets of X, indexed by the finite subsets of X such that $A \subset B$ implies $\Gamma_A \subset \Gamma_B$.

A subset D of X is called H-convex, if for every finite subset A of D, it follows that $\Gamma_A \subset D$.

A subset D of X is called weakly H-convex, if for every finite subset A of D, it results that $\Gamma_A \cap D$ is non-empty and contractible. This is equivalent to saying that the pair $(D, \{\Gamma_A \cap D\})$ is an H-space.

A subset K of X is called H-compact, if for every finite subset A of X, there exists a compact, weakly H-convex set D of X such that $K \cup A \subset D$.

Definition 4.2. Let $(X, \{\Gamma_A\})$ be an H-space. A multi-valued mapping $F : X \to 2^X$ is called H-KKM if $\Gamma_A \subset \bigcup_{u \in A} F(u)$, for every finite subset A of X.

Theorem 4.3. *[7]. Let $(X, \{\Gamma_A\})$ be an H-space and $F : X \to 2^X$ be an H-KKM multi-valued mapping such that*

(1) *for each $u \in X, F(u)$ is compactly closed, that is $B \cap F(u)$ is closed in B for every compact set $B \subset X$,*
(2) *there exists a compact set $L \subset X$ and an H-compact set $K \subset X$ such that for every weakly H-convex set D with $K \subset D \subset X$, we have*

$$\bigcap_{u \in D} (F(u) \cap D) \subset L.$$

Then $\bigcap_{u \in X} F(u) \neq \emptyset$.

Theorem 4.4. *Let K be a non-empty compact subset of a topological space X, Y be a topological space and $(K, \{\Gamma_B\})$ be an H-space. Let $g : K \times X \to Y, f : K \to K$ be the continuous mappings and $\eta : K \times K \to X$ be a mapping*

(1) *for each $y \in K, N_y = \{u \in K : g(\lambda u + (1-\lambda)z, \eta(y, f(u))) \in -\mathrm{int}_Y C(u)\}$ is H-convex or empty.*
(2) *$g(\lambda u + (1-\lambda)z, \eta(u, f(u))) \notin -\mathrm{int}_Y C(u)$, for all $u \in K$.*
(3) *$A : K \to 2^K$ a multi-valued mapping such that $cl_K A : K \to 2^K$ is uppersemicontinuous.*
(4) *There exists a compact subset L of K and an H-compact subset $E \subset K$ such that for every weakly H-convex set D with $E \subset D \subset K$,*

$$\{y \in D : g(\lambda u + (1-\lambda)z, \eta(y, f(u))) \notin -\mathrm{int}_Y C(u), \text{ for each } u \in K\} \subset L.$$

Then there exists $x_0 \in K$ such that for all $z \in K, \lambda \in (0,1], x_0 \in cl_K A(x_0)$ and

$$g(\lambda x_0 + (1-\lambda)z, \eta(y, f(x_0))) \notin -\mathrm{int}_Y C(x_0), \text{ for all } y \in K.$$

Proof. Let $S(y) = \{u \in K : g(\lambda u + (1-\lambda)z, \eta(y, f(u))) \notin -\mathrm{int}_Y C(u)\}$, for all $y \in K, z \in K$ and $\lambda \in (0,1]$. First we prove that S is an H-KKM mapping. Suppose that S is not an H-KKM mapping. Then there exists a finite subset B of K such that

$$\Gamma_B \not\subset \bigcap_{y \in B} S(y).$$

Thus there exists $v \in \Gamma_B$ such that

$$v \notin S(y), \text{ for all } y \in B,$$

that is, $g(\lambda v + (1-\lambda)z, \eta(y, f(v))) \in -\mathrm{int}_Y C(v)$, for all $y \in B$.
By Assumption (1), $B \subset N_v$ and $\Gamma_B \subset N_v$. Since N_v is H-convex, that is, $v \in N_v$ such that

$$g(\lambda v + (1-\lambda)z, \eta(v, f(v))) \in -\mathrm{int}_Y C(v),$$

which contradicts assumption (2). Thus $\Gamma_B \subset \bigcup_{y \in B} S(y)$, for every finite subset B of K. Hence S is an H-KKM mapping.

On the other hand, for any $y \in K, S(y)$ is closed. Indeed, let $\{x_n\}$ is a sequence in $S(y)$ such that $x_n \to x_0 \in K$. Since $x_n \in S(y)$, for all n, we have

$$g(\lambda x_n + (1-\lambda)z, \eta(y, f(x_n))) \notin -\mathrm{int}_Y C(x_n).$$

Since g and f are continuous mappings and $x_n \to x_0$, we have

$$g(\lambda x_n + (1-\lambda)z, \eta(y, f(x_n))) \to g(\lambda x_0 + (1-\lambda)z, \eta(y, f(x_0))) \notin -\mathrm{int}_Y C(x_0).$$

It follows that $x_0 \in S(y)$ and so $S(y)$ is closed for any $y \in K$, that is, condition (1) of Theorem 4.3 holds. It is easy to see that the assumption (4) of this theorem and condition (2) of Theorem 4.3 are same. Thus we have

$$\bigcap_{y \in K} S(y) \neq \emptyset.$$

It implies that $x_0 \in K$ such that for all $z \in K, \lambda \in (0,1]$, we have

$$g(\lambda x_0 + (1-\lambda)z, \eta(y, f(x_0))) \notin -\mathrm{int}_Y C(x_0), \text{ for all } y \in K.$$

Since $cl_K A(x_0)$ is uppersemicontinuous with compact values, there exists $x_0 \in K$ such that for all $z \in K, \lambda \in (0,1], x_0 \in cl_K A(x_0)$ and

$$g(\lambda x_0 + (1-\lambda)z, \eta(y, f(x_0))) \notin -\mathrm{int}_Y C(x_0).$$

□

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