ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 41 (2015), No. 6, pp. 1423–1431

Title:

Finite groups have even more centralizers

Author(s):

S. M. Jafarian Amiri, M. Amiri, H. Madadi and H. Rostami

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 6, pp. 1423–1431
Online ISSN: 1735-8515

FINITE GROUPS HAVE EVEN MORE CENTRALIZERS

S. M. JAFARIAN AMIRI*, M. AMIRI, H. MADADI AND H. ROSTAMI

(Communicated by Ali Reza Ashrafi)

Abstract. For a finite group G, let $Cent(G)$ denote the set of centralizers of single elements of G. In this note we prove that if $|G| \leq \frac{3}{2}|Cent(G)|$ and G is 2-nilpotent, then $G \cong S_3, D_{10}$ or $S_3 \times S_3$. This result gives a partial and positive answer to a conjecture raised by A. R. Ashrafi [On finite groups with a given number of centralizers, *Algebra Colloq.* 7 (2000), no. 2, 139–146].
Keywords: Finite groups, Centralizer, involution.
MSC(2010): Primary: 20D60; Secondary: 20D10.

1. Introduction

Throughout this paper G is a non-trivial finite group. Let $Cent(G)$ denote the set of centralizers of single elements of G and let $|Cent(G)|$ be its cardinality. The group G is called n-centralizer if $|Cent(G)| = n$. Starting with Belcastro and Sherman [4], many authors have studied the influence of $|Cent(G)|$ on the structure of the group G (see [1–3] and [11]).

In [4], Belcastro and Sherman raised the question whether or not there exists a finite $n-centralizer$ group G other than Q_8 and D_{2p} ($p > 2$ is a prime) such that $|G| \leq 2n$. Ashrafi in [2] showed that there are several groups satisfying the given properties. Then Ashrafi raised the following conjecture (conjecture 2.4 of [2]):

Conjecture 1.1. *Suppose that G is an n-centralizer group. If $|G| \leq \frac{3n}{2}$, then G is isomorphic to $S_3, S_3 \times S_3$ or D_{10}, the dihedral group of order 10.*

In [11], it is proved that if $|G| \leq \frac{3n}{2}$ then G is solvable. In this paper first we confirm Conjecture 1.1 for groups whose Sylow 2-subgroups have order at most 4 (see Propositions 2.6 and 3.7) and then we prove the following main result:

Article electronically published on December 15, 2015.
Received: 27 January 2014, Accepted: 3 September 2014.
*Corresponding author.

Theorem 1.2. *Suppose that G is a n-centralizer group such that $|G| \leq \frac{3n}{2}$. If G is 2-nilpotent, then $G \cong S_3, D_{10}$ or $S_3 \times S_3$.*

Throughout this paper all groups mentioned are assumed to be finite. $Z(G)$ denotes the center of the group G and D_{2n} denotes the dihedral group of order $2n$. Also $C_G(x)$ is the centralizer of x in G, elements of order 2 are called involutions and $I(G) = \{a \in G | a^2 = 1\} = \{a \in G | a = a^{-1}\}$. We denote by $o(x)$ the order of the element x in G. $Cl(x)$ is the conjugacy class of x in G. Most notations are standard and they are taken mainly from [8–10].

2. **Frobenious groups**

The following lemma will be used in the proof of some results.

Lemma 2.1. *Let $G = K \rtimes H$ be a Frobenius group with kernel K. Then*

(i) *If $Z(H) = 1$, then $|Cent(K)| + |K|(|Cent(H)| - 1) + 1 = |Cent(G)|$.*
(ii) *If $Z(H) \neq 1$, then $|Cent(K)| + |K||Cent(H)| + 1 = |Cent(G)|$.*

Proof. The proof is trivial by definition of Frobenius groups. □

Lemma 2.2. *If G is a nonabelian n-centralizer group such that $|G| \leq \frac{3n}{2}$, then $Z(G) = 1$ and $|I(G)| \geq \frac{|G|}{3}$. Also G is a solvable group of order $2^r m$, $r \geq 1$ and m is an odd integer greater than 1.*

Proof. It is easy to see that $n \leq |\frac{G}{Z(G)}|$. Therefore $n|Z(G)| \leq \frac{3n}{2}$ which implies that $Z(G) = 1$. It follows from Lemma 2.3 of [11] that $n \leq \frac{|G|+|I(G)|}{2}$. Thus $2n \leq |G| + |I(G)| \leq \frac{3n}{2} + |I(G)|$ which gives the result. Finally G is solvable by Theorem 2.6 of [11]. □

Lemma 2.3. *Let $G = S \ltimes K$ be an n-centralizer Frobenius group with kernel K and S be a Sylow 2-subgroup of G. If $|G| \leq \frac{3n}{2}$, then $|S| \leq 4$.*

Proof. By Lemma 2.1, we have

$$\begin{aligned} \frac{2}{3}|K||S| &\leq |Cent(G)| \\ &\leq |Cent(K)| + |K||Cent(S)| + 1 \\ &\leq |K| - 1 + |K|\frac{|S|}{2} + 1. \end{aligned}$$

It follows that $|S| \leq 4$, as wanted. □

Lemma 2.4. *Let G be a dihedral group of order $2m$ such that m is odd. If G is an n-centralizer group such that $|G| \leq \frac{3}{2}n$, then $m = 3$ or 5.*

Proof. According to Lemma 2.1(ii), we have $|Cent(G)| = m + 2$. Thus $m = 3$ or $m = 5$ by assumption. □

Proposition 2.5. *Let G be a Frobenius group such that $|I(G)| \geq \frac{|G|}{3}$. Then $G \cong S_3, D_{10}, C_3 \rtimes C_4$ or $(C_2 \times C_2 \times \cdots \times C_2) \rtimes C_3$. In particular, if $|Cent(G)| \geq \frac{2}{3}|G|$, then $G \cong S_3$ or D_{10}.*

Proof. Suppose that $G = K \rtimes H$ is Frobenius with kernel K and $|H|$ is of even order. Then by Theorem 6.3 of [9], K is abelian of odd order and H contains a unique element of order 2 and so G has exactly one conjugacy class of involutions. Therefore we have $|I(G)| = |K| + 1 \geq \frac{|H||K|}{3}$. So we obtain $|H| \leq 4$ and $|K| = 3$. Thus $G \cong S_3$ or $C_3 \rtimes C_4$.

Now suppose that $|H|$ is of odd order. Then $|K|$ is even. Since K is normal in G, we have $I(G) = I(K)$. Consequently $\frac{|H||K|}{3} \leq |I(K)| \leq |K|$. Hence $|I(K)| = |K|$ and $|H| = 3$. Therefore K is elementary abelian and so we have the result.

If $|Cent(G)| \geq \frac{2}{3}|G|$, then the result follows by Lemma 2.1 and Lemma 2.4. This completes the proof. $\square$

In what follows we confirm the Conjecture 1.1 for groups whose Sylow 2-subgroups have order 2.

Proposition 2.6. *Let G be an n-centralizer group of order $2m$ where m is an odd integer. If $|G| \leq \frac{3n}{2}$, then $G \cong S_3$ or D_{10}.*

Proof. If $|G| \leq 12$, then $G \cong D_{10}$ or S_3. So we can assume that $|G| > 12$. Then G has a normal subgroup K of index 2 by hypothesis and so $G = K\langle a \rangle$ for some $a \in G$ of order 2. Therefore G has only one conjugacy class of involutions. Since $|I(G)| \geq \frac{|G|}{3}$, we have $|\{a^g : g \in G\}| + 1 = |G : C_G(a)| + 1 \geq \frac{|G|}{3}$ which yields that $|C_G(a)| \leq \frac{3|G|}{|G|-3} < 4$. So $|C_G(a)| = 2$. Thus $\langle a \rangle$ acts fixed point freely on K and so G is a Frobenius group with kernel K. The result follows by Proposition 2.5. $\square$

3. Groups with Sylow 2-subgroups of order 4

Recall that a group G is a CA-group if $C_G(x)$ is abelian for every $x \in G \setminus Z(G)$.

Lemma 3.1. *Let G be an n-centralizer group of order $2^r m$ such that $r > 1$ and m is odd. If $|G| \leq \frac{3n}{2}$, then G is not a CA- group.*

Proof. Suppose for a contradiction that G is a CA- group, we proceed by Theorem A of [5]. In the first case we assume that G has an abelian normal subgroup of index p such as K. Then by Lemma 4.6 of [9], $K = G'$. Now by Theorem 2.3 of [3], we have $|Cent(G)| = |K| + 2$. Therefore $|G| \leq \frac{3}{2}(|K| + 2) = \frac{3}{2}(\frac{|G|}{p} + 2)$. So $2p|G| - 3|G| \leq 6p$ which yields that $|G| \leq \frac{6p}{2p-3}$. On the other hand $6 < |G|$. Consequently $6p > 12p - 18$ and $p = 2$ which is a contradiction because in this case $|G| \leq 12$. Now by Theorem A of [5], Lemma 2.2 and Proposition 2.5 we have the result. □

Remark 3.2. *If G is a group of order $2^r m$ such that $|G| \leq \frac{3}{2}|Cent(G)|$ and $r > 1$, then by Lemma 3.1 and Corollary 2.6 of [3] we can reduce problem to groups with $\frac{1}{3} \leq \frac{|I(G)|}{|G|} \leq \frac{1}{2}$.*

Lemma 3.3. *Let G be an n-centralizer group of order $4m$ such that m is an odd integer. If $|G| \leq \frac{3n}{2}$, then Sylow 2-subgroups of G are not cyclic.*

Proof. Suppose, for a contradiction, that a Sylow 2-subgroup S of G is cyclic.Then $|I(G)| \leq m+1$ since the number of Sylow 2-subgroups of G is at most m. On the other hand by Lemma 2.2, $|I(G)| \geq \frac{4m}{3}$ and so $\frac{4m}{3} \leq m+1$. Therefore $|G| \leq 12$ which is impossible. □

Proposition 3.4. *Let G be a group of order $2^r m$ such that m is odd, $r \geq 1$ and S be a Sylow 2-subgroup of G. If $\frac{|G|}{3} \leq |I(G)|$, then either $N_G(S) = S$ or $G \cong C_2^r \rtimes C_3$. In particular, if $|G| \leq \frac{3}{2}|Cent(G)|$, then $N_G(S) = S$.*

Proof. Since every involution of G lies in some Sylow 2-subgroup of G, we have $I(G) \subseteq \cup_{x \in G} x^{-1}Sx$. But S has $|G : N_G(S)|$ conjugates in G and so $|I(G)| \leq |G : N_G(S)|(|S|-1)+1 \leq \frac{|G||S|}{|N_G(S)|}$. Therefore $|N_G(S)| \leq 3|S|$ by hypothesis. If $N_G(S) \neq S$, then $k = |G : N_G(S)| = \frac{m}{3} < m$. Also by Proposition 4.3 of [6] we have $|I(G)| \leq (2^r-1)k+1$. We conclude that $k = 1$ and so $S \lhd G$ and $m = 3$. But in this case we have $2^r \leq |I(G)| = |I(S)| \leq 2^r$ which means that S is an elementary abelian group and hence $G \cong C_2^r \rtimes C_3$.

Now if $|G| \leq \frac{3}{2}|Cent(G)|$, then $G \ncong C_2^r \rtimes C_3$ by Proposition 2.5. This completes the proof.

□

Lemma 3.5. *Let G be an n-centralizer group of order $4m$ such that m is an odd integer and $|G| \leq \frac{3n}{2}$. If $S = \{1, a, b, ab\}$ is a Sylow 2-subgroup of G, then G is 2-nilpotent and there exists $1 \neq x \in S$ such that $S = C_G(x)$. Also if $C_G(ab) = S$ then $C_G(a)$ and $C_G(b)$ are not equal to S.*

Proof. It follows from Lemma 3.3 that $S \cong C_2 \times C_2$ and by Proposition 3.4, we deduce $S = C_G(S) = N_G(S)$. Therefore by Burnside's Theorem, S has a normal complement in G say K and so $G = S \ltimes K$. Now if $S = C_G(b) = C_G(a) = C_G(ab)$, then the action of S on K is Frobenius and we reach to a contradiction by Proposition 2.5. On the other hand if $C_G(b)$, $C_G(a)$ and $C_G(ab)$ are not equal to S, then there are $g_1, g_2, g_3 \in K$ such that $(ab)^{g_3} = ab, b^{g_2} = b, a^{g_1} = a$ where $o(g_i) \geq 3$. Therefore every involution of G is contained in at least three distinct conjugates of S. So G has at most $\frac{3m}{3} = m$ involutions which contradicts to Lemma 2.2. So for at least one element of S say ab we have $C_G(ab) = S$. We claim that it is the only element of S with this property. For if $C_G(a) = S$, then $Cl(a) \cup Cl(ab)$ has $2m$ elements and so $|I(G)| \geq 2m + 2 > \frac{4m}{2}$, which is a contradiction by Remark 3.2. □

Lemma 3.6. *Let G be an n-centralizer group of order $4m$, where $|G| \leq \frac{3n}{2}$. Then $|I(G)| \leq \frac{5m}{3} + 1$.*

Proof. Let $S = \{1, a, b, ab\}$ be a Sylow 2-subgroup of G. By Lemma 3.5, we can assume that $C_G(ab) = S$. On the other hand, the number of involutions of G is equal to $|Cl(ab) \bigcup Cl(a) \bigcup Cl(b)|$. So we have $|I(G)| \leq m + \frac{m}{3} + \frac{m}{3} + 1 = \frac{5m}{3} + 1$, as wanted. □

In what follows we confirm the Conjecture 1.1 for groups whose Sylow 2-subgroups have order 4.

Proposition 3.7. *Let G be an n-centralizer group such that $|G| \leq \frac{3n}{2}$. If $|G| = 4m$, where m is an odd integer, then $G \cong S_3 \times S_3$.*

Proof. Let $S = \{1, a, b, ab\}$ be a Sylow 2-subgroup of G. By Lemma 3.5 we can assume that $G = S \ltimes K$, $C_G(ab) = S$ and $S < C_G(a), C_G(b)$. Now we define the automorphism ϕ of K as $\phi(g) = g^{ab}$. It is clear that ϕ is a fixed-point-free automorphism of order 2. Thus by Lemma 1.1 of [8], $g^{ab} = g^{-1}$ for all $g \in K$ which shows that K is abelian. Therefore $C_G(g) = K, \langle K, b\rangle$ or $\langle K, a\rangle$ for each $1 \neq g \in K$.

Now if $x \in G - (I(G) \bigcup K)$, then $x^{-1} \in G - (I(G) \bigcup K)$ and $C_G(x) = C_G(x^{-1})$ while $x^{-1} \neq x$. Therefore we have

$$|Cent(G)| \leq 3 + |I(G)| + \frac{3m - |I(G)|}{2} = 3 + \frac{3m}{2} + \frac{|I(G)|}{2}.$$

By Lemma 3.6 we have $|Cent(G)| \leq 3 + \frac{3m}{2} + \frac{1}{2}(\frac{5m}{3} + 1)$. Hence $\frac{8m}{3} \leq 3 + \frac{1}{2} + \frac{7m}{3}$ which yields that $m \leq 10$ and $|G| \leq 40$. Now by considering groups of order at most 40 by GAP [7], we reach to $S_3 \times S_3$ as only group which satisfies the hypotheses of the proposition. □

4. **Proof of the main result**

The following lemma is very useful in the sequel.

Lemma 4.1. *Let G be an n-centralizer group of order $2^r m$, where m is odd, $r > 2$ and let S be a Sylow 2-subgroup of G. Suppose that $|G| \leq \frac{3n}{2}$ and G is 2-nilpotent. Then we have:*

(i) *The normal Hall complement K of S in G is abelian.*
(ii) *There exits exactly one involution in S such as u which $C_G(u) \bigcap K = 1$ and for all $x \in K$, $x^u = x^{-1}$.*
(iii) *S is elementary abelian and there is $u \in S$ such that $C_G(u) = S$.*

Proof. (i) Let $J(S) = \{x \in S | o(x) = 2\}$, $T = \{u \in J(S) | C_G(u) \cap K = 1\}$ and $t = |T|$. Since K is a normal subgroup of G, for every $u \in J(S)$, the map $\varphi_u : K \longrightarrow K$ with $\varphi_u(x) = x^u$ is an automorphism of K. If $u \in J(S) \setminus T$, then there exists $1 \neq x \in K$ such that $u^x = u^{x^2} = u$. We conclude that $u \in S \cap S^x \cap S^{x^2}$. Since $x \in K$, the order of x is odd and so S, S^x and S^{x^2} are distinct, by Proposition 3.4. Therefore we have $\frac{2^r m}{3} \leq |I(G)| \leq 1 + mt + \frac{|J(S)|-t}{3}m$, which is equivalent to

$$t \geq 2^{r-1} - \frac{|J(S)|}{2} - \frac{3}{2m}. \ (*)$$

Since $|J(S)| \leq 2^r - 1$, we see that $t \geq \frac{1}{2} - \frac{3}{2m}$.

Now if $m = 3$, then K is abelian and $t \geq 1$. If $m > 3$, then $t \geq 1$, i.e, there exists $u \in J(S)$ such that $C_G(u) \bigcap K = 1$. Therefore $\varphi_u \in Aut(K)$ is a fixed-point-free and since $\varphi_u^2 = 1$, K is abelian and $\varphi_u(x) = x^{-1}$ for every $x \in K$, by Theorem 10.5.1(iv) of [10].

(ii) As in the proof of the part(i), suppose that u is an involution of S such that $C_G(u) \bigcap K = 1$. Then for every $x \in K$ we have that $x^u = x^{-1}$. Now if b is an another involution of S such that $C_G(b) \bigcap K = 1$, then we have for every $x \in K$, $x^b = x^u$ which follows that $bu \in Cor_G(S)$. On the other hand by definition of system normalizer and Theorem 9.2.8 of [10], $Cor_G(S) = 1$ and so $b = u$ which is a contradiction to our choice of b and u.

(iii) By substituting $t = 1$ in inequality (*), we have $|J(S)| \geq 2^r - 2 - \frac{3}{m}$. If $|S| \geq 16$, then $|I(S)| \geq 2^r - 1 - \frac{3}{m} > \frac{3}{4}|S|$ and so S is elementary abelian. Now Let $|S| = 8$. Then we conclude that $|I(S)| \geq 6$ and so $S = C_2 \times C_2 \times C_2$.

□

Lemma 4.2. *Let G be an n-centralizer group of order $2^r m$ such that $r > 2$, m be odd and $|G| \leq \frac{3n}{2}$. If G is 2-nilpotent, then there exists an involution such as t where $|C_G(t)| = 3|S|$.*

Proof. Suppose, for a contradiction, that $|C_G(t)| \geq 5.2^r$ for every $t \in S$ such that $C_G(t) \neq S$. It follows from Lemma 4.1(ii) that there exists only one element a in S such that $C_G(a) = S$. If $S = \{t_1, \cdots, t_{2^r}\}$, then we have

$$\begin{aligned} \frac{|G|}{3} &\leq |I(G)| \leq \sum_{i=1}^{2^r} |Cl(t_i)| \\ &= \sum_{i=1}^{2^r} |G : C_G(t_i)| \\ &\leq 1 + m + \frac{(2^r - 2)m}{5} \end{aligned}$$

and hence $2^r \leq \frac{3\times 5}{m(5-3)} + \frac{3\times 5-6}{5-3} \leq 7$, a contradiction. $\square$

Lemma 4.3. *Let G be an n-centralizer group such that $|G| \leq \frac{3n}{2}$. If G is 2-nilpotent, then the order of a Sylow 2-subgroup of G is not 8.*

Proof. Suppose, for a contradiction, that $|G| = 8m$ such that $(8, m) = 1$. Suppose that S is a Sylow 2-subgroup of G. Using arguments similar to that of Lemma 3.6, one may to prove that $|I(G)| \leq 1 + 3m$. On the other hand there is an element $b \in S$ such that $C_G(b) = S$ and for every $x \in K$ we have $C_G(x) = HK$ where $H \leq S$ and $b \notin H$. The number of such subgroups is 11, which means that if $T = \{C_G(x)|x \in K\}$, then $|T| \leq 11$. Also if $y \in G - (I(G) \bigcup K)$, then $y^{-1} \in G - (I(G) \bigcup K)$ and $C_G(y) = C_G(y^{-1})$ while $y \neq y^{-1}$. Therefore we have

$$\begin{aligned}
\frac{2\times 8m}{3} &\leq |Cent(G)| \\
&\leq |I(G)| + 11 + \frac{7m - |I(G)|}{2} \\
&= 11 + \frac{7m}{2} + \frac{|I(G)|}{2} \\
&\leq 11 + \frac{7m}{2} + \frac{1+3m}{2} \\
&= \frac{23}{2} + 5m
\end{aligned}$$

Thus $\frac{16m}{3} \leq \frac{23}{2} + 5m$ and hence $m \leq \frac{3}{2} \times 23$. So we deduce that $|G| \in \{264, 216, 168, 120, 72, 24\}$. It can be checked by GAP [7] that none of these groups satisfy the hypothesis. $\square$

Now we are ready to prove the main result.

Theorem 4.4. *Let G be an n-centralizer group such that $|G| \leq \frac{3n}{2}$. If G is 2-nilpotent, then $G \cong S_3, D_{10}$ or $S_3 \times S_3$.*

Proof. Suppose that $|G| = 2^r.m$ such that m is odd. If $r = 1$ or $r = 2$, then we have the result by Propositions 2.6 and 3.7 respectively. So assume that $r > 2$ and S is a Sylow 2-subgroup of G. Then $G = S \ltimes K$ by hypothesis and S is elementary abelian by Lemma 4.1(iii). Also there is a unique element $b \in S$ such that $C_G(b) = S$ by Lemma 4.1. According to Lemma 4.2, there exist $s_1 \in S$ and $y_1 \in K$ such that $C_G(s_1) = S\langle y_1\rangle$ where $o(y_1) = 3$ and so $\langle y_1\rangle \trianglelefteq C_G(s_1)$. Therefore there is $T_1 \leq S$ such that $|T_1| = \frac{|S|}{2}$ and $y_1 \in C_G(T_1)$. Now we consider two following cases:

Case(1): Suppose that there exists $s_2 \in S - T_1$ such that $C_G(s_2) = S\langle y_2\rangle$ and $o(y_2) = 3$. Then there is $T_2 \leq S$ such that $|T_2| = \frac{|S|}{2}$ and $y_2 \in C_G(T_2)$.

Subcase(1): Suppose that $\langle y_1\rangle = \langle y_2\rangle$. Then $y_1 \in C_G(\langle T_1, s_2\rangle)$ and since T_1 is maximal in S, we have $y_1 \in C_G(S)$ which is a contradiction.

Subcase(2): Suppose that $\langle y_1 \rangle \neq \langle y_2 \rangle$. Then we have $|K| \geq 9$ since K is abelian by Lemma 4.1(i).

For each $s \in T_1 \bigcap T_2 = R$ we have $|C_G(s)| \geq 9|S|$. Thus we have

$$\begin{aligned} |I(G)| &\leq 1 + \frac{|G|}{|C_G(b)|} + \sum_{1 \neq s \in R} \frac{|G|}{|C_G(s)|} + \sum_{s \in S-(R \cup \{b\})} \frac{|G|}{|C_G(s)|} \\ &\leq 1 + \frac{|G|}{|S|} + |\frac{|S|}{4} - 1| \times \frac{|G|}{9|S|} + (|S| - \frac{|S|}{4} - 1)\frac{|G|}{3|S|} \\ &= \frac{|G|}{|S|}(1 + \frac{|S|-4}{36} + \frac{3|S|-4}{12}) + 1 \\ &= \frac{|G|}{|S|}(\frac{20+10|S|}{36}) + 1 \end{aligned}$$

Since $\frac{|G|}{3} \leq |I(G)|$, we find that $|S| \leq 10 + \frac{18|S|}{|G|}$. It follows that $|S| = 8$, contrary to Lemma 4.3.

Case 2: For every $b, 1 \neq s_2 \in S - T_1$ we have $C_G(s_2) > 3|S|$.

Subcase(1): Suppose that there exists $s_2 \in S - T_1$ such that $C_G(s_2) = S\langle y_2 \rangle$ and $o(y_2) = 5 \, or \, 7$. Then by the similar way, there exists $T_2 \leq S$ such that $|T_2| = \frac{|S|}{2}$ and $y_1, y_2 \in C_G(T_1 \bigcap T_2)$. So $|C_G(T_1 \bigcap T_2)| \geq 15|S|$. Now if $R = T_1 \bigcap T_2$, then

$$\begin{aligned} |I(G)| &\leq 1 + \frac{|G|}{|C_G(b)|} + \sum_{1 \neq s \in R} \frac{|G|}{|C_G(s)|} + \sum_{s \in S-(R \cup \{b\})} \frac{|G|}{|C_G(s)|} \\ &\leq 1 + |\frac{G}{S}| + |\frac{|S|}{4} - 1| \times \frac{|G|}{15|S|} + (|S| - \frac{|S|}{4} - 1)\frac{|G|}{3|S|} \\ &= \frac{|G|}{|S|}(1 + \frac{|S|-4}{60} + \frac{3|S|-4}{12}) + 1 \\ &= \frac{|G|}{|S|}(\frac{9+4|S|}{15}) + 1. \end{aligned}$$

Since $\frac{|G|}{3} \leq |I(G)|$, we see that $|S| \leq 9 + \frac{15|S|}{|G|} \leq 9 + \frac{15}{9}$, a contradiction.

Subcase(2): Assume that $C_G(s_2) \geq 9|S|$ for every $b \neq s_2 \in S - T_1$. Then we have

$$\begin{aligned} |I(G)| &\leq 1 + \frac{|G|}{|C_G(b)|} + \sum_{1 \neq s \in T_1} \frac{|G|}{|C_G(s)|} + \sum_{s \in S-(T_1 \cup \{b\})} \frac{|G|}{|C_G(s)|} \\ &\leq 1 + \frac{|G|}{|S|} + |\frac{|S|}{2} - 1|\frac{|G|}{3|S|} + |\frac{|S|}{2} - 1|\frac{|G|}{9|S|} \\ &= \frac{|G|}{|S|}(\frac{4|S|+10}{18}) + 1. \end{aligned}$$

Since $\frac{|G|}{3} \leq |I(G)|$, we find that $|S| \leq 5 + \frac{9|S|}{|G|} \leq 5 + 1 = 6$. This is our final contradiction.

$\square$

Corollary 4.5. *Let G be an n-centralizer group such that $|G| \leq \frac{3n}{2}$. If a Sylow 2-subgroup S of G is abelian, then $G \cong S_3, D_{10}$ or $S_3 \times S_3$.*

Proof. It follows from Proposition 3.4 that $N_G(S) = S$. Since S is abelian, G is 2-nilpotent by Burnside's theorem and the result follows from Theorem 4.4. □

Acknowledgments

The authors would like to thank the referee for the careful reading and valuable comments.

References

[1] A. Abdollahi, S. M. Jafarian Amiri and A. Mohammadi Hassanabadi, Groups with specific number of centralizers, *Houston J. Math.* **33** (2007), no. 1, 43–57.
[2] A. R. Ashrafi, On finite groups with a given number of centralizers, *Algebra Colloq.* **7** (2000), no. 2, 139–146.
[3] S. J. Baishya, On finite groups with specific number of centralizers, *Int. Electron. J. Algebra* **13** (2013) 53–62.
[4] S. M. Belcastro and G. J. Sherman, Counting centralizers in finite groups, *Math. Mag.* **5** (1994), no. 5, 366–374.
[5] S. Dolfi, M. Herzog and E. Jabara, Finite groups whose noncentral commuting elements have centralizers of equal size, *Bull. Aust. Math. Soc.* **82** (2010), no. 2, 293–304.
[6] A. L. Edmonds, The partition problem for equifacetal simplices, *Beiträge Algebra Geom.* **50** (2009), no. 1, 195–213.
[7] The GAP Group, GAP-Groups, Algoritms, and Programming, version 4.4.10, 2007,(http://www.gap-system.org).
[8] D. Gorenstein, Finite Groups, Harper & Row, New York-London, 1968.
[9] I. M. Isaacs, Finite Group Theory, Grad. Stud. Math, vol. 92, Amer. Math. Soc., Providence, 2008.
[10] D. J. S. Robinson, A course in the theory of groups, Graduate Texts in Mathematics, Springer-Verlag New York, 1996.
[11] M. Zarrin, On solubility of groups with finitely many centralizers, *Bull. Iranian Math. Soc.* **39** (2013), no. 3, 517–521.

(Seyyed Majid Jafarian Amiri) Department of Mathematics, Faculty of Sciences, University of Zanjan, P.O. Box 45371-38791, Zanjan, Iran
E-mail address: sm_jafarian@znu.ac.ir

(Mohsen Amiri) Department of Mathematics, Faculty of Sciences, University of Zanjan, P.O. Box 45371-38791, Zanjan, Iran
E-mail address: m.amiri@znu.ac.ir

(Halimeh Madadi) Department of Mathematics, Faculty of Sciences, University of Zanjan, P.O. Box 45371-38791, Zanjan, Iran
E-mail address: halime_madadi@yahoo.com

(Hojjat Rostami) Department of Mathematics, Faculty of Sciences, University of Zanjan, P.O. Box 45371-38791, Zanjan, Iran
E-mail address: h.rostami5991@gmail.com