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ON THE EXISTENCE OF SOLUTION FOR A k-DIMENSIONAL SYSTEM OF THREE POINTS NABLA FRACTIONAL FINITE DIFFERENCE EQUATIONS

SH. REZAPOUR* AND S. SALEHI

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Abstract. In this paper, we investigate the existence of solution for a k-dimensional system of three points nabla fractional finite difference equations. Also, we present an example to illustrate our result.
Keywords: Fixed point, fractional finite difference equation, k-dimensional system.
MSC(2010): Primary: 39A12; Secondary: 34A25, 26A33.

1. Introduction

There are a lot of works on discrete fractional calculus for special equations via distinct boundary conditions (see for example, [5–8, 13–19, 21] and [22]). Also, there are some published papers on the nabla operator and fractional finite difference inclusion (see for example, [1, 3, 4, 9–11] and [20]). In this paper, we investigate the existence of solutions for the k-dimensional system of nabla fractional finite difference equations

$$\begin{cases} \nabla^{\mu_1}_{\mu_1-3}x_1(t) + h_1\big(x_1(t), x_2(t), \cdots, x_k(t)\big) = 0, \\ \nabla^{\mu_2}_{\mu_2-3}x_2(t) + h_2\big(x_1(t), x_2(t), \cdots, x_k(t)\big) = 0, \\ \vdots \\ \nabla^{\mu_k}_{\mu_k-3}x_k(t) + h_k\big(x_1(t), x_2(t), \cdots, x_k(t)\big) = 0, \quad (1.1) \end{cases}$$

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via the boundary conditions

$$\begin{cases} x_1(-3) = x_1(b) = x_1(\beta) = 0, \\ x_2(-3) = x_2(b) = x_2(\beta) = 0, \\ \vdots \\ x_k(-3) = x_k(b) = x_k(\beta) = 0, \ (1.2) \end{cases}$$

where $b \in \mathbb{N}_0$, $2 < \mu_i \leq 3$, $\beta \in \mathbb{N}_1^{b-1}$ and the mappings $h_i : \mathbb{R}^k \to \mathbb{R}$ are continuous functions for all $i = 1, 2, \cdots, k$. We show that the problem (1.1) via the conditions (1.2) is equivalent to a summation equation and by using the Kranoselskii's fixed point theorem, we investigate solutions of the problem. We present an example to illustrate our result.

2. Preliminaries

The nabla operator ∇ on a function f acts by $\nabla f(t) = f(t) - f(t-1)$. Some known information on the nabla operator to facilitate the analysis of results can be found in [1,3,4,10,11] and [20]. Now, define $t^{\overline{\nu}} := \frac{\Gamma(t+\nu)}{\Gamma(t)}$ for all $t, \nu \in \mathbb{R}$ whenever the right-hand side is defined ([4]). We define $a^{\overline{\nu}} = 0$ whenever a is a non-positive integer and ν is not an integer. We can simply conclude that $t^{\overline{\nu}} = (t + \nu - 1)^{\underline{\nu}}$. Also, it is easy to check that $\nabla_t (t-a)^{\overline{\nu}} = \nu (t-a)^{\overline{\nu - 1}}$ and

$$\nabla_t (a-t)^{\overline{\nu}} = -\nu (a - \rho(t))^{\overline{\nu-1}}.$$

Similar to other works, we use the notations $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$ for all $a \in \mathbb{R}$ and $\mathbb{N}_a^b = \{a, a+1, a+2, \dots, b\}$ for all real numbers a and b whenever $b - a$ is a natural number.

Now, suppose that $\nu > 0$ with $m - 1 < \nu \leq m$ for some natural number m. Then the ν-th nabla fractional sum of f based at a is defined by

$$\nabla_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{k=a+1}^{t} (t - \rho(k))^{\overline{\nu-1}} f(k)$$

for all $t \in \mathbb{N}_a$ ([4]). Similarly, one can define

$$\nabla_a^{\nu} f(t) = \frac{1}{\Gamma(-\nu)} \sum_{k=a+1}^{t} (t - \rho(k))^{\overline{-\nu-1}} f(k)$$

for all $t \in \mathbb{N}_{a+m}$. In 2009, Atici and Eloe proved next result about relations of the operators delta and nabla ([4]).

Lemma 2.1. *Let $a \in \mathbb{R}$, m a natural number, $0 \leq m-1 < \nu \leq m$ and y a map on $\mathbb{N}_a$. Then, $\Delta_a^{\nu} y(t-\nu) = \nabla_a^{\nu} y(t)$ for all $t \in \mathbb{N}_{m+a}$ and $\Delta_a^{-\nu} y(t+\nu) = \nabla_a^{-\nu} y(t)$ for all $t \in \mathbb{N}_a$.*

Let $a \in \mathbb{R}$, m a natural number, $0 \leq m-1 < \nu \leq m$ and $h : \mathbb{N}_a \to \mathbb{R}$ a mapping. It is known that the general solution for the fractional finite difference equation $\Delta_a^\nu y(t) = h(t)$ is given by

$$y(t) = \sum_{i=1}^{m} c_i (t-a)^{\underline{\nu-i}} + \Delta_a^{-\nu} h(t)$$

$$= \sum_{i=1}^{m} c_i (t-a)^{\underline{\nu-i}} + \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} h(s)$$

for all $t \in \mathbb{N}_a$, where $c_1, \cdots, c_m$ are arbitrary constants. Here, we provide a similar result by using the nabla operator which can be proved by using Lemma 2.1.

Lemma 2.2. *Let $a \in \mathbb{R}$, m a natural number, $m-1 < \nu \leq m$ and $h : \mathbb{N}_0 \to \mathbb{R}$ a mapping. Then the general solution for the equation $\nabla_{\nu-m}^{\nu} y(t) = h(t)$ is given by*

$$y(t) = \sum_{i=1}^{m} c_i (t+i+1)^{\overline{\nu-i}} + \nabla^{-\nu} h(t)$$

$$= \sum_{i=1}^{m} c_i (t+i+1)^{\overline{\nu-i}} + \frac{1}{\Gamma(\nu)} \sum_{s=0}^{t} (t-\rho(s))^{\overline{\nu-1}} h(s),$$

for all $t \in \mathbb{N}_{-m}$, where $\rho(s) = s-1$ and $c_1, \cdots, c_m$ are arbitrary constants.

Let $P \neq \{0\}$ be a non-empty closed subset of a topological vector space E. Then P is called a cone whenever $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b and $P \cap (-P) = \{0\}$ (for more details and examples see [23] and references therein). One can find next result in [2].

Lemma 2.3. *Let X be a Banach space K a cone in X. Assume that Ω_1 and Ω_2 are open subsets of X such that $0 \in \Omega_1$ and $\overline{\Omega_1} \subseteq \Omega_2$. Suppose that $T : K \cap (\overline{\Omega_2} \backslash \Omega_1) \to K$ is a completely continuous operator.*
If either $\|Ty\| \leq \|y\|$ for all $y \in K \cap \partial\Omega_1$ and $\|Ty\| \geq \|y\|$ for all $y \in K \cap \partial\Omega_2$ or
$\|Ty\| \geq \|y\|$ for all $y \in K \cap \partial\Omega_1$ and $\|Ty\| \leq \|y\|$ for all $y \in K \cap \partial\Omega_2$,
then T has at least one fixed point in $K \cap (\overline{\Omega_2} \backslash \Omega_1)$.

3. Main Result

First, we investigate the problem (1.1).

Lemma 3.1. *Let $b \in \mathbb{N}_0$, $2 < \mu_i \leq 3$, $\beta \in \mathbb{N}_1^{b-1}$ and let the mappings $h_i : \mathbb{R}^k \to \mathbb{R}$ be continuous functions for all $i = 1, 2, \cdots, k$. Then, for the mappings $x_i : \mathbb{N}_{-3}^{b} \to \mathbb{R}$, the nabla fractional finite difference equation*

$$\nabla_{\mu_i-3}^{\mu_i} x_i(t) + h_i\big(x_1(t), x_2(t), \cdots, x_k(t)\big) = 0 \tag{3.1}$$

via the boundary conditions $x_i(-3)=0$, $x_i(b)=0$ and $x_i(\beta)=0$ has a solution x_i^* if and only if x_i^* is a solution of the summation equation

$$x_i(t)=\sum_{s=0}^{b} G_i(t,s,\beta)h_i\big(x_1(t),x_2(t),\cdots,x_k(t)\big),$$

where

$$G_i(t,s,\beta)=\frac{(t+2)^{\overline{\mu_i-1}}-(\beta+2)(t+3)^{\overline{\mu_i-2}}}{(b-\beta)(b+3)^{\overline{\mu_i-2}}\Gamma(\mu_i)}(b-\rho(s))^{\overline{\mu_i-1}}$$

$$-\frac{(t+2)^{\overline{\mu_i-1}}-(b+2)(t+3)^{\overline{\mu_i-2}}}{(b-\beta)(\beta+3)^{\overline{\mu_i-2}}\Gamma(\mu_i)}(\beta-\rho(s))^{\overline{\mu_i-1}}-\frac{1}{\Gamma(\mu_i)}(t-\rho(s))^{\overline{\mu_i-1}}$$

whenever $0\le s\le\beta\le b$ and $0\le s\le t\le b$,

$$G_i(t,s,\beta)=\frac{(t+2)^{\overline{\mu_i-1}}-(\beta+2)(t+3)^{\overline{\mu_i-2}}}{(b-\beta)(b+3)^{\overline{\mu_i-2}}\Gamma(\mu_i)}(b-\rho(s))^{\overline{\mu_i-1}}$$

$$-\frac{(t+2)^{\overline{\mu_i-1}}-(b+2)(t+3)^{\overline{\mu_i-2}}}{(b-\beta)(\beta+3)^{\overline{\mu_i-2}}\Gamma(\mu_i)}(\beta-\rho(s))^{\overline{\mu_i-1}}$$

whenever $0\le t<s\le\beta\le b$,

$$G_i(t,s,\beta)=\frac{(t+2)^{\overline{\mu_i-1}}-(\beta+2)(t+3)^{\overline{\mu_i-2}}}{(b-\beta)(b+3)^{\overline{\mu_i-2}}\Gamma(\mu_i)}(b-\rho(s))^{\overline{\mu_i-1}}$$

$$-\frac{1}{\Gamma(\mu_i)}(t-\rho(s))^{\overline{\mu_i-1}}$$

whenever $0\le\beta<s\le t\le b$ and

$$G_i(t,s,\beta)=\frac{(t+2)^{\overline{\mu_i-1}}-(\beta+2)(t+3)^{\overline{\mu_i-2}}}{(b-\beta)(b+3)^{\overline{\mu_i-2}}\Gamma(\mu_i)}(b-\rho(s))^{\overline{\mu_i-1}}$$

whenever $0\le\beta<s\le b$ and $0\le t<s\le b$ for all $s\in\mathbb{N}_0^b$. Here, $i\in\{1,2,\cdots,k\}$.

Proof. Let $i\in\{1,2,\cdots,k\}$, $h_i(t):=h_i\big(x_1(t),x_2(t),\cdots,x_k(t)\big)$ and x_i^* be a solution of the nabla fractional finite difference equation

$$\nabla^{\mu_i}_{\mu_i-3}x_i(t)+h_i\big(x_1(t),x_2(t),\cdots,x_k(t)\big)=0.$$

By using Lemma 2.2, we get

$$x_i^*(t)=c_1(t+2)^{\overline{\mu_i-1}}+c_2(t+3)^{\overline{\mu_i-2}}+c_3(t+4)^{\overline{\mu_i-3}}$$

$$-\frac{1}{\Gamma(\mu_i)}\sum_{s=0}^{t}(t-\rho(s))^{\overline{\mu_i-1}}h_i(s).$$

By using the boundary condition $x_i^*(-3) = 0$, we obtain

$$0 = c_1(-1)^{\overline{\mu_i - 1}} + c_2(0)^{\overline{\mu_i - 2}} + c_3(1)^{\overline{\mu_i - 3}} - \frac{1}{\Gamma(\mu_i)} \sum_{s=0}^{-3} (-3 - \rho(s))^{\overline{\mu_i - 1}} h_i(s)$$

and so $c_3 = 0$. Now by using the boundary condition $x_i^*(b) = 0$, we get

$$0 = c_1(b+2)^{\overline{\mu_i - 1}} + c_2(b+3)^{\overline{\mu_i - 2}} - \frac{1}{\Gamma(\mu_i)} \sum_{s=0}^{b} (b - \rho(s))^{\overline{\mu_i - 1}} h_i(s).$$

Since $(b+2)^{\overline{\mu_i - 1}} = (b+2)(b+3)^{\overline{\mu_i - 2}}$, we have

$$0 = c_1(b+2) + c_2 - \frac{1}{(b+3)^{\overline{\mu_i - 2}}\Gamma(\mu_i)} \sum_{s=0}^{b} (b - \rho(s))^{\overline{\mu_i - 1}} h_i(s).$$

Similarly, by using the boundary condition $x_i^*(\beta) = 0$, we get

$$0 = c_1(\beta+2) + c_2 - \frac{1}{(\beta+3)^{\overline{\mu_i - 2}}\Gamma(\mu_i)} \sum_{s=0}^{\beta} (\beta - \rho(s))^{\overline{\mu_i - 1}} h_i(s)$$

and so

$$\begin{aligned} c_1 = & \frac{1}{(b-\beta)(b+3)^{\overline{\mu_i - 2}}\Gamma(\mu_i)} \sum_{s=0}^{b} (b - \rho(s))^{\overline{\mu_i - 1}} h_i(s) \\ & - \frac{1}{(b-\beta)(\beta+3)^{\overline{\mu_i - 2}}\Gamma(\mu_i)} \sum_{s=0}^{\beta} (\beta - \rho(s))^{\overline{\mu_i - 1}} h_i(s), \end{aligned}$$

and

$$\begin{aligned} c_2 = & \frac{-(\beta+2)}{(b-\beta)(b+3)^{\overline{\mu_i - 2}}\Gamma(\mu_i)} \sum_{s=0}^{b} (b - \rho(s))^{\overline{\mu_i - 1}} h_i(s) \\ & + \frac{(b+2)}{(b-\beta)(\beta+3)^{\overline{\mu_i - 2}}\Gamma(\mu_i)} \sum_{s=0}^{\beta} (\beta - \rho(s))^{\overline{\mu_i - 1}} h_i(s). \end{aligned}$$

Hence

$$\begin{aligned} x_i^*(t) = & \frac{(t+2)^{\overline{\mu_i - 1}} - (\beta+2)(t+3)^{\overline{\mu_i - 2}}}{(b-\beta)(b+3)^{\overline{\mu_i - 2}}\Gamma(\mu_i)} \sum_{s=0}^{b} (b - \rho(s))^{\overline{\mu_i - 1}} h_i(s) \\ & - \frac{(t+2)^{\overline{\mu_i - 1}} - (b+2)(t+3)^{\overline{\mu_i - 2}}}{(b-\beta)(\beta+3)^{\overline{\mu_i - 2}}\Gamma(\mu_i)} \sum_{s=0}^{\beta} (\beta - \rho(s))^{\overline{\mu_i - 1}} h_i(s) \\ & - \frac{1}{\Gamma(\mu_i)} \sum_{s=0}^{t} (t - \rho(s))^{\overline{\mu_i - 1}} h_i(s) \\ = & \sum_{s=0}^{b} G_i(t, s, \beta) h_i\big(x_1(s), x_2(s), \cdots, x_k(s)\big). \end{aligned}$$

Now, let x_i^* be a solution of the fractional sum equation

$$x_i(t) = \sum_{s=0}^{b} G_i(t,s,\beta) h_i\big(x_1(s), x_2(s), \cdots, x_k(s)\big).$$

Then, we have

$$x_i^*(t) = \frac{(t+2)^{\overline{\mu_i-1}} - (\beta+2)(t+3)^{\overline{\mu_i-2}}}{(b-\beta)(b+3)^{\overline{\mu_i-2}}\Gamma(\mu_i)} \sum_{s=0}^{b} (b-\rho(s))^{\overline{\mu_i-1}} h_i(s)$$

$$- \frac{(t+2)^{\overline{\mu_i-1}} - (b+2)(t+3)^{\overline{\mu_i-2}}}{(b-\beta)(\beta+3)^{\overline{\mu_i-2}}\Gamma(\mu_i)} \sum_{s=0}^{\beta} (\beta-\rho(s))^{\overline{\mu_i-1}} h_i(s)$$

$$- \frac{1}{\Gamma(\mu_i)} \sum_{s=0}^{t} (t-\rho(s))^{\overline{\mu_i-1}} h_i(s) = c_1(t+2)^{\overline{\mu_i-1}} + c_2(t+3)^{\overline{\mu_i-2}}$$

$$- \frac{1}{\Gamma(\mu_i)} \sum_{s=0}^{t} (t-\rho(s))^{\overline{\mu_i-1}} h_i(s).$$

Since $(-1)^{\overline{\mu_i-1}} = 0$ and $0^{\overline{\mu_i-2}} = 0$, we get $x_i^*(-3) = 0$. Similarly, we obtain $x_i^*(b) = 0$ and $x_i^*(\beta) = 0$. On the other hand, one can check that $x_i^*(t)$ is a solution for the equation (3.1). This completes the proof. □

By using a similar proof of Proposition 2.2.2 in [7], one can check that the Green function in the last result satisfies $G_i(t,s,\beta) \leq 0$ for all $s,t \in \mathbb{N}_0^b$. Let $M_i \subseteq \mathbb{N}_0^b$ be such that $G_i(t,s,\beta) \neq 0$ for all $t \in M_i$. Since the Green function is bounded, there exist $\lambda_i \in (0,1)$ such that

$$\min_{t \in M_i} |G_i(t,s,\beta)| \geq \lambda_i \max_{t \in M_i} |G_i(t,s,\beta)|$$

for all $s \in \mathbb{N}_0^b$. Now, suppose that $\mathcal{A}_i$ is the Banach space of the maps $u : \mathbb{N}_{-3}^b \to \mathbb{R}$ via the usual norm $\|u\| = \max\{|u(t)| : t \in \mathbb{N}_{-3}^b\}$. Consider the space $\mathcal{X} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_k$ via the norm $\|(x_1, x_2, \cdots, x_k)\|_{\mathcal{X}} = \|x_1\| + \|x_2\| + \cdots + \|x_k\|$. Its clear that, $(\mathcal{X}, \|.\|_{\mathcal{X}})$ is a Banach space. Define the map $T : \mathcal{X} \to \mathcal{X}$ by

$$T(x_1, x_2, \ldots, x_k) \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_k \end{pmatrix} = \begin{pmatrix} T_1(x_1, x_2, \ldots, x_k)(t_1) \\ T_2(x_1, x_2, \ldots, x_k)(t_2) \\ \vdots \\ T_k(x_1, x_2, \ldots, x_k)(t_k) \end{pmatrix}, \tag{3.2}$$

where

$$T_i(x_1, x_2, \ldots, x_k)(t) = \sum_{s=0}^{b} |G_i(t,s,\beta)| h_i\big(x_1(s), x_2(s), \cdots, x_k(s)\big)$$

for $i = 1, 2, \cdots, k$. Also, consider the cone

$$\mathcal{K} = \Big\{ (x_1, x_2, \cdots, x_k) \in \mathcal{X} : x_i \geq 0, \min_{(t_1, t_2, \cdots, t_k) \in M_1 \times M_2 \times \cdots \times M_k} [x_1(t_1) + x_2(t_2) + \cdots + x_k(t_k)] \geq \lambda \|(x_1, x_2, \cdots, x_k)\|_{\mathcal{X}} \Big\},$$

where $\lambda = \min_{1 \leq i \leq k} \lambda_i$. First, we show that $T(\mathcal{K}) \subseteq \mathcal{K}$ whenever the functions h_i are non-negative for $i = 1, 2, \cdots, k$. Let $(x_1, x_2, \ldots, x_k) \in \mathcal{K}$. Then, we have

$$\begin{aligned}
&\min_{(t_1, t_2, \cdots, t_k) \in M_1 \times M_2 \times \cdots \times M_k} \sum_{n=1}^{k} T_n(x_1, x_2, \cdots, x_k)(t_n) \\
&\geq \sum_{n=1}^{k} \min_{t_n \in M_n} T_n(x_1, x_2, \cdots, x_k)(t_n) \\
&= \sum_{n=1}^{k} \min_{t_n \in M_n} \sum_{s=0}^{b} |G_n(t_n, s, \beta)| h_n(x_1(s), x_2(s), \cdots, x_k(s)) \\
&\geq \sum_{n=1}^{k} \lambda_n \max_{t_n \in M_n} \sum_{s=0}^{b} |G_n(t_n, s, \beta)| h_n(x_1(s), x_2(s), \cdots, x_k(s)) \\
&= \sum_{n=1}^{k} \lambda_n \|T_n(x_1, x_2, \cdots, x_k)\| \geq \lambda \sum_{n=1}^{k} \|T_n(x_1, x_2, \cdots, x_k)\| \\
&= \lambda \|T(x_1, x_2, \cdots, x_k)\|_{\mathcal{X}},
\end{aligned}$$

where $\lambda = \min_{1 \leq n \leq k} \lambda_n$. Hence, $T(x_1, x_2, \cdots, x_k) \in \mathcal{K}$ and so $T(\mathcal{K}) \subseteq \mathcal{K}$. Now, we are ready to present our main result.

Theorem 3.2. *Suppose that $h_1, \cdots, h_k \in C([0, \infty)^k)$ and there exists $0 < \epsilon < \min\{B_i : 1 \leq i \leq k\}$ such that*

$$\sum_{s=0}^{b} \max_{t \in M_i} |G_i(t, s, \beta)|(A_i + \epsilon) \leq \frac{1}{k} \text{ and } \sum_{s=0}^{b} \lambda \max_{t \in M_i} |G_i(t, s, \beta)|(B_i - \epsilon) \geq \frac{1}{k}$$

for all $i \in \{1, 2, \cdots, k\}$, where G_i is the related Green function for the equation (3.1), $\lambda = \min_{1 \leq i \leq k} \lambda_i$,

$$\lim_{(x_1, x_2, \cdots, x_k) \to (0^+, 0^+, \cdots, 0^+)} \frac{h_i(x_1, x_2, \cdots, x_k)}{x_1 + x_2 + \cdots + x_k} = A_i$$

and

$$\lim_{(x_1, x_2, \cdots, x_k) \to (+\infty, +\infty, \cdots, +\infty)} \frac{h_i(x_1, x_2, \cdots, x_k)}{x_1 + x_2 + \cdots + x_k} = B_i$$

for all $i \in \{1, 2, \cdots, k\}$. Then the k-dimensional system of nabla fractional finite difference equations (1.1) has at least one solution.

Proof. Consider the operator $T:\mathcal{K}\to\mathcal{K}$ defined by (3.2) and the cone $\mathcal{K}$. It is clear that T is completely continuous because it is a summation operator on a finite set. Choose $\delta_1>0$ such that

$$h_i(x_1,x_2,\cdots,x_k)\le(A_i+\epsilon)(x_1+x_2+\cdots+x_k)$$

for all $(x_1,x_2,\cdots,x_k)\in\mathcal{X}$ with $\|(x_1,x_2,\cdots,x_k)\|_{\mathcal{X}}<\delta_1$. Now, put $\Omega_1=\big\{(x_1,x_2,\cdots,x_k)\in\mathcal{X}:\|(x_1,x_2,\cdots,x_k)\|_{\mathcal{X}}<\delta_1\big\}$. Then, $0\in\Omega_1$ and $\|(x_1,x_2,\cdots,x_k)\|_{\mathcal{X}}=\delta_1$ for all $(x_1,x_2,\cdots,x_k)\in\mathcal{K}\cap\partial\Omega_1$. Also, we have

$$\|T_i(x_1,x_2,\cdots,x_k)\|=\max_{t_i\in M_i}\sum_{s=0}^{b}|G_i(t_i,s,\beta)|h_i\big(x_1(s),x_2(s),\cdots,x_k(s)\big)$$

$$\le\sum_{s=0}^{b}\max_{t_i\in M_i}|G_i(t_i,s,\beta)|(A_i+\epsilon)(x_1+x_2+\cdots+x_k)$$

$$\le\|(x_1,x_2,\cdots,x_k)\|_{\mathcal{X}}\sum_{s=0}^{b}\max_{t_i\in M_i}|G_i(t_i,s,\beta)|(A_i+\epsilon)$$

$$\le\frac{1}{k}\|(x_1,x_2,\cdots,x_k)\|_{\mathcal{X}}$$

for all $(x_1,x_2,\cdots,x_k)\in\mathcal{K}\cap\partial\Omega_1$. Hence,

$$\|T(x_1,x_2,\cdots,x_k)\|_{\mathcal{X}}=\sum_{i=1}^{k}\|T_i(x_1,x_2,\cdots,x_k)\|$$

$$\le k\times\frac{1}{k}\|(x_1,x_2,\cdots,x_k)\|_{\mathcal{X}}=\|(x_1,x_2,\cdots,x_k)\|_{\mathcal{X}}$$

for all $(x_1,x_2,\cdots,x_k)\in\mathcal{K}\cap\partial\Omega_1$. Now, choose $\beta>\delta_1$ such that

$$h_i(x_1,x_2,\cdots,x_k)\ge(B_i-\epsilon)(x_1+x_2+\cdots+x_k)\tag{3.3}$$

for all $(x_1,x_2,\cdots,x_k)\in\mathcal{X}$ with $\|(x_1,x_2,\cdots,x_k)\|_{\mathcal{X}}\ge\beta$. Since $\beta\ge1$, $\sum_{s=0}^{b}\lambda\max_{t_i\in M_i}|G_i(t_i,s,\beta)|(B_i-\epsilon)\ge\frac{1}{k}$ implies

$$\beta\lambda\sum_{s=0}^{b}\max_{t_i\in M_i}|G_i(t_i,s,\beta)|(B_i-\epsilon)\ge\beta\frac{1}{k}>0$$

for all $i=1,\cdots,k$. Thus, we can choose $\delta_2>0$ such that

$$\frac{1}{k}\beta\le\delta_2\le\lambda\beta\min_{1\le i\le k}\sum_{s=0}^{b}\max_{t_i\in M_i}|G_i(t_i,s,\beta)|(B_i-\epsilon).$$

Now, put $\Omega_2=\big\{(x_1,x_2,\cdots,x_k)\in\mathcal{X}:\|(x_1,x_2,\cdots,x_k)\|_{\mathcal{X}}<k\delta_2\big\}$. Then, $\overline{\Omega_1}\subseteq\Omega_2$ and

$$x_1(t_1)+x_2(t_2)+\cdots+x_k(t_k)$$
$$\ge\min_{(t_1,t_2,\cdots,t_k)\in M_1\times M_2\times\cdots\times M_k}[x_1(t_1)+x_2(t_2)+\cdots+x_k(t_k)]$$

$$\geq \lambda \|(x_1, x_2, \cdots, x_k)\|_{\mathcal{X}}$$

for all $(x_1, x_2, \cdots, x_k) \in \mathcal{K} \cap \partial\Omega_2$. Thus by using (3.3), we get

$$\|T_i(x_1, x_2, \cdots, x_k)\| = \max_{t_i \in M_i} \sum_{s=0}^{b} |G_i(t_i, s, \beta)| h_i\big(x_1(s), x_2(s), \cdots, x_k(s)\big)$$

$$\geq \sum_{s=0}^{b} \max_{t_i \in M_i} |G_i(t_i, s, \beta)|(B_i - \epsilon)(x_1 + x_2 + \cdots + x_k)$$

$$\geq \lambda \|(x_1, x_2, \cdots, x_k)\|_{\mathcal{X}} \sum_{s=0}^{b} \max_{t_i \in M_i} |G_i(t_i, s, \beta)|(B_i - \epsilon)$$

$$\geq \frac{1}{k} \|(x_1, x_2, \cdots, x_k)\|_{\mathcal{X}}$$

for all $(x_1, x_2, \cdots, x_k) \in \mathcal{K} \cap \partial\Omega_2$. Hence,

$$\|T(x_1, x_2, \cdots, x_k)\|_{\mathcal{X}} = \sum_{i=1}^{k} \|T_i(x_1, x_2, \cdots, x_k)\|$$

$$\geq k \times \frac{1}{k} \|(x_1, x_2, \cdots, x_k)\|_{\mathcal{X}} = \|(x_1, x_2, \cdots, x_k)\|_{\mathcal{X}}$$

for all $(x_1, x_2, \cdots, x_k) \in \mathcal{K} \cap \partial\Omega_2$. Therefore by using Lemma 2.3, T has at least one fixed point $(x_1^*, x_2^*, \cdots, x_k^*)$ in $\mathcal{K} \cap (\overline{\Omega_2} \backslash \Omega_1)$. Hence by using Lemma 3.1, the k-dimensional system of nabla fractional finite difference equations (1.1) has at least one solution. □

Example 3.3. *Consider the 2-dimensional nabla fractional finite difference equation system*

$$\begin{cases} \nabla_{-0.5}^{2.5} x_1(t) + 200 e^{\frac{-10}{x_1(t)+x_2(t)+1}} (x_1(t) + x_2(t)) = 0, \\ \nabla_{-0.9}^{2.1} x_2(t) = \begin{cases} -180(x_1(t) + x_2(t)) e^{\frac{-8 \sin x_2(t)}{x_2(t)}} & x_2(t) > 0, \\ -180(x_1(t) + x_2(t)) e^{-8} & x_2(t) = 0, \end{cases} \end{cases} \quad (3.4)$$

via the boundary conditions $x_1(-3) = x_1(3) = x_1(4) = 0$ *and* $x_2(-3) = x_2(3) = x_2(4) = 0$. *We show that the problem has at least one solution. Let* $\mu_1 = 2.5$, $\mu_2 = 2.1$, $b = 4$, $\beta = 3$, $k = 2$, $h_1(x_1, x_2) = 200 e^{\frac{-10}{x_1+x_2+1}} (x_1 + x_2)$ *and*

$$h_2(x_1, x_2) = (x_1 + x_2) \begin{cases} 180 e^{\frac{-8 \sin x_2}{x_2}} & x_2 > 0, \\ 180 e^{-8} & x_2 = 0. \end{cases}$$

Thus, the system (3.4) is a special case of the system (1.1). Its easy to check that $h_i \in C([0, \infty)^2)$ *for* $i = 1, 2$. *Now by some calculation, we give values of the Green function* G_1 *in next table.*

t	*0*	*1*	*2*	*3*	*4*
$G_1(t,0,3)$	*-0.1258*	*-0.0524*	*-0.0152*	*0*	*0*
$G_1(t,1,3)$	*-0.4662*	*-0.1841*	*-0.0524*	*0*	*0*
$G_1(t,2,3)$	*-1.4652*	*-0.4662*	*-0.1258*	*0*	*0*
$G_1(t,3,3)$	*-2.0246*	*-1.3053*	*-0.2797*	*0*	*0*
$G_1(t,4,3)$	*-1.9180*	*-1.4918*	*-0.8391*	*0*	*0*

Table 3.1: Values of the Green function G_1 *for* $\mu_1 = 2.5$

One can check that $M_1 = \mathbb{N}_0^2$, $\min_{t\in M_1} |G_1(t,s,3)| = 0.0152$ *and*

$$\max_{t\in M_1} |G_1(t,s,3)| = 2.0246$$

for all $s \in \mathbb{N}_0^4$ *and so* $\lambda_1 = 0.0075$. *Similarly by some calculation, we give values of the Green function* G_2 *in next table.*

t	*0*	*1*	*2*	*3*	*4*
$G_2(t,0,3)$	*-0.0212*	*-0.0074*	*-0.0019*	*0*	*0*
$G_2(t,1,3)$	*-0.0987*	*-0.0302*	*-0.0074*	*0*	*0*
$G_2(t,2,3)$	*-1.1153*	*-0.0987*	*-0.0212*	*0*	*0*
$G_2(t,3,3)$	*-2.0333*	*-1.0819*	*-0.0642*	*0*	*0*
$G_2(t,4,3)$	*-2.7313*	*-1.8816*	*-0.9643*	*0*	*0*

Table 3.2: Values of the Green function G_2 *for* $\mu_2 = 2.1$

It is easy to see that $M_2 = \mathbb{N}_0^2$, $\min_{t\in M_2} |G_2(t,s,3)| = 0.0019$ *and* $\max_{t\in M_2} |G_2(t,s,3)| = 2.7313$ *for all* $s \in \mathbb{N}_0^4$. *Hence,* $\lambda_2 = 0.0007$ *and* $\lambda = \min\{\lambda_1, \lambda_2\} = 0.0007$. *On the other hand by calculation of some limits, one can get that* $A_1 = 200e^{-10}$, $B_1 = 200$, $A_2 = 180e^{-8}$ *and* $B_2 = 180$. *Moreover, we have*

$$\sum_{s=0}^{b} \max_{t\in M_1} |G_1(t,s,3)| = \sum_{s=0}^{4} \max_{t\in \mathbb{N}_0^2} |G_1(t,s,3)| = 5.9998$$

and $\sum_{s=0}^{b} \max_{t\in M_2} |G_2(t,s,3)| = 5.9998$. *Put* $\epsilon = 0.0001$. *Thus, we have* $0 < \epsilon < \min\{B_1, B_2\}$,

$$\sum_{s=0}^{b} \max_{t\in M_1} |G_1(t,s,3)|(A_1+\epsilon) = 5.9998(200e^{-10} + 0.0001) \le \frac{1}{2},$$

$$\sum_{s=0}^{b} \lambda \max_{t\in M_1} |G_1(t,s,\beta)|(B_1-\epsilon) = 0.0007 \times 5.9998(200 - 0.0001) \ge \frac{1}{2},$$

$$\sum_{s=0}^{b} \max_{t\in M_2} |G_2(t,s,3)|(A_2+\epsilon) = 5.9998(180e^{-8} + 0.0001) \le \frac{1}{2},$$

and

$$\sum_{s=0}^{b} \lambda \max_{t\in M_2} |G_2(t,s,\beta)|(B_2-\epsilon) = 0.0007 \times 5.9998(180 - 0.0001) \ge \frac{1}{2}.$$

Now by using Theorem 3.2, the 2-dimensional system of nabla fractional finite difference equations (3.4) has at least one solution.

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