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Author(s):

L. X. Peng and C. Yang

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# A NOTE ON VOLTERRA AND BAIRE SPACES

L. X. PENG* AND C. YANG

(Communicated by Fariborz Azarpanah)

Abstract. In Proposition 2.6 in (G. Gruenhage, A. Lutzer, Baire and Volterra spaces, *Proc. Amer. Math. Soc.* 128 (2000), no. 10, 3115–3124) a condition that every point of $D$ is $G_\delta$ in $X$ was overlooked. So we proved some conditions by which a Baire space is equivalent to a Volterra space. In this note we show that if $X$ is a monotonically normal $T_1$-space with countable pseudocharacter and $X$ has a $\sigma$-discrete dense subspace $D$, then $X$ is a Baire space if and only if $X$ is Volterra. We show that if $X$ is a metacompact normal sequential $T_1$-space and $X$ has a $\sigma$-closed discrete dense subset, then $X$ is a Baire space if and only if $X$ is Volterra. If $X$ is a generalized ordered (GO) space and has a $\sigma$-closed discrete dense subset, then $X$ is a Baire space if and only if $X$ is Volterra. And also some known results are generalized.

**Keywords:** Volterra space, Baire space, monotonically normal, metacompact.
**MSC(2010):** Primary: 54D20; Secondary: 54G99.

## 1. **Introduction**

Recall that a topological space $X$ is a *Baire space* if the intersection of any sequence of dense open subsets of $X$ is dense. It follows immediately that the intersection of countably many dense $G_\delta$-subsets of a Baire space $X$ must be dense in $X$. It is well known that any Čech-complete space is a Baire space. A weaker condition is that the intersection of any two dense $G_\delta$-sets of $X$ must be dense in $X$, and that is the definition of a *Volterra space* [5]. Obviously, any Baire space is Volterra. In [4], Gruenhage and Lutzer described broad classes of spaces for which the Baire space property is equivalent to Volterra. In [1] Cao and Junnila showed that every regular stratifiable Volterra space is Baire.

In Proposition 2.6 in [4], the authors did not notice that it was necessary to require that points of $D$ have countable pseudocharacter in $X$, i.e., points of $D$

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*Corresponding author.

are $G_\delta$ in $X$. If we add this requirement, then Proposition 2.6 is true and also follows from Theorem 2.12 in this note. However, parts of [4, Corollary 2.8], do not follow as claimed. In particular, one must also add that points of $D$ are $G_\delta$ in Corollary 2.8(d), which states that Volterra implies Baire in the class of regular metacompact sequential spaces with a $\sigma$-closed discrete dense subset $D$. In this paper we show that 2.8(d) also holds if instead of assuming points are $G_\delta$, we assume that the space is normal; that is to say, we prove that Volterra implies Baire in the class of normal metacompact sequential spaces with a $\sigma$-closed discrete dense subset. We also prove that if $X$ is a monotonically normal $T_1$-space with countable pseudocharacter and a $\sigma$-discrete dense subset, then $X$ is Baire if and only if $X$ is Volterra. This has as a corollary that if $X$ is a generalized ordered (GO) space and has a $\sigma$-closed discrete dense subset, then $X$ is Baire if and only if $X$ is Volterra.

The set of all natural numbers is denoted by $\mathbb{N}$ and $\omega$ is $\mathbb{N} \cup \{0\}$.

## 2. Main results

A space $X$ is *discretely generated* [2] if for every $A \subset X$ and $x \in \overline{A}$ there exists a discrete subset $D \subset A$ such that $x \in \overline{D}$. In [2], it was proved that every Hausdorff sequential space is discretely generated. Recall that a space $X$ is called *sequential* if $A \subset X$ and $A \neq \overline{A}$ implies that there is a sequence $\{a_n : n \in \mathbb{N}\} \subset A$ such that $\{a_n : n \in \mathbb{N}\}$ converges to some $y \in \overline{A} \setminus A$. Given a space $X$ and $C \subset X$, we say that $C$ is *strongly discrete* if there exists a disjoint family $\{U_x : x \in C\}$ of open subsets of $X$ such that $x \in U_x$ for each $x \in C$. If $X$ is a Hausdorff space and a sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to a point $x$ of $X$, then the set $\{x_n : n \in \mathbb{N}\}$ is strongly discrete. A space $X$ is called *strongly discretely generated* if for every $A \subset X$ and $x \in \overline{A}$ there exists a strongly discrete set $D \subset A$ such that $x \in \overline{D}$ [8]. In what follows, we show that every Hausdorff sequential space is strongly discretely generated.

**Theorem 2.1.** *A space $X$ is strongly discretely generated if and only if $A \subset X$ and $A \neq \overline{A}$ implies there exists a strongly discrete subset $D \subset A$ such that there $\overline{D} \setminus A$ is not empty.*

*Proof.* ($\Rightarrow$) Obvious.

($\Leftarrow$) Let $A \subset X$. Denote $B = \bigcup\{\overline{D} : D \subset A$ and $D$ is strongly discrete$\}$. So $A \subset B \subset \overline{A}$. Suppose $B$ is not closed in $X$; then there exists a strongly discrete set $C \subset B$ and a point $y \in \overline{C} \setminus B$. For each $x \in C$ there is a strongly discrete set $D_x \subset A$ such that $x \in \overline{D_x}$. Since the set $C$ is strongly discrete, there is a disjoint family $\{U_x : x \in C\}$ of open subsets of $X$ such that $x \in U_x$ for each $x \in C$. Denote $D_x^* = D_x \cap U_x$. If $D = \bigcup\{D_x^* : x \in C\}$, then $D \subset A$ and $D$ is strongly discrete. Since $y \in \overline{C}$ and $x \in \overline{D_x^*}$ for each $x \in C$, we know that $y \in \overline{D}$. Thus $y \in B$. This is a contradiction with $y \notin B$. Thus the set $B$ is closed. $\square$

**Theorem 2.2.** *Let $X$ be a Hausdorff sequential space. If $A \subset X$ and $x \in \overline{A}$, then there is a countable strongly discrete set $D \subset A$ such that $x \in \overline{D}$.*

*Proof.* Let $B = \bigcup\{\overline{D} : D \subset A$ and $D$ is countable strongly discrete$\}$. By sequential property and a proof similar to Theorem 2.1, we can show that the set $B$ is closed. Thus $\overline{A} = B$. □

**Corollary 2.3.** *Every Hausdorff sequential space is strongly discretely generated.*

Let $X$ be a topological space and let $Y$ be a subspace of $X$. If for every $A \subset Y$ and $x \in \overline{A} \cap Y$ there exists a strongly discrete set $D$ of $X$ such that $D \subset A$ and $x \in \overline{D}$, then the subspace $Y$ of $X$ is called *strongly discretely generated in $X$*.

**Lemma 2.4.** *Let $X$ be a strongly discretely generated space. If $Y \subset X$, then $Y$ is strongly discretely generated in $X$.*

*Proof.* Let $A \subset Y$ and $x \in \overline{A} \cap Y$. Since $X$ is strongly discretely generated, there is a strongly discrete set $D$ of $X$ such that $D \subset A$ and $x \in \overline{D}$. Thus $Y$ is strongly discretely generated in $X$. □

By Corollary 2.3 and Lemma 2.4, we have

**Corollary 2.5.** *Every subspace of a Hausdorff sequential space $X$ is strongly discretely generated in $X$.*

**Corollary 2.6.** *If a subspace $D$ of a topological space $X$ is homeomorphic to a subspace of a Hausdorff sequential space $Y$, then $D$ is strongly discretely generated.*

Since every strongly discrete set $A$ of a dense subspace $D$ of a space $X$ is strongly discrete in $X$, we have

**Proposition 2.7.** *If a dense subspace $D$ of a topological space $X$ is strongly discretely generated, then $D$ is strongly discretely generated in $X$.*

**Lemma 2.8.** *[4, Lemma 2.3], Suppose $\mathcal{U}$ is a point-finite collection of open subsets of $X$ and that for each $U \in \mathcal{U}$ we have a $G_\delta$-subset $G(U) \subset U$. Then $S = \bigcup\{G(U) : U \in \mathcal{U}\}$ is a $G_\delta$-subset of $X$.*

By a similar proof with Lemma 2.4 of [4], we have

**Theorem 2.9.** *Suppose $X$ is a $T_1$-space and $X$ has a dense subspace $D = \bigcup\{D_n : n \geq 1\}$ satisfying:*

*(a) Every point of $D$ is $G_\delta$ in $X$;*

*(b) $D$ is strongly discretely generated in $X$;*

*(c) for each $n \geq 1$ there is a collection $\{V(d,n) : d \in D_n\}$ of open subsets of $X$ that is point-finite in $X$ and has $\{d\} = V(d,n) \cap D_n$ for each $d \in D_n$.*

*If $X$ is of the first category in itself, then $D$ contains a subspace $E$ that is dense in $X$ and is a $G_\delta$-subset of $X$.*

*Proof.* Since $X$ is a first category space, there is a sequence $\{G_n : n \geq 1\}$ of dense open subsets of $X$ such that $\bigcap\{G_n : n \geq 1\} = \emptyset$. We may assume $G_{n+1} \subset G_n$ for each $n \in \mathbb{N}$. Then $X$ has no isolated point, so neither does the set $D$. Let $n \in \mathbb{N}$. For each $d \in D_n$, the set $V(d,n) \cap G_n \cap D$ is dense in $V(d,n)$. Thus $d \in \overline{(V(d,n) \cap G_n \cap D) \setminus \{d\}}$. So there is a strongly discrete set $M(d,n) \subset (V(d,n) \cap G_n \cap D) \setminus \{d\}$ such that $d \in \overline{M(d,n)}$. Since every point of $D$ is $G_\delta$ in $X$ and $M(d,n)$ is strongly discrete, the set $M(d,n)$ is a $G_\delta$-set in $X$ by Lemma 2.8.

Since $\{V(d,n) : d \in D_n\}$ is point-finite, the set $\bigcup\{M(d,n) : d \in D_n\} = K_n$ is a $G_\delta$-set in $X$ and $K_n \subset G_n$. Since $\bigcap\{G_n : n \geq 1\} = \emptyset$, the family $\{G_n : n \in \mathbb{N}\}$ is point-finite, the set $E = \bigcup\{K_n : n \in \mathbb{N}\}$ is a $G_\delta$-set of $X$ and $E \subset D$. If $O$ is any non-empty open subset of $X$, then there is some $d \in D_n$ for some $n \in \mathbb{N}$ such that $d \in O$. So $O \cap M(d,n) \neq \emptyset$. Thus $O \cap E \neq \emptyset$. Thus the set $E$ is dense in $X$. □

By Corollary 2.6 and Theorem 2.9, we can get Proposition 2.6 of [4].

Recall from [6] that a space $X$ is *resolvable* if $X$ contains two disjoint dense subsets.

**Lemma 2.10.** *([9]) Any dense-in-itself subspace of a Hausdorff sequential space is resolvable.*

**Theorem 2.11.** *Suppose $X$ is a $T_1$-space and $X$ has a dense subspace $D = \bigcup\{D_n : n \geq 1\}$ satisfying:*

*(a) Every dense-in-itself open subspace $M$ of $D$ is resolvable;*

*(b) $D$ is strongly discretely generated;*

*(c) Every point of $D$ is $G_\delta$ in $X$;*

*(d) for each $n \geq 1$ there is a collection $\{V(d,n) : d \in D_n\}$ of open subsets of $X$ that is point-finite in $X$ and has $\{d\} = V(d,n) \cap D_n$ for each $d \in D_n$.*

*Then $X$ is a Baire space if and only if $X$ is Volterra.*

*Proof.* We just need to prove that $X$ is a Baire space if $X$ is Volterra. Suppose $X$ is not a Baire space. There is a sequence $\{G_n : n \in \mathbb{N}\}$ of dense open subsets of $X$ such that $\bigcap\{G_n : n \in \mathbb{N}\}$ is not dense in $X$. There is an non-empty open subset $Y$ of $X$ such that $Y \cap (\bigcap\{G_n : n \in \mathbb{N}\}) = \emptyset$. The set $D \cap Y$ is dense in $Y$ and satisfying (a), (b), (c) and (d). So we assume that $Y = X$. Hence $X$ is the first category and $X$ is dense-in-itself. Thus the set $D$ is also dense-in-itself. There are two dense subset $M_1$ and $M_2$ of $D$ such that $M_1 \cap M_2 = \emptyset$ by (a). By Lemma 2.4 we know that $M_1$ and $M_2$ are strongly discretely generated in $D$. By Proposition 2.7 we know that $M_1$ and $M_2$ are strongly discretely generated in $X$. So $M_i$ contains a subspace $E_i$ that is dense in $X$ and is a $G_\delta$-set of $X$ by Theorem 2.9 for $i = 1, 2$. So $E_1 \cap E_2 = \emptyset$. This contradicts that $X$ is a Volterra space. □

**Theorem 2.12.** *Suppose $X$ is a $T_1$-space and $X$ has a dense subspace $D = \bigcup\{D_n : n \geq 1\}$ satisfying:*

*(a) $D$ is homeomorphic to a subspace of a Hausdorff sequential space $Y$;*

*(b) Every point of $D$ is $G_\delta$ in $X$;*

*(c) for each $n \geq 1$ there is a collection $\{V(d,n) : d \in D_n\}$ of open subsets of $X$ that is point-finite in $X$ and has $V(d,n) \cap D_n = \{d\}$ for each $d \in D_n$.*

*Then $X$ is a Baire space if and only if $X$ is Volterra.*

*Proof.* Since every dense-in-itself open subspace $M$ of $D$ is also homeomorphic to a subspace of $Y$. Thus $M$ is resolvable by Lemma 2.10. By Corollary 2.6 we know that the set $D$ is strongly discretely generated. So $X$ is a Baire space if and only if $X$ is Volterra by Theorem 2.11. □

Proposition 2.6 of [4] with the $G_\delta$ condition added is the same as Theorem 2.12, except that Theorem 2.12 assumes only $T_1$ instead of regular.

A space $X$ is said to be *monotonically normal* [7] if there is a function $G$ which assigns to each ordered pair $(H, K)$ of disjoint closed subsets of $X$ an open set $G(H, K)$ such that

(1) $H \subset G(H,K) \subset \overline{G(H,K)} \subset X \setminus K$;

(2) if $(H', K')$ is a pair of disjoint closed subsets having $H \subset H'$ and $K \supset K'$, then $G(H,K) \subset G(H', K')$.

Recall that every generalized ordered (GO) space is monotonically normal.

**Lemma 2.13.** *( [1, Lemma 2.3]) Every dense-in-itself monotonically normal Hausdorff space is resolvable.*

**Lemma 2.14.** *([3, Theorem 5.18 and Theorem 5.20]) Every monotonically normal Hausdorff space is collectionwise normal and every subspace of a monotonically normal space is monotonically normal.*

The Hausdorff property in Lemma 2.14 can be replaced by $T_1$ separation axiom. In [2, Theorem 3.10], it was proved that any monotonically normal $T_1$-space is discretely generated. With the same proof, we can prove the following conclusion.

**Lemma 2.15.** *Every monotonically normal $T_1$-space is strongly discretely generated.*

**Theorem 2.16.** *Let $X$ be a monotonically normal $T_1$-space with countable pseudocharacter. If $X$ has a $\sigma$-discrete dense subspace $D$ of $X$, then $X$ is a Baire space if and only if $X$ is Volterra.*

*Proof.* Let $D = \bigcup\{D_n : n \in \mathbb{N}\}$ be a $\sigma$-discrete dense subspace of $X$ and let $n \in \mathbb{N}$. For each $d \in D_n$ there is an open neighborhood $V_d$ of $d$ in $X$ such that $V_d \cap D_n = \{d\}$. Denote $V_n = \bigcup\{V_d : d \in D_n\}$. Thus $V_n$ is open in $X$. So $V_n$ is monotonically normal by Lemma 2.14. Since $D_n$ is closed discrete in $V_n$ and $V_n$

is collectionwise normal, there is a disjoint family $\{V(d,n) : d \in D_n\}$ of open sets of $X$ such that $d \in V(d,n) \subset V_d$ for each $d \in D_n$. Thus $V(d,n) \cap D_n = \{d\}$.

Every subspace of $X$ is monotonically normal by Lemma 2.14. Thus every dense-in-itself open subspace $M$ of $D$ is resolvable by Lemma 2.13. The subspace $D$ of $X$ is strongly discretely generated by Lemma 2.15. Thus the conditions of Theorem 2.11 are satisfied. So $X$ is a Baire space if and only if $X$ is Volterra. □

It is well known that a GO space with a $\sigma$-closed discrete dense set is first countable. So we have

**Corollary 2.17.** *Let $X$ be a GO space. If $X$ has a $\sigma$-closed discrete dense subset, then $X$ is a Baire space if and only if $X$ is Volterra.*

A space $X$ is *semi-stratifiable* [3] if there is a function $G$ which assigns to each $n \in \mathbb{N}$ and closed set $H \subset X$, an open set $G(n,H)$ containing $H$ such that

(1) $H = \bigcap\{G(n,H) : n \in \mathbb{N}\}$;

(2) $H \subset K \Rightarrow G(n,H) \subset G(n,K)$.

If also

(3) $H = \bigcap\{\overline{G(n,H)} : n \in \mathbb{N}\}$,

then $X$ is *stratifiable*.

It is easy to see from condition (3) that stratifiable spaces are regular. It is well known that every stratifiable (regular and $T_1$) space is a $\sigma$-space (see, e.g., [3, Theorem 5.9]). So a stratifiable $T_1$-space has a $\sigma$-closed discrete dense subset. Thus we have

**Corollary 2.18.** *([1, Theorem 2.5]) Let $X$ be a stratifiable $T_1$-space. Then $X$ is a Baire space if and only if $X$ is Volterra.*

In [4, Corollary 2.8], it is pointed out that if $X$ is a regular metacompact sequential space that has a $\sigma$-closed discrete dense subset $D$, then $X$ is a Baire space if and only if $X$ is Volterra. However, this does not follow from the corrected Proposition 2.6 unless the condition that points of $D$ are $G_\delta$ in $X$ is added. In fact, the conclusion (d) of Corollary 2.8 in [4] also holds if we add a condition that the space $X$ is normal. Thus we have the following theorem.

**Theorem 2.19.** *Let $X$ be a metacompact normal sequential $T_1$-space. If $X$ has a $\sigma$-closed discrete dense subset, then $X$ is a Baire space if and only if $X$ is Volterra.*

*Proof.* Let $D = \bigcup\{D_n : n \in \mathbb{N}\}$ be a $\sigma$-closed discrete dense subspace of $X$. Suppose $X$ is not a Baire space. There is a sequence $\{G_n : n \in \mathbb{N}\}$ of dense open subsets of $X$ such that $\bigcap\{G_n : n \in \mathbb{N}\}$ is not dense in $X$. Thus there is a non-empty open subset $Y$ of $X$ such that $Y \cap (\bigcap\{G_n : n \in \mathbb{N}\}) = \emptyset$. For each $n \in \mathbb{N}$, $Y \cap G_n$ is dense in $Y$. We can assume $Y = X$. Thus $\bigcap\{G_n : n \in \mathbb{N}\}) = \emptyset$. So $X$ has no isolated points. Thus the set $D$ is dense in itself. Since $X$ is sequential,

$D$ is resolvable by Lemma 2.10. Hence there are disjoint subsets $M_1$ and $M_2$ of $D$ such that $\overline{M_1} = \overline{M_2} = X$.

Let $M_1 = \bigcup\{D_n^1 : n \in \mathbb{N}\}$ and $M_2 = \bigcup\{D_n^2 : n \in \mathbb{N}\}$, where $D_n^1 \subset D_n$ and $D_n^2 \subset D_n$. For each $n \in \mathbb{N}$ we denote $D_n^{1*} = D_n^1 \setminus \bigcup\{D_m^1 : m < n\}$ and $D_1^{1*} = D_1^1$, $D_n^{2*} = D_n^2 \setminus \bigcup\{D_m^2 : m < n\}$ and $D_1^{2*} = D_1^2$. Thus $\mathcal{F} = \{D_n^{1*}, D_n^{2*} : n \in \mathbb{N}\}$ is a countable family of closed discrete subsets of $X$ and $\mathcal{F}$ is a pairwise disjoint family.

We can assume $\mathcal{F} = \{F_i : i \in \mathbb{N}\}$. Since $X$ is normal and $F_1 \cap F_2 = \emptyset$, there are disjoint open sets $O_{21}$ and $O_{22}$ such that $F_1 \subset O_{21}$, $F_2 \subset O_{22}$. Similarly, there are disjoint open sets $O_{31}$, $O_{32}$ and $O_{33}$ such that $F_1 \subset O_{31}$, $F_2 \subset O_{32}$, $F_3 \subset O_{33}$. In this manner, we can get disjoint open sets $O_{in}$ for each $i \leq n$ such that $F_i \subset O_{in}$, where $n \in \mathbb{N}$. If $P_n = \bigcap\{O_{mn} : m \geq n\}$, then $\mathcal{U} = \{P_n : n \in \mathbb{N}\}$ is a family of disjoint $G_\delta$-sets of $X$ and $F_n \subset P_n$ for each $n$.

We can assume $\mathcal{U} = \{A_n, B_n : n \in \mathbb{N}\}$ of $G_\delta$-sets of $X$ and $\mathcal{U}$ is pairwise disjoint such that $D_n^{1*} \subset A_n$ and $D_n^{2*} \subset B_n$ for each $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. The set $D_n^{1*}$ is a closed discrete subset of $X$. For each $d \in D_n^{1*}$ there is an open set $O_d^1$ of $X$ such that $d \in O_d^1$ and $O_d^1 \cap D_n^{1*} = \{d\}$. If $\mathcal{U}_n^1 = \{O_d^1 : d \in D_n^{1*}\} \cup \{X \setminus D_n^{1*}\}$, then $\mathcal{U}_n^1$ is an open cover of $X$. The space $X$ is metacompact, so $\mathcal{U}_n^1$ has a point-finite open refinement $\mathcal{V}_n^1$. For each $d \in D_n^{1*}$, there is some $V_n^1(d) \in \mathcal{V}_n^1$ such that $d \in V_n^1(d)$ and $V_n^1(d) \cap D_n^{1*} = \{d\}$. Denote $\mathcal{V}_n^{1*} = \{V_n^1(d) : d \in D_n^{1*}\}$. Thus $\mathcal{V}_n^{1*}$ is point-finite. Similarly, we have an open family $\mathcal{V}_n^{2*} = \{V_n^2(d) : d \in D_n^{2*}\}$ such that $V_n^2(d) \cap D_n^{2*} = \{d\}$ and $\mathcal{V}_n^{2*}$ is point-finite for each $n \in \mathbb{N}$.

Since $X$ is a sequential Hausdorff space, the space $X$ is strongly discretely generated by Corollary 2.3. Let $n \in \mathbb{N}$ and $d \in D_n^{1*}$. So $d \in V_n^1(d)$ and $d \in \overline{V_n^1(d) \cap G_n \setminus \{d\}}$. Since $\{d\}$ is not open and $\{d\}$ is closed, the set $V_n^1(d) \cap G_n \setminus \{d\}$ is an non-empty open subset of $X$. Since $\overline{M_1} = X$, we have $d \in \overline{(V_n^1(d) \cap G_n \setminus \{d\}) \cap M_1}$. By strongly discretely generated property of $X$, there is a discrete subspace $C_d \subset (V_n^1(d) \cap G_n \setminus \{d\}) \cap M_1$ and a pairwise disjoint open family $\mathcal{V}_d = \{V_x : x \in C_d\}$ such that $x \in V_x$. We can assume that $V_x \subset V_n^1(d) \cap G_n$. For each $x \in C_d$ there is some $m_x \in \mathbb{N}$ such that $x \in D_{m_x}^{1*}$. Let $G_x = A_{m_x} \cap V_x$. So $G_x$ is a $G_\delta$-set of $X$ and $x \in G_x \subset V_x \subset G_n$. So $\bigcup\{G_x : x \in C_d\}$ is a $G_\delta$-set of $X$ and $C_d \subset \bigcup\{G_x : x \in C_d\}$. Hence $d \in \overline{\bigcup\{G_x : x \in C_d\}}$ and $\bigcup\{G_x : x \in C_d\} \subset V_n^1(d) \cap G_n$. The family $\{V_n^1(d) : d \in D_n^{1*}\}$ is a point-finite family of open subsets of $X$. If $K_n^1 = \bigcup\{\bigcup_{x \in C_d} G_x : d \in D_n^{1*}\}$, then $K_n^1$ is a $G_\delta$-set of $X$ and $K_n^1 \subset G_n$. For each $m \in \mathbb{N}$, we know that $A_i \cap B_m = \emptyset$ for each $i \in \mathbb{N}$. Since for each $x \in C_d$, the set $G_x \subset A_i$ for some $i \in \mathbb{N}$, we know that $G_x \cap B_m = \emptyset$. Thus $K_n^1 \cap B_m = \emptyset$. Since $\bigcap\{G_n : n \in \mathbb{N}\} = \emptyset$, the family $\{G_n : n \in \mathbb{N}\}$ is point-finite. Thus $\bigcup\{K_n^1 : n \in \mathbb{N}\} = L_1$ is a $G_\delta$-set of $X$ by Lemma 2.8. We know that $L_1 \subset \bigcup\{A_n : n \in \mathbb{N}\}$.

Let $p \in X$ and let $O_p$ be any open neighborhood of $p$ in $X$. There is some $d \in M_1$ such that $d \in O_p$. So there is some $n \in \mathbb{N}$ such that $d \in D_n^{1*}$.

Since $d \in \overline{\bigcup\{G_x : x \in C_d\}}$ and $\bigcup\{G_x : x \in C_d\} \subset K_n^1$, we have $d \in \overline{K_n^1}$. So $O_p \cap K_n^1 \neq \emptyset$. Thus $O_p \cap L_1 \neq \emptyset$. Hence $L_1$ is a dense $G_\delta$-set of $X$ and $L_1 \cap B_m = \emptyset$ for each $m \in \mathbb{N}$.

Similarly, we have a $G_\delta$-set $L_2$ of $X$ such that $L_2$ is a dense subset of $X$, $L_2 \cap A_n = \emptyset$ for each $n \in \mathbb{N}$ and $L_2 \subset \bigcup\{B_m : m \in \mathbb{N}\}$.

So $L_1$ and $L_2$ are two disjoint dense $G_\delta$-sets of $X$, this contradicts that $X$ is a Volterra space. Thus $X$ is a Bare space. □

By the proof of the above theorem, we know that the sequential property in the above theorem is just to make the space $X$ to be strongly discretely generated and every dense-in-itself subspace of $X$ is resolvable. The metacompact property in the above theorem is just to make every closed discrete subspace of $X$ to be separated by a point-finite open family.

By Lemmas 2.13 and 2.14 and a proof similar to Theorem 2.19, we have

**Theorem 2.20.** *Let $X$ be a monotonically normal $T_1$-space. If $X$ has a $\sigma$-closed discrete dense subset, then $X$ is a Baire space if and only if $X$ is Volterra.*

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(Liang-Xue Peng) College of Applied Science, Beijing University of Technology, Beijing 100124, China

*E-mail address*: pengliangxue@bjut.edu.cn

(Chong Yang) College of Applied Science, Beijing University of Technology, Beijing 100124, China

*E-mail address*: yangchong@emails.bjut.edu.cn