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A NUMERICAL SCHEME FOR SOLVING NONLINEAR BACKWARD PARABOLIC PROBLEMS

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Abstract. In this paper a nonlinear backward parabolic problem in one dimensional space is considered. Using a suitable iterative algorithm, the problem is converted to a linear backward parabolic problem. For the corresponding problem, the backward finite differences method with suitable grid size is applied. It is shown that if the coefficients satisfy some special conditions, this algorithm not only is convergent, but also is conditionally stable. Moreover, it is proved that the estimated values converge to the exact solution of the problem. All these approaches examined in some numerical examples. corresponding theorems for the convergency and stability of the solution are studied.
Keywords: Nonlinear backward parabolic problem, linear backward parabolic problem, iterative algorithm, stability, convergency. **MSC(2010):** Primary: 35K55; Secondary: 65N21, 65N20, 65N12.

1. Introduction

In this paper we consider a nonlinear backward problem of the form

$$\begin{aligned} \partial_t u(x,t) - \partial_x(D(u(x,t))\partial_x u(x,t)) &= 0, && (x,t) \in [0,1]\times(0,T), \\ u(0,t) &= u(1,t) = 0, && t \in (0,T), \\ u(x,T) &= u_T(x), && x \in [0,1], \end{aligned} \tag{1.1}$$

where $T > 0$ is a known constant, u_T is a nonnegative known function in $C[0,1]$ and the conductivity term is given as $D(u) = au + b$ such that $a(\neq 0)$, b, and all of their first partial derivatives are known functions and belong to $C^1([0,1]\times[0,T])$. The solution, $u(x,t)$, is unknown and it is computed for $(x,t) \in (0,1)\times[0,T]$. It should be mentioned that the above assumptions are necessary for the proof of the next sections' theorems. One may apply these problems to find the distribution of temperature from the final time data. In the simplest case $D \equiv 1$, Tikhonov in [16] showed that the problem is ill-posed.

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In fact, for a given final data, u_T, there is no guarantee that a solution for the problem (1.1) exists it depends continuously on the final data (see [16]).

In recent years, some new approaches for solving backward parabolic problems have been developed in both theory and practice, for example, Tikhonov in [16] proposed a regularization method based on generalization of the above inverse problem. Tuan et. al. in [17], have introduced a modified integral equation method for semilinear backward heat conduction problem and have estimated the error of Hölder type of regularized solutions. The Galerkin method is applied for forward-backward parabolic problems by Aziz et. al in [2]. Cheng et. al in [3] have applied the finite difference method for the forward-backward heat equation and have used a coarse mesh-second central difference at the middle line mesh points. In addition, some numerical methods have been applied to solve backward parabolic stochastic partial differential equations, such as using a numerical method for backward parabolic problems with nonselfadjoint operators (see [9]), applying Galerkin and weighted Galerkin methods for the numerical solution of parabolic partial differential equations (see [13]), introducing space-time finite element method to solve a model forward-backward heat equation (see [5]).

In this paper, based on an iterative algorithm and a suitable finite difference method, an approach is proposed to find a conditionally stable approximation of the solution (1.1), such that it converges to the exact solution. It is proved if $a \neq 0$, b and their first partial derivatives respect to the independent variables belong to $C^1([0,1] \times [0,T])$, and $D(u)\partial_t D(u) > 0$. In this situation, the obtained solutions of iterative algorithm are unique for each iteration, conditionally stable and they converge to the exact solution of (1.1). For this purpose, an iterative algorithm is suggested in section two. Using finite difference method with suitable grid size, one can obtain an approximation of the solution (1.1) in section three. Meanwhile some stability and convergency theorems are given. Finally, we give a numerical example in the last section.

2. **Reducing the nonlinear problem to a linear one**

If $D > 0$ is constant or independent of u, the problem (1.1) is a linear backward parabolic problem. In this case, several approaches are investigated in some books and papers, which provide analytical and numerical schemes for solving it, such as applying a method of fundamental solutions for inverse heat conduction problems in an anisotropic medium (see [4]). Using a semi-implicit finite difference method for backward inverse heat conduction problems in one-dimensional space (see [15]), numerical iterative algorithm, using boundary element method in order to solve the one dimensional backward heat conduction problem (see [14]), a heatlets approach to regularize the backward heat equation and, more generally, ill-posed Cauchy problems (see [10]) and non-overlap domain decomposition method for the numerical solution of the forward-backward heat equation (see [8]), are typical samples of ill-posed inverse problems. If D depends on u, then the problem is either forward-backward or backward. As in [11], to reach the backward problem, we impose some suitable conditions on the problem. Then, the nonlinear problem is converted to a sequence of linear parabolic problems by an iterative algorithm. We first introduce some practical notations.

First, assume that $\Omega \subseteq \mathbb{R}^m$ is an open set, $L_2(\Omega)$ is the space of functions with the finite norm $\|u\|_2^2 = \int_\Omega |u|^2 d\Omega$, and $H_{(k)}$ is the space of functions on Ω with partial derivatives of an order less than k in $L_2(\Omega)$. The norm of this space is defined by $\|.\|_{2,2} = \sum_{|\rho|\leq k} \|\partial^\rho u\|_2^2$, where ρ is a multi-index with nonnegative integer components ρ_j $(j = 1, 2, ..., m)$ and the sum is over $|\rho| = \rho_1 + \rho_2 + ... + \rho_m$. Also, suppose that $\mathring{H}_{(1)}(\Omega)$ is the completion of $C_\circ^k(\Omega)$, where $C_\circ^k(\Omega)$ is the space of functions with compact support in $C^k(\Omega)$.

Isakov in [11] states a stability estimation theorem as follows.

Theorem 2.1. *Suppose that A is a linear operator with domain $H_{(2)}(\Omega) \cap \mathring{H}_{(1)}(\Omega) \subset L_2(\Omega)$. If $u(t) \in C(\overline{\Omega} \times [0, T])$ is a solution of*

$$\|\partial_t u + Au\|_2 \leq \alpha \|u\|_2, \tag{2.1}$$

then A satisfies the following inequalities

$$\|A_- u\|_2^2 \leqslant \alpha(\|A_+ u\|_2 + \|u\|_2)\|u\|_2 \tag{2.2}$$

and

$$\partial_t (A_+ u, u) \leqslant 2(\partial_t u, A_+ u) + \alpha(\|A_+ u\|_2 + \|u\|_2)\|u\|_2 \tag{2.3}$$

for some constant α, where A_+ and A_- are symmetric and skew-symmetric operators of A respectively, such that $A = A_+ + A_-$, and $(.,.)$ denotes the scaler product in $L_2(\Omega)$. Moreover, the following stability estimation holds.

$$\|u(t)\|_2 \leqslant C^* \|u(0)\|_2^{1-\theta} \|u(T)\|_2^\theta,$$

where $C^* \leqslant e^{((2\alpha+2)T + 2e^{CT}/C)}$, and $\theta = (1-e^{-Ct})/(1-e^{-CT})$ or $\theta = (e^{C(t-T)} - e^{-CT})/(1-e^{-CT})$ and C depends on α. When $\alpha = 0$ then, $C^* = 1$ and $\theta = t/T$ is chosen.

Proof. See [11]. □

Now put $u^{(0)}(x, t) = u_T(x)$ and $u^{(0)}(0, t) = u^{(0)}(1, t) = 0$. Then for $r = 1, 2, ...,$ consider the linear backward problems of the form

$$\partial_t u^{(r)}(x,t) + \partial_x (D(u^{(r-1)}(x,t))\partial_x u^{(r)}(x,t)) = 0, \qquad (x,t) \in [0,1] \times (0,T), \tag{2.4}$$

$$u^{(r)}(0,t) = u^{(r)}(1,t) = 0, \qquad t \in (0,T), \tag{2.5}$$

$$u^{(r)}(x,T) = u_T(x), \qquad x \in [0,1]. \tag{2.6}$$

where D and u_T are as mentioned before. If there exists a constant $M^{(r)} \in \mathbb{R}$, such that $\|u^{(r)}\|_2 \leq M^{(r)}$, in $(0,T)$, then an upper bound of the solution $u^{(r)}(r \in \mathbb{N})$ is determinable. According to [11], it is immediately obtained:

$$\|u^{(r)}\|_2 \leq (M^{(r)})^{1-t/T} \|u_T\|_2^{t/T}.$$

In addition, let $\|\partial_t u^{(r)}\|_2 \leq M_1^{(r)}$ in $(0,T)$, where $M_1^{(r)}$ is known constant for $r \in \mathbb{N}$. Then, using Macleurant formula for $\|u^r\|_2^2$ and [11], the following inequality is yield.

$$\|u^{(r)}(0)\|_2^2 \leq -M^{(r)} M_1^{(r)} T(1 - ln(-M_1^{(r)} T/Mln(\varepsilon^{(r)})))/ln(\varepsilon^{(r-1)}),$$

where $\varepsilon^{(r)} = \|u_T\|_2 / M^{(r)}$.

Therefore, applying maximum principal (see [6]), homogeneous Dirichlet conditions (2.5) and above inequality simultaneously, an upper bound for $\|u^{(r)}\|$ in $[0,T]$, is gained. So, the following theorem, related to stability estimation and uniqueness, is proposed.

Theorem 2.2. *Consider the problem* (2.4)-(2.6)*. Suppose that for any* $t \in [0,T]$*, a fixed* $r \in \mathbb{N}$*,* $\partial_x^2 u^{(r)} \in L_2([0,1] \times [0,T])$*, and* $\varepsilon > 0$*, just one of the statements*

(i) $au^{(r-1)} + b \geq \varepsilon$ *on* $[0,1]$*,*
(ii) $\partial_t(au^{(r-1)} + b) \leq 0$ *on* $[0,1]$

holds. Then $u^{(r)}$ *is unique and conditionally stable. Moreover* $\{u^{(r)}\}_{r=1}^{\infty}$ *is convergent to* $u \in L_2([0,1] \times [0,T])$ *and* $u^{(r)}$ *satisfies in (2.1), and we have*

$$\begin{aligned} \|\partial_t u + \partial_x((au+b)\partial_x u)\|_2 &= \lim_{r\to\infty} \|\partial_t u^{(r)} + \partial_x((au^{(r-1)}+b)\partial_x u^{(r)})\|_2 \\ &\leq (\limsup \alpha^{(r)})_{r\to\infty} (\limsup M^{(r)})_{r\to\infty}. \end{aligned}$$

Furthermore, according to Theorem 2.1, the following estimation of solution $u(t)$ *on* $[0,1]$ *exists*

$$\|u(t)\| \leq C^* \|M^*\|^{\theta} \|u_T\|^{1-\theta},$$

where $\theta = t/T$*,* $C^* = limsup(C^{*(r)})_{r\to\infty}$*,* $M^* = limsup(M^{(r)})_{r\to\infty}$ *and* $M^{(r)} = Max_{r\in\mathbb{N}} \|u^{(r)}\|$ *on* $[0,1] \times (0,1)$*.* $C^{*(r)}$ *and* $\alpha^{(r)}$ *are the bounds of* $u^{(r)}$ *which satisfy in Theorem 2.1 for all* $r \in \mathbb{N}$*.*

Proof. First, we show that the assumptions of Theorem 2.1 are hold. For this purpose, we trivially have

$$A_+^{(r-1)} = -\partial_x \left((au^{(r-1)} + b)\partial_x u^{(r)} \right),$$

and inequality (2.2) is satisfied for any choice of $\alpha^{(r)} \geq 0$. Now, put $w^{(r-1)} = au^{(r-1)} + b$, and assume $\partial_x^2 u^{(r)} \in L_2([0,1] \times [0,T])$. Using integration by parts and (2.5), it is obtained

$$\begin{aligned}\partial_t(A_+^{(r-1)}, u^{(r)}) &= \partial_t \int_0^1 -\partial_x \left(w^{(r-1)}\partial_x u^{(r)} \right) u^{(r)} dx \\ &= \int_0^1 \partial_t w^{(r-1)} (\partial_x u^{(r)})^2 dx + \int_0^1 2w^{(r-1)}\partial_x u^{(r)} \partial_t \partial_x u^{(r)} dx \\ &\leq \int_0^1 \left(\partial_t w^{(r-1)} (\partial_x u^{(r)})^2 - 2\partial_x (w^{(r-1)}\partial_x u^{(r)})\partial_t u^{(r)} \right) dx,\end{aligned}$$

Obviously, the second term in the last integral is

$$\int_0^1 \partial_x (w^{(r-1)}\partial_x u^{(r)})\partial_t u^{(r)} dx = \left(Au^{(r)}, \partial_t u^{(r)} \right).$$

If $w^{(r-1)} \geq \varepsilon > 0$, and since $\partial_x^2 u \in L_2([0,1]\times[0,T])$, the operator $A^{(r-1)}$ is uniformly elliptic and $\|w^{(r-1)}\|_{2,2} \leq \alpha^{(r)} \|A^{(r-1)}\|_2$. Otherwise, if $\partial_t(au^{(r)}+b) \leq 0$, then $\partial_t w^{(r-1)}$ is not positive and by applying integration by parts to the first term of the above integral, conditions (i), (ii) and Cauchy-Schwartz inequality (see [11]), we get

$$-\int_0^1 \partial_x (\partial_t w^{(r-1)}\partial_x u^{(r)}) u^{(r)} dx \leq M_2^{(r)} \|u^{(r)}\|_{2,2} \|u^{(r)}\|_2 \leq \alpha^{(r)} \|Au^{(r)}\|_2 \|u^{(r)}\|_2.$$

Therefore, (2.3) is hold.
To prove the convergency, it is noticed that $L_2([0,1]\times[0,T])$ is a complete space and for all $r \in \mathbb{N}$, $u^{(r)} \in L_2([0,1]\times[0,T])$ so, $\{u^{(r-1)}\}_{r=0}^{\infty}$ uniquely converges to $u \in L_2([0,1]\times[0,T])$. □

In general, by ignoring (i) and (ii), the conditions of the Theorem 2.1 are still satisfied (see [11]). We have $A_+^{(r-1)} u^{(r)} = -\partial_x \left((au^{(r-1)} + b)\partial_x u^{(r)} \right)$ so, we deal with a linear backward parabolic problem. When Theorem 2.1 applies for this problem, stability is guaranteed. Since

$$\|\partial_t u + \partial_x \left((au+b)\partial_x u \right)\|_2 = \lim_{r\to\infty} \|\partial_t u^{(r)} + \partial_x \left((au^{(r-1)} + b)\partial_x u^{(r)} \right)\|_2,$$

then $u^{(r)} \to u$, where u is a solution of (1.1). Consequently, for all $r \in \mathbb{N}$, $u^{(r)} \in L_2([0,1]\times[0,T])$, any subsequence of $\{u^{(r-1)}\}_{r=0}^{\infty}$ tends to an element of $L_2([0,1]\times[0,T])$. Inequality (2.2) satisfies for all arbitrary $\alpha \geq 0$. To prove (2.3), we use symmetric properties of $A_+^{(r-1)}$ and integration by parts. We have

$$\begin{aligned}\partial_t \int_0^1 (A_+^{(r-1)} u^{(r)}) u^{(r)} dx &= \int_0^1 \partial_t \left(w^{(r-1)} (\partial_x u^{(r)})^2 \right) dx \\ &= \int_0^1 \left(\partial_t w^{(r-1)} (\partial_x u^{(r)})^2 + 2A_+^{(r-1)} u^{(r)} \partial_t u^{(r)} \right) dx.\end{aligned}$$

Applying a simultaneous process, it is realized that $\int_0^1 \partial_t w^{(r-1)}(\partial_x u^{(r)})^2 dx$ is bounded, and (2.3) is yield. Now, using finite difference method or some approximation operators, which satisfy in conditions (2.5) and (2.6), a conditionally stable solution, $u^{(r)}$ is obtained.

In the next section a numerical method, based on a finite difference scheme, is used to find an estimated solution of problem (2.4)-(2.6).

3. **A numerical approach**

Many numerical methods are widely used in solving backward parabolic problems. Due to ill-posedness, usually when the solutions of the numerical method exist, they are exponentially unstable. Recent years discretizing the differential equations and finite difference methods are used to solve linear and nonlinear parabolic problems and, in many cases, are accompanied by regularization techniques. In this section, we use a finite difference scheme to find an approximated solution for the reduced backward parabolic problem (2.4)-(2.6). Consider $N \times M$ nods on the rectangle $[0,1] \times [0,T]$, which formed by points $(x_i, t_j) = (ih, jk)$ for $0 \leq i \leq N,\ 0 \leq j \leq M$, where $h = \frac{1}{N}$ and $k = \frac{T}{M}$. Put $u_{i,j} = u(x_i, t_j)$. In this section, notation $(.)_{i,j}$ refers to the value of function at point (x_i, t_j). Using finite difference scheme, the Eq. (2.4) is written as follows

$$\begin{aligned} u_{i,j+1}^{(r)} &= u_{i,j}^{(r)} + k\left(a_{i,j}u_{i,j}^{(r-1)} + b_{i,j}\right)\frac{u_{i-1,j}^{(r)} - 2u_{i,j}^{(r)} + u_{i+1,j}^{(r)}}{h^2} \\ &+ k\left(\partial_x a_{i,j}u_{i,j}^{(r-1)} + \partial_x b_{i,j} + a_{i,j}\frac{u_{i+1,j}^{(r-1)} - u_{i,j}^{(r-1)}}{h}\right)\left(\frac{u_{i+1,j}^{(r)} - u_{i,j}^{(r)}}{h}\right) \end{aligned} \tag{3.1}$$

where $u_{i,j}^{(r)}$ is the solution $u^{(r)}$ at the node (x_i, t_j) and $r^{th}(r = 1, 2, ...)$ iteration. The stability of this approach was already investigated in [8] and [9]. With some elementary calculations, we reduce Eq. (3.1) to following form (see [1])

$$\begin{aligned} u_{i,j+1}^{(r)} &= (c_1^{(r-1)})_{i,j}u_{i+1,j}^{(r)} + (c_0^{(r-1)})_{i,j}u_{i,j}^{(r)} + (c_{-1}^{(r-1)})_{i,j}u_{i-1,j}^{(r)}, \\ (c_{-1}^{(r-1)})_{i,j} &= \frac{k}{h^2}D_{i,j}^{(r-1)}, \\ (c_0^{(r-1)})_{i,j} &= 1 - 2\frac{k}{h^2}D_{i,j}^{(r)} - \frac{k}{h}B_{i,j}^{(r-1)}, \\ (c_1^{(r-1)})_{i,j} &= \frac{k}{h^2}D_{i,j}^{(r-1)} + \frac{k}{h}B_{i,j}^{(r-1)}, \end{aligned} \tag{3.2}$$

where

$$D_{i,j}^{(r-1)} = a_{i,j}u_{i,j}^{(r-1)} + b_{i,j},$$

and

$$B_{i,j}^{(r-1)} = \left(\partial_x a_{i,j} u_{i,j}^{(r-1)} + \partial_x b_{i,j} + a_{i,j} \frac{u_{i+1,j}^{(r-1)} - u_{i,j}^{(r-1)}}{h} \right).$$

Remark 1. *The following finite differences' standard equations are used to obtain the above equations.*

$$\begin{aligned} \partial_t u(x_i, t_j) &\simeq \frac{u_{i,j+1} - u_{i,j}}{k}, \\ \partial_x u(x_i, t_j) &\simeq \frac{u_{i+1,j} - u_{i,j}}{h}, \\ \partial_{xx} u(x_i, t_j) &\simeq \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}. \end{aligned}$$

Obviously, Eq. (3.2) is of positive type (see [1]), and is written as a special form

$$u_{i,j} = \sum_s (c_s)_{i,j} u_{i+s,j+1} + k d_{i,j}. \tag{3.3}$$

According to [1] and [7], if $D^{(r)}(x,t) > 0, R = \frac{k}{h^2} < \frac{1}{2D^{(r)}(x,t)}$, then the finite difference scheme (3.2) is conditionally stable.

Now, assume that coefficients c_s are twice continuously differentiable with respect to h. John in [12] proved, if $u \in C^{2,1}$ be the solution of equation $u_t = au_{xx} + bu_x + cu + d$ subject to $u(x,0) = f(x)$ and U be the solution of Eq. (3.3) with initial condition $U_{i,0} = f_{i,0}$, and the following consistency conditions are hold, then the solution of the finite differences equation (3.3) converges uniformly to u as $h \to 0$ ($k = Rh^2$) (see [12] and [7]).

$$\begin{aligned} c_{i,j} &= \lim_{h \to 0} \frac{1}{k} \Big(\sum_s (c_s)_{i,j} - 1 \Big), \\ b_{i,j} &= \lim_{h \to 0} \frac{h}{k} \sum_s s (c_s)_{i,j}, \\ a_{i,j} &= \lim_{h \to 0} \frac{h^2}{2k} \sum_s s^2 (c_s)_{i,j}. \end{aligned}$$

Therefore, if all the above conditions are hold, then convergency is guaranteed.

In next section, we give a numerical experiment. Using the iterative algorithm and finite difference method, an approximated solution for a nonlinear backward problem is obtained.

4. Numerical experiment

Consider the following nonlinear backward equation

$$\partial_t u = \partial_x \left(\left(\frac{e^t}{6(1+e^t)^2} u + \frac{1}{12(1+e^t)} \right) \partial_x u \right), \qquad (x,t) \in [0,1] \times (0,1),$$

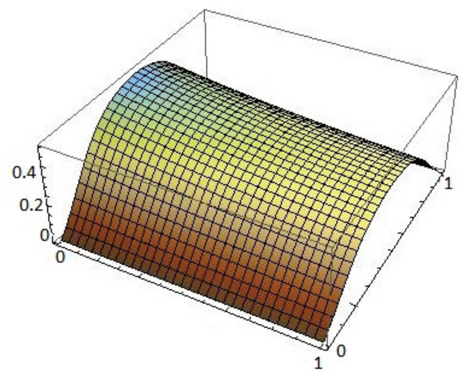


Figure 1. Exact solution in $[0,1]\times[0,1]$.

with the Dirichlet boundary condition

$$u(0,t) \quad = \quad u(1,t)=0, \qquad\qquad t\in(0,1),$$

and final data

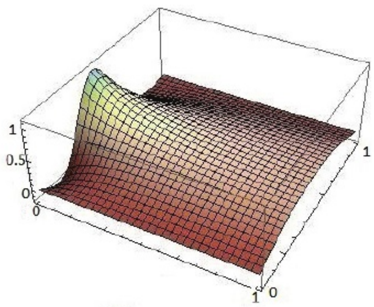
$$u(x,1) \quad = \quad e^{-1}(1-x)x+(1-x)x, \qquad x\in[0,1].$$

The exact solution of the backward parabolic problem is

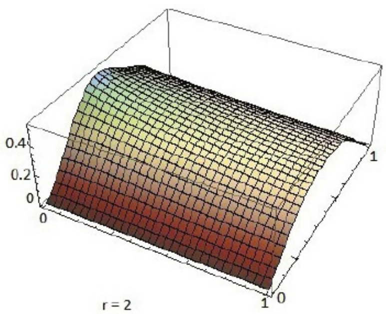
$$u(x,t) \quad = \quad x(1-x)(e^{-t}+1), \qquad (x,t)\in[0,1]\times(0,1)$$

First the iterative algorithm (2.4)-(2.6) and then the finite difference scheme (3.2) are applied to solve this problem with five steps. To avoid discontinuity, we choose $u^{(0)}=e^{-1}(1-x)x+(1-x)x$. Results are obtained by Mathematica 8.2 software on a common dual-core CPU personal computer with $3.07GHz$ frequency. Approximate solutions are computed for $r=1,2,...5$. According to section (3), we have $R=\frac{k}{h^2}\ll 6$ (the bounds of R for these 5 steps are not bigger than 6) with 5×600 mesh points ($h=\frac{1}{5}$, and $k=\frac{1}{600}$) to discretize this problem. To apply finite difference approach, explained in previous section, on a backward problem, we need to solve some systems of linear equations. We obtained good results however, due to severely ill-posedness of the problem, errors at initial points are eligible.

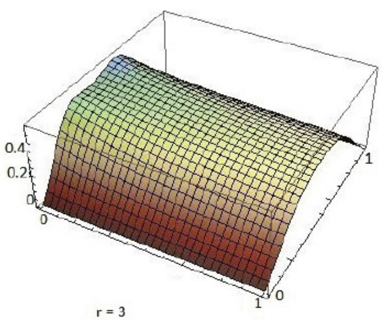
The exact solution, approximation solution, relative error and absolute error for some points in 5^{th} step are represented in table 1. The exact solution, approximation solutions and relative error functions are shown in pictures 1, 2 and 3.



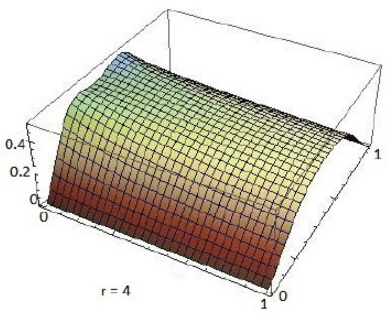
r = 1



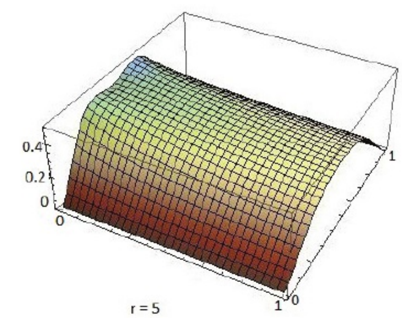
r = 2



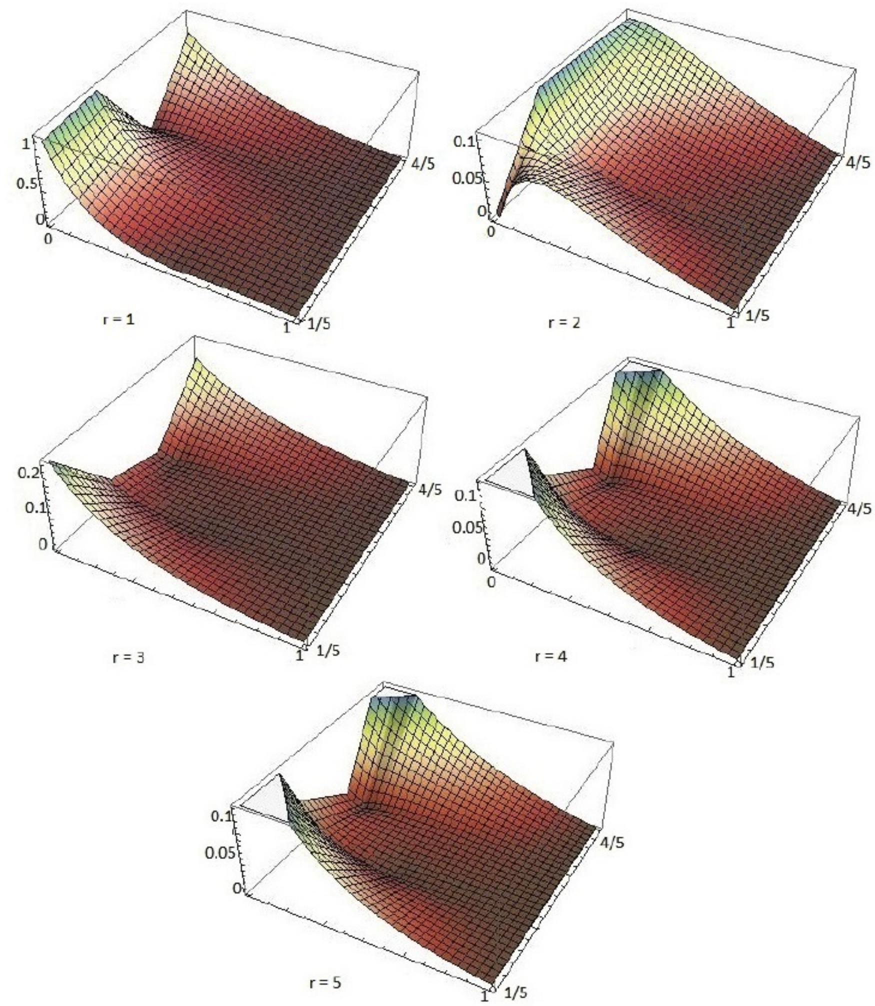
r = 3



r = 4



r = 5


1
0.5
0
0
1
1/5
4/5
r = 1
0.1
0.05
0
0
1
1/5
4/5
r = 2
0.2
0.1
0
1
1/5
4/5
r = 3
0.1
0.05
0
0
1
1/5
4/5
r = 4
0.1
0.05
0
1
1/5
4/5
r = 5

(i,j)	exact solution	approximated solution	relative error	absolute error
(2,575)	0.332048	0.332155	3.24021×10^{-4}	1.0759×10^{-4}
(3,550)	0.335964	0.335322	1.91062×10^{-3}	6.41898×10^{-4}
(4,525)	0.224275	0.224275	1.06871×10^{-2}	2.42275×10^{-3}
(1,500)	0.232116	0.232116	1.12423×10^{-2}	2.58075×10^{-3}
(2,475)	0.348741	0.347006	1.59866×10^{-3}	5.57519×10^{-4}
(3,450)	0.353368	0.351235	6.03685×10^{-3}	2.13323×10^{-4}
(4,425)	0.238794	0.232046	2.82593×10^{-2}	6.74817×10^{-3}
(1,400)	0.242147	0.248824	2.7576×10^{-2}	6.67746×10^{-3}
(2,375)	0.368463	0.369333	2.36294×10^{-3}	8.70657×10^{-4}
(3,350)	0.373928	0.370088	1.02699×10^{-2}	3.84022×10^{-3}
(4.325)	0.253084	0.240191	5.24296×10^{-2}	1.28932×10^{-2}
(1,300)	0.257045	0.270522	5.24296×10^{-2}	1.34768×10^{-2}
(2,275)	0.391761	0.386685	1.10012×10^{-3}	4.30983×10^{-4}
(3,250)	0.398218	0.392804	1.35951×10^{-2}	5.41381×10^{-3}
(4,225)	0.269966	0.24809	8.10324×10^{-2}	2.1876×10^{-2}
(1,200)	0.274645	0.30009	9.26464×10^{-2}	2.54449×10^{-2}
(2,175)	0.419284	0.416669	6.2366×10^{-3}	2.61491×10^{-3}
(3,150)	0.429612	0.421318	1.00192×10^{-2}	5.59391×10^{-3}
(4,125)	0.28991	0.254341	1.22689×10^{-2}	3.55688×10^{-2}
(2,100)	0.443156	0.433535	2.1687×10^{-2}	9.61073×10^{-3}
(1,75)	0.3012	0.358037	1.88702×10^{-1}	5.6831×10^{-2}
(2,50)	0.460811	0.441591	4.17085×10^{-2}	1.92197×10^{-2}
(3,25)	0.470205	0.473256	6.48868×10^{-3}	3.05101×10^{-3}
(4,0)	0.32	0.254408	2.04976×10^{-1}	6.55922×10^{-2}

Table 1 Exact solution, approximated solution, relative error and absolute error at point (x_i, t_j) in 5^{th} iteration.

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