ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 41 (2015), No. 6, pp. 1465–1475

Title:

On centralizers of prime rings with involution

Author(s):

S. Ali and N. A. Dar

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 6, pp. 1465–1475
Online ISSN: 1735-8515

ON CENTRALIZERS OF PRIME RINGS WITH INVOLUTION

S. ALI AND N. A. DAR*

(Communicated by Bernhard Keller)

Abstract. Let R be a ring with involution $*$. An additive mapping $T : R \to R$ is called a left(respectively right) centralizer if $T(xy) = T(x)y$ (respectively $T(xy) = xT(y)$) for all $x, y \in R$. The purpose of this paper is to examine the commutativity of prime rings with involution satisfying certain identities involving left centralizers.
Keywords: Prime ring, normal ring, involution, left centralizer, centralizer.
MSC(2010): Primary: 16N60; Secondary: 16W10.

1. Introduction

Throughout this article, R will represent an associative ring with centre $Z(R)$. A ring R is said to be 2-torsion free if $2a = 0$ (where $a \in R$) implies $a = 0$. A ring R is called a prime ring if $aRb = (0)$ (where $a, b \in R$) implies $a = 0$ or $b = 0$. We write $[x, y]$ for $xy - yx$ and xoy for $xy + yx$, respectively. An additive map $x \mapsto x^*$ of R into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ holds for all $x, y \in R$. A ring equipped with an involution is known as ring with involution or $*$-ring. An element x in a ring with involution $*$ is said to hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. If R is 2-torsion free then every $x \in R$ can be uniquely represented in the form $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$. Note in this case x is normal, i.e., $xx^* = x^*x$, if and only if h and k commute. If all elements in R are normal, then R is called a normal ring. An example is the ring of quaternions. A description of such rings can be found in [5], where further references can be found.

Following [14], an additive mapping $T : R \to R$ is said to be a left (respectively right) centralizer (multiplier) of R if $T(xy) = T(x)y$ (respectively $T(xy) = xT(y)$) for all $x, y \in R$. An additive mapping T is called a centralizer

Article electronically published on December 15, 2015.
Received: 13 October 2012, Accepted: 12 September 2014.
*Corresponding author.

in case T is a left and a right centralizer of R. Considerable work has been done on left (respectively right) centralizers in prime and semiprime rings during the last few decades (see for example [1, 4, 7, 8, 11, 12, 13] and [14]) where further references can be found. Over the last 30 years, several authors have investigated the relationships between the commutativity of the ring R and certain specific types of maps on R. The first result in this direction is due to Devinsky [3] who proved that a simple artinian ring is commutative if it has a commuting non-trivial automorphims. This result was subsequently refined and extended by a number of authors in various directions (viz., [2, 6, 9, 10]). Recently, first author together with Ashraf [1] proved that if a prime ring admits a left centralizer (multiplier) $T : R \to R$ such that $T([x, y]) = [x, y]$ with $T(x) \neq x$ for all $x, y \in I$, a nonzero ideal of R, then R is commutative. Moreover, in [1] some related results involving left centralizers have also been discussed. In [7] Oukhtite established similar problems for certain situations involving left centralizers acting on Lie ideals.

In this paper, we shall consider similar problems when the ring R is equipped with a fixed involution $*$ even in more general setting by replacing y by x^*. More precisely, we prove that if a prime ring with involution such that $char(R) \neq 2$ admits a left centralizer $T : R \to R$ satisfying any one of the following conditions: (i) $T([x, x^*]) = 0$, (ii) $T(xox^*) = 0$, (iii) $T([x, x^*]) \pm [x, x^*] = 0$, (iv) $T(xox^*) \pm (xox^*) = 0$ for all $x \in R$, then R is commutative.

We shall restrict our attention on left centralizers, since all results presented in this article are also true for right centralizers because of left-right symmetry.

2. **Preliminaries**

We shall do a great deal of calculations with commutators and anticommutators and routinely use the following basic identities: For all $x, y, z \in R$, we have

$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = [x, y]z + y[x, z].$$

Moreover,

$$xo(yz) = (xoy)z - y[x, z] = y(x \circ z) + [x, y]z$$

and

$$(xy)oz = (xoz)y + x[y, z] = x(y \circ z) - [x, z]y.$$

We begin this section with the following lemmas which are essential for developing the proof of our main results.

Lemma 2.1. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$. If R is normal, then R is commutative.*

Proof. By the hypothesis, we have R is normal that is,

$$[h,k] = 0 \textit{ for all } h \in H(R) \textit{ and } k \in S(R). \tag{2.1}$$

Let s be any nonzero element of $S(R) \cap Z(R)$. Then for any $h_1 \in H(R)$ and $k_1 \in S(R)$, $sh_1 \in S(R)$ and $sk_1 \in H(R)$. Therefore in view of (2.1), we have

$$s[h,h_1] = [h,sh_1] = 0 \textit{ for all } h,h_1 \in H(R), \tag{2.2}$$

and

$$s[k_1,k] = [sk_1,k] = 0 \textit{ for all } k,k_1 \in S(R). \tag{2.3}$$

Now since every $x \in R$ can be uniquely represented as $2x = h + k$, where $h \in H(R)$, $k \in S(R)$, so in view of equations (2.1), (2.2) and (2.3), we obtain $4s[x,y] = s[2x,2y] = s[h+k,h_1+k_1] = 0$ for all $x,y \in R$. Since $char(R) \neq 2$, we have $s[x,y] = 0$ for all $x,y \in R$. This further implies that $sR[x,y] = (0)$ for all $x,y \in R$. The primeness of R forces that $[x,y] = 0$ for all $x,y \in R$. Which completes the proof of the lemma.

□

Lemma 2.2. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$. If $xox^* = 0$ for all $x \in R$ or $[x,x^*] = 0$ for all $x \in R$, then R is commutative.*

Proof. We have $xox^* = 0$ for all $x \in R$. Linearization of the above relation yields that $xoy^* + yox^* = 0$ for all $x,y \in R$. Replacing y by x^2 in the last relation and using the given hypothesis, we obtain

$$\begin{array}{rcl} 0 & = & xo(x^*)^2 + x^*ox^2 \\ & = & (xox^*)x^* - x^*[x,x^*] + (x^*ox)x - x[x^*,x] \\ & = & x[x,x^*] - x^*[x,x^*] \\ & = & (x-x^*)[x,x^*] \end{array}$$

for all $x \in R$. Substituting $h+k$ for x, where $h \in H(R)$, $k \in S(R)$, we get $2k([k,h]-[h,k]) = 0$ and hence $4k[h,k] = 0$ for all $h \in H(R)$ and $k \in S(R)$. Since $char(R) \neq 2$, the above relation forces that $k[h,k] = 0$ for all $h \in H(R)$ and $k \in S(R)$. Replacing k by $k+k_1$ where $k_1 \in S(R) \cap Z(R)$, we obtain $k_1[h,k] = 0$ for all $h \in H(R)$, $k \in S(R)$ and $k_1 \in S(R) \cap Z(R)$. Since the centre of a prime ring is free from zero divisors we have either $k_1 = 0$ for all $k_1 \in S(R) \cap Z(R)$ or $[h,k] = 0$ for all $h \in (R)$ and $k \in S(R)$. But $S(R) \cap Z(R) \neq 0$, we conclude that $[h,k] = 0$ for all $h \in H(R)$ and $k \in S(R)$. That is R is normal. Hence, application of Lemma 2.1 yields the required result. On the other hand, if $[x,x^*] = 0$ for all $x \in R$, then R is normal. Hence, R is commutative by Lemma 2.1. This proves the lemma. □

3. Main results

We begin our investigations with the following theorem.

Theorem 3.1. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$. If R admits a nonzero left centralizer $T : R \to R$ such that $T([x, x^*]) = 0$ for all $x \in R$, then R is commutative.*

Proof. By the given assumption we have

$$T([x, x^*]) = 0 \tag{3.1}$$

for all $x \in R$. Linearizing (3.1) and using it, we obtain

$$T([x, y^*] + [y, x^*]) = 0 \tag{3.2}$$

for all $x, y \in R$. Replacing y by xx^* in (3.2) and using (3.1), we get

$$\begin{aligned} 0 &= T([x, xx^*] + [xx^*, x^*]) \\ &= T(x[x, x^*] + [x, x^*]x^*) \\ &= T(x)[x, x^*] + T([x, x^*])x^* \\ &= T(x)[x, x^*] \end{aligned}$$

for all $x \in R$. The last relation forces that

$$T(x)[x, x^*] = 0 \tag{3.3}$$

for all $x \in R$. Replacing x by $h + k$, where $h \in H(R)$, $k \in S(R)$, we obtain

$$T(h)[k, h] - T(h)[h, k] + T(k)[k, h] - T(k)[h, k] = 0$$

for all $h \in H(R)$ and $k \in S(R)$. This implies that

$$2T(h)[h, k] + 2T(k)[h, k] = 0$$

for all $h \in H(R)$ and $k \in S(R)$. Since $char(R) \neq 2$, the above expression forces that

$$T(h)[h, k] + T(k)[h, k] = 0 \tag{3.4}$$

for all $h \in H(R)$ *and* $k \in S(R)$. Replacing h by $-h$ in (3.4), we get

$$T(h)[h, k] - T(k)[h, k] = 0 \tag{3.5}$$

for all $h \in H(R)$ *and* $k \in S(R)$. Adding (3.4) and (3.5) and using the fact that $char(R) \neq 2$, we obtain

$$T(h)[h, k] = 0 \tag{3.6}$$

for all $h \in H(R)$ *and* $k \in S(R)$. Replacing h by $h + h^{'}$ in (3.6), where $h^{'} \in H(R) \cap Z(R)$ we obtain

$$T(h^{'})[h, k] = 0 \tag{3.7}$$

for all $h \in H(R), h^{'} \in H(R) \cap Z(R)$ *and* $k \in S(R)$. Replacing k by $h_1k^{'}$ in (3.7), where $h_1 \in H(R)$ and $k^{'} \in S(R) \cap Z(R)$, we get $T(h^{'})[h, h_1]k^{'} = 0$ for all

$h, h_1 \in H(R)$, $h' \in H(R) \cap Z(R)$ and $k^{'} \in S(R) \cap Z(R)$. Using the fact that the centre of a prime ring is free from zero divisors and $S(R) \cap Z(R) \neq 0$, we obtain

$$T(h^{'})[h, h_1] = 0 \tag{3.8}$$

for all $h, h_1 \in H(R)$ and $h^{'} \in H(R) \cap Z(R)$. Now since every $x \in R$ can be uniquely represented as $2x = h + k$, where $h \in H(R)$, $k \in S(R)$. Thus, in view of equations (3.7) and (3.8), we obtain $0 = T(h^{'})[h, 2x] = 2T(h^{'})[h, x]$. Since $char(R) \neq 2$, we arrive at $T(h^{'})[h, x] = 0$ for all $x \in R$, $h \in H(R)$ and $h^{'} \in H(R) \cap Z(R)$. Replacing x by yx in the above equation and using it, we get $T(h^{'})y[h, x] = 0$. Using the primeness of R, we have either $T(h^{'}) = 0$ for all $h^{'} \in H(R) \cap Z(R)$ or $[h, x] = 0$ for all $x \in R$ and $h \in H(R)$. Suppose $T(h^{'}) = 0$ for all $h^{'} \in H(R) \cap Z(R)$. Replacing $h^{'}$ by $(k^{'})^2$, where $k^{'} \in S(R) \cap Z(R)$, we get $T(k^{'})k^{'} = 0$ for all $k^{'} \in S(R) \cap Z(R)$. Since R is prime, the last expression yields that either $T(k^{'}) = 0$ or $k^{'} = 0$. Since $k^{'} = 0$ also implies $T(k^{'}) = 0$, so finally we arrive at $T(k^{'}) = 0$ for all $k^{'} \in S(R) \cap Z(R)$. Thus we have

$$T(h^{'}) = 0 \ for\ all\ h^{'} \in H(R) \cap Z(R). \tag{3.9}$$

$$T(k^{'}) = 0 \ for\ all\ k^{'} \in S(R) \cap Z(R). \tag{3.10}$$

Let $x_1 \in Z(R)$. Since $char(R) \neq 2$, every $x_1 \in Z(R)$ can be uniquely represented as $2x_1 = h_1 + k_1$ where $h_1 \in H(R) \cap Z(R)$ and $k_1 \in S(R) \cap Z(R)$. This implies that $T(2x_1) = T(h_1 + k_1) = T(h_1) + T(k_1) = 0$. This implies that $T(x_1) = 0$ for all $x_1 \in Z(R)$. Now $x_1 \in Z(R)$ implies $x_1 y = yx_1$ for all $y \in R$. This yields $T(x_1)y = T(y)x_1$ for all $y \in R$. This give $T(y)x_1 = 0$ for all $x_1 \in Z(R)$ and $y \in R$. Thus the primeness of R yields that either $x_1 = 0$ for all $x_1 \in Z(R)$ or $T(y) = 0$ for all $y \in R$. Which gives a contradiction since $Z(R) \neq 0$ and T is nontrivial. Thus the only possibility is $[h, x] = 0$ for all $x \in R$ and $h \in H(R)$. That is, R is normal. Hence R is commutative by Lemma 2.1. This completes the proof of the theorem. □

If we replace commutator by anti-commutator in Theorem 3.1, the corresponding result also holds.

Theorem 3.2. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$. If R admits a nonzero left centralizer $T : R \to R$ such that $T(xox^*) = 0$ for all $x \in R$, then R is commutative.*

Proof. We are given that

$$T(xox^*) = 0 \tag{3.11}$$

for all $x \in R$. Replacing x by $x + y$ in (3.11) and using it, we obtain

$$T(xoy^* + yox^*) = 0 \tag{3.12}$$

for all $x, y \in R$. Substituting x^2 for y in (3.12) and using (3.11), we get

$$\begin{aligned} 0 &= T(xo(x^*)^2 + x^2ox^*) \\ &= T((xox^*)x^* - x^*[x,x^*] + (x^*ox)x - x[x^*,x]) \\ &= -T(x^*)[x,x^*] - T(x)[x^*,x] \\ &= T(x)[x,x^*] - T(x^*)[x,x^*]. \end{aligned}$$

The last relation forces that

$$T(x - x^*)[x, x^*] = 0$$

for all $x \in R$. Replacing x by $h + k$, where $h \in H(R)$ and $k \in S(R)$ and using the fact that $char(R) \neq 2$ we get

$$T(k)[h,k] = 0 \tag{3.13}$$

for all $h \in H(R)$ and $k \in S(R)$. Substituting $k + k^{'}$ for k, where $k^{'} \in S(R) \cap Z(R)$ in (3.13), we find that

$$T(k^{'})[h,k] = 0 \tag{3.14}$$

for all $h \in H(R)$, $k \in S(R)$ and $k^{'} \in S(R) \cap Z(R)$. Replacing h by $k_1k_2^{'}$ in (3.14), where $k_1 \in S(R)$ and $k_2^{'} \in S(R) \cap Z(R)$, we get $T(k^{'})[k_1,k]k_2^{'} = 0$ for all $k, k_1 \in S(R)$, $k', k_2^{'} \in S(R) \cap Z(R)$. Using the fact that the centre of a prime ring is free from zero divisors and $S(R) \cap Z(R) \neq 0$, we obtain

$$T(k^{'})[k_1,k] = 0 \tag{3.15}$$

for all $k, k_1 \in S(R)$ and $k^{'} \in S(R) \cap Z(R)$. Now since every $x \in R$ can be uniquely represented as $2x = h + k$, where $h \in H(R)$, $k \in S(R)$, in view of equations (3.14) and (3.15), we obtain $0 = T(k^{'})[2x,k] = 2T(k^{'})[x,k]$. Since $char(R) \neq 2$, we arrive at $T(k^{'})[x,k] = 0$ for all $x \in R$, $k \in S(R)$ and $k^{'} \in S(R) \cap Z(R)$. Replacing x by yx in the above equation and using it, we get $T(k^{'})y[x,k] = 0$. Using the primeness of R, we have either $T(k^{'}) = 0$ for all $k^{'} \in S(R) \cap Z(R)$ or $[x,k] = 0$ for all $x \in R$ and $k \in S(R)$. Suppose $T(k^{'}) = 0$ for all $k^{'} \in S(R) \cap Z(R)$. Substituting $h^{'}k^{'}$ for $k^{'}$, where $h^{'} \in H(R) \cap Z(R)$, we get $T(h^{'}k^{'}) = 0$ i.e., $T(h^{'})k^{'} = 0$ for all $h^{'} \in H(R) \cap Z(R)$ and $k^{'} \in S(R) \cap Z(R)$. Again using the fact that the centre of a prime ring is free from zero divisors we have either $k^{'} = 0$ for all $k^{'} \in S(R) \cap Z(R)$ or $T(h^{'}) = 0$ for all $h^{'} \in H(R) \cap Z(R)$. But $S(R) \cap Z(R) \neq 0$. So we have the only possibility that $T(h^{'}) = 0$ for all $h^{'} \in H(R) \cap Z(R)$. Therefore, we have

$$T(h^{'}) = 0 \; for \; all \; h^{'} \in H(R) \cap Z(R). \tag{3.16}$$

$$T(k^{'}) = 0 \; for \; all \; k^{'} \in S(R) \cap Z(R). \tag{3.17}$$

Henceforth using similar approach as we have used after equations (3.9) and (3.10) in the proof of Theorem 3.1, we get the required result. This finishes the proof of the theorem. □

Theorem 3.3. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$. If R admits a left centralizer $T : R \to R$ such that $T([x, x^*]) \pm [x, x^*] = 0$ for all $x \in R$ and $T(x) \neq \pm x$, then R is commutative.*

Proof. First we consider the case

$$T([x, x^*]) - [x, x^*] = 0 \tag{3.18}$$

for all $x \in R$. Linearization of (3.18) yields that

$$T([x, y^*]) - [x, y^*] + T([y, x^*]) - [y, x^*] = 0 \tag{3.19}$$

for all $x, y \in R$. Replacing y by x^2 in (3.19) and using (3.18), we obtain

$$T(x^*)[x, x^*] + T(x)[x, x^*] - x^*[x, x^*] - x[x, x^*] = 0$$

for all $x \in R$. The above relation can be further written as

$$(T(x + x^*) - (x + x^*))[x, x^*] = 0 \tag{3.20}$$

for all $x \in R$. Taking $x = h + k$ where $h \in H(R)$ and $k \in S(R)$ in (3.20) and using the fact that $char(R) \neq 2$, we get

$$(T(h) - h)[h, k] = 0 \tag{3.21}$$

for all $h \in H(R), k \in S(R)$. Let $h^{'} \in H(R) \cap Z(R)$. Replacing h by $h + h^{'}$ in (3.21), we obtain

$$(T(h^{'}) - h^{'})[h, k] = 0 \tag{3.22}$$

for all $h^{'} \in H(R) \cap Z(R), h \in H(R)$ *and* $k \in S(R)$. Replacing k by $h_1 k^{'}$ in (3.22), where $h_1 \in H(R)$ and $k^{'} \in S(R) \cap Z(R)$, we get $(T(h^{'}) - h^{'})[h, h_1]k^{'} = 0$ for all $h, h_1 \in H(R)$, $h' \in H(R) \cap Z(R)$ and $k^{'} \in S(R) \cap Z(R)$. Using the fact that the centre of a prime ring is free from zero divisors and $S(R) \cap Z(R) \neq (0)$, we obtain

$$(T(h^{'}) - h^{'})[h, h_1] = 0 \tag{3.23}$$

for all $h, h_1 \in H(R)$ and $h^{'} \in H(R) \cap Z(R)$. Now since every $x \in R$ can be uniquely represented as $2x = h + k$, where $h \in H(R)$, $k \in S(R)$, in view of equations (3.22) and (3.23), we obtain $0 = (T(h^{'}) - h^{'})[h, 2x] = 2(T(h^{'}) - h^{'})[h, x]$. Since $char(R) \neq 2$, we arrive at $(T(h^{'}) - h^{'})[h, x] = 0$ for all $x \in R$, $h \in H(R)$ and $h^{'} \in H(R) \cap Z(R)$. Replacing x by yx in the above equation and using it, we get $(T(h^{'}) - h^{'})y[h, x] = 0$. Using the primeness of R, we have either $T(h^{'}) - h^{'} = 0$ for all $h^{'} \in H(R) \cap Z(R)$ or $[h, x] = 0$ for all $x \in R$ and $h \in H(R)$. Suppose $T(h^{'}) = h^{'}$ for all $h^{'} \in H(R) \cap Z(R)$. Replacing $h^{'}$ by $(k^{'})^2$ where $k^{'} \in S(R) \cap Z(R)$, we have $T((k^{'})^2) - (k^{'})^2 = 0$. This implies $(T(k^{'}) - k^{'})k^{'} = 0$ for all $k^{'} \in S(R) \cap Z(R)$. Using the primeness of R we have either $k^{'} = 0$ or $T(k^{'}) = k^{'}$. Since $k^{'} = 0$ implies $T(k^{'}) = k^{'}$, we have $T(k^{'}) = k^{'}$ for all $k^{'} \in S(R) \cap Z(R)$. Thus we have

$$T(h^{'}) = h^{'} \textit{ for all } h^{'} \in H(R) \cap Z(R). \tag{3.24}$$

$$T(k^{'}) = k^{'} \textit{ for all } k^{'} \in S(R) \cap Z(R). \tag{3.25}$$

Let $x_1 \in Z(R)$. Since $char(R) \neq 2$, every $x_1 \in Z(R)$ can be uniquely represented as $2x_1 = h_1 + k_1$ where $h_1 \in H(R) \cap Z(R)$ and $k_1 \in S(R) \cap Z(R)$. This implies that $T(2x_1) = T(h_1 + k_1) = T(h_1) + T(k_1) = h_1 + k_1 = 2x_1$. Thus we obtain

$$T(x_1) = x_1 \textit{ for all } x_1 \in Z(R). \tag{3.26}$$

But $x_1 \in Z(R)$ implies $x_1 y = y x_1$ for all $y \in R$. This yields $T(x_1)y = T(y)x_1$ for all $y \in R$. Using (3.26) we obtain $(T(y) - y)x_1 = 0$ for all $x_1 \in Z(R)$ and $y \in R$. Using the primeness of R we have $x_1 = 0$ for all $x_1 \in Z(R)$ or $T(y) = y$ for all $y \in R$. Which gives a contradiction, since $Z(R) \neq 0$ and $T(x) \neq x$. Thus the only possibility is that, $[h, k] = 0$ for all $h \in H(R)$ *and* $k \in S(R)$ and hence R is normal. In view of Lemma 2.1, we conclude that R is commutative.

By the same argument, we obtain the similar conclusion in the case $T([x, x^*]) + [x, x^*] = 0$ for all $x \in R$. This proves the theorem completely. □

Theorem 3.4. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$. If R admits a left centralizer $T : R \to R$ such that $T(xox^*) \pm (xox^*) = 0$ for all $x \in R$ and $T(x) \neq \pm x$, then R is commutative.*

Proof. First we consider the case

$$T(xox^*) - (xox^*) = 0 \tag{3.27}$$

for all $x \in R$. Linearizing the above relation, we get

$$T(xoy^*) - xoy^* + T(yox^*) - yox^* = 0 \tag{3.28}$$

for all $x, y \in R$. Replacing y by x^2 in (3.28) and using (3.27), we obtain

$$-T(x^*)[x, x^*] + T(x)[x, x^*] + x^*[x, x^*] - x[x, x^*] = 0$$

for all $x \in R$. This can be further written as

$$(T(x - x^*) - (x - x^*))[x, x^*] = 0 \tag{3.29}$$

for all $x \in R$. Replacing x by $h + k$, where $h \in H(R)$ and $k \in S(R)$ and using the fact that $char(R) \neq 2$, we get

$$(T(k) - k)[h, k] = 0 \tag{3.30}$$

for all $h \in H(R), k \in S(R)$. Let $k^{'} \in S(R) \cap Z(R)$. Replacing k by $k + k^{'}$ in (3.30), we obtain

$$(T(k^{'}) - k^{'})[h, k] = 0 \tag{3.31}$$

for all $k^{'} \in S(R) \cap Z(R), h \in H(R)$ *and* $k \in S(R)$. Replacing h by $k_1k_2^{'}$ in (3.31), where $k_1 \in S(R)$ and $k_2^{'} \in S(R) \cap Z(R)$, we get $(T(k^{'}) - k^{'})[k_1, k]k_2^{'} = 0$ for all $k, k_1 \in S(R)$, $k', k_2^{'} \in S(R) \cap Z(R)$. Using the fact that the centre of a prime ring is free from zero divisors and $S(R) \cap Z(R) \neq (0)$, we obtain

$$(T(k^{'}) - k^{'})[k_1, k] = 0 \tag{3.32}$$

for all $k, k_1 \in S(R)$ and $k^{'} \in S(R) \cap Z(R)$. Now since every $x \in R$ can be uniquely represented as $2x = h + k$, where $h \in H(R)$, $k \in S(R)$, in view of equations (3.31) and (3.32), we obtain $0 = (T(k^{'}) - k^{'})[2x, k] = 2(T(k^{'}) - k^{'})[x, k]$. Since $char(R) \neq 2$, we arrive at $(T(k^{'}) - k^{'})[x, k] = 0$ for all $x \in R$, $k \in S(R)$ and $k^{'} \in S(R) \cap Z(R)$. Replacing x by yx in the above equation and using it, we get $(T(k^{'}) - k^{'})y[x, k] = 0$. Using the primeness of R, we have either $T(k^{'}) = k^{'}$ for all $k^{'} \in S(R) \cap Z(R)$ or $[x, k] = 0$ for all $x \in R$ and $k \in S(R)$. Suppose $T(k^{'}) = k^{'}$ for all $k^{'} \in S(R) \cap Z(R)$. Replacing $k^{'}$ by $h^{'}k^{'}$, where $h^{'} \in H(R) \cap Z(R)$ we get $T(h^{'})k^{'} = h^{'}k^{'}$. This implies $(T(h^{'}) - h^{'})k^{'} = 0$ for all $h^{'} \in H(R) \cap Z(R)$ and $k^{'} \in S(R) \cap Z(R)$. Again using the fact that the centre of a prime ring is free from zero divisors, we have $T(h^{'}) = h^{'}$ for $h^{'} \in H(R) \cap Z(R)$ or $k^{'} = 0$ for all $k^{'} \in S(R) \cap Z(R)$. But since $S(R) \cap Z(R) \neq (0)$ we have $T(h^{'}) = h^{'}$ *for all* $h^{'} \in H(R) \cap Z(R)$. Therefore we find that

$$T(h^{'}) = h^{'} \ for \ all \ h^{'} \in H(R) \cap Z(R). \tag{3.33}$$

$$T(k^{'}) = k^{'} \ for \ all \ k^{'} \in S(R) \cap Z(R). \tag{3.34}$$

Hence using the same approach as we have used after equations (3.24) and (3.25) in the proof of the Theorem 3.3 we get the required result.

By the same argument, we obtain the similar conclusion in the case $T(xox^*) + (xox^*) = 0$ for all $x \in R$. This proves the theorem. □

Theorem 3.5. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$. If R admits a left centralizer $T : R \to R$ such that $T([x, x^*]) \pm (xox^*) = 0$ for all $x \in R$ and $T(x) \neq \pm x$, then R is commutative.*

Proof. First we consider the case

$$T([x, x^*]) - (xox^*) = 0 \tag{3.35}$$

for all $x \in R$. Linearizing the above relation, we get

$$T([x, y^*]) - xoy^* + T([y, x^*]) - yox^* = 0 \tag{3.36}$$

for all $x, y \in R$. Replacing y by x^2 in (3.36) and using (3.35), we obtain

$$T(x^*)[x, x^*] + T(x)[x, x^*] + x^*[x, x^*] + x[x, x^*] = 0$$

for all $x \in R$. This can be further written as

$$(T(x + x^*) + (x + x^*))[x, x^*] = 0 \tag{3.37}$$

for all $x \in R$. Replacing x by $h+k$, where $h \in H(R)$ and $k \in S(R)$ and using the fact that $char(R) \neq 2$, we get

$$(T(h)+h)[h,k]=0 \tag{3.38}$$

for all $h \in H(R), k \in S(R)$. Henceforth using the similar approach with necessary variations as we have used after equation (3.21) in the proof of the Theorem 3.3, we are forced to conclude that R is normal. Further in view of Lemma 2.1, we get the commutativity of R.

By the same argument, we obtain the similar conclusion in case $T([x,x^*])+(xox^*)=0$ for all $x \in R$. This completes the proof of theorem. □

Acknowledgments

The authors are greateful to the learned referees for carefully reading the manuscript. The valuable suggestions have simplified and clarified the paper greatly. Moreover, the proof of Lemma 2.1 is based on referee's observations. This research is partially supported by a Major Research Project funded by U.G.C. (Grant No. 39-37/2010(SR)).

References

[1] M. Ashraf and S. Ali, On left multipliers and the commutativity of prime rings, *Demonstratio Math.* **41** (2008), no. 4, 764–771.
[2] A. Ali and M. Yasen, A note on automorphisms of prime and semiprime rings, *J. Math. Kyoto Univ.* **45** (2005), no. 2, 243–246.
[3] N. Divinsky, On commuting automorphisms of rings, *Trans. Roy. Soc. Canada. Sect. III. (3)* **49** (1955), 19–22.
[4] I. R. Hentzel and T. El-Sayiad, Left centralizers of rings that are not semiprime, *Rocky Mountain J. Math.* **41** (2011), no. 5, 1471–1482.
[5] I. N. Herstein, Rings with Involution, Chicago Lectures in Mathematics, The University of Chicago Press, Chicago, 1976.
[6] J. Luh, A note on commuting automorphisms of rings, *Amer. Math. Monthly* **77** 1970 61–62.
[7] L. Oukhtite, Left Multipliers and Lie ideals in rings with involution, *Int. J. Open Probl. Comput. Sci. Math.* **3** (2010), no. 3, 267–277.
[8] L. Oukhtite, Left multipliers and Jordan ideals in rings with involution, *Afr. Diaspora J. Math.* **11** (2011), no. 1, 24–28.
[9] E. C. Posner, Derivations in prime rings, *Proc. Amer. Math. Soc.* **8** (1957) 1093–1100.
[10] M. F. Smiley, Remarks on commuting automorphisms, *Amer. Math. Monthly* **63** (1956) 466–470.
[11] J. Vukman, An identity related to centralizer in semiprime rings, *Comment. Math. Univ. Carolin.* **40** (1999), no. 3, 447–456.
[12] J. Vukman, Centralizer on semiprime rings, *Comment. Math. Univ. Carolin* **42** (2001), no. 2, 237–245.
[13] J. Vukman and K. U. Irena, On centralizers of semiprime rings with involution, *Studia Scientiarum Math. Hungarica* **43** (2006), no. 1, 61–67.
[14] B. Zalar, On centralizer of semiprime rings, *Comment. Math. Univ. Carolinae,* **32** (1991), no. 4, 609–614.

(Shakir Ali) Department of mathematics, Faculty of science & Arts-Rabigh, King Abdulaziz University, Jaddeh 21589, Saudi Arabia

E-mail address: shakir50@rediffmail.com; shshah@kau.edu.sa

(Nadeem Ahmad Dar) Department of mathematics, Aligarh Muslim University, Aligarh-202002, Aligarh, India

E-mail address: ndmdarlajurah@gmail.com