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GROUPS WITH ONE CONJUGACY CLASS OF NON-NORMAL SUBGROUPS - A SHORT PROOF

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Abstract. For a finite group G let $\nu(G)$ denote the number of conjugacy classes of non-normal subgroups of G. We give a short proof of a theorem of Brandl, which classifies finite groups with $\nu(G) = 1$.
Keywords: Dedekind groups; conjugacy classes of non-normal subgroups; classification of finite groups.
MSC(2010): Primary: 20D25; Secondary: 20E45.

1. Introduction

Let G be a finite group. We denote by $\nu(G)$ the number of conjugacy classes of non-normal subgroups of G. Obviously, $\nu(G) = 0$ if and only if G is Dedekind. In 1995, Brandl [1] classified finite groups with $\nu(G) = 1$. In this paper we give a proof for Brandl's theorem much shorter.

Our notation is standard; for example, $\mathbb{Z}_n$, D_{2n} and Q_{2^n} denote the cyclic group of order n, the dihedral group of order $2n$ and the generalized quaternion group of order 2^n, respectively.

2. Proof of the main Theorem

Theorem 2.1 (Brandl)**.** *Let G be a finite group. Then $\nu(G) = 1$ if and only if*

(i) *$G \cong Q \rtimes P$ is a non-abelian split extension with $[Q, \Phi(P)] = 1$, where $Q \cong \mathbb{Z}_q$ and $P \cong \mathbb{Z}_{p^n}$, p and q primes with $p \mid q - 1$; or*

(ii) *$G \cong \langle g, h | g^{p^n} = h^p = 1,\ g^h = g^{1+p^{n-1}} \rangle$, where p is a prime, $n \geq 2$ and $n \geq 3$ if $p = 2$.*

Proof. We assume first that G is a finite group such that $\nu(G) = 1$. We prove the necessary condition by distinguishing the non-nilpotent and nilpotent cases.

Case 1: G is a non-nilpotent group.
Let P be a non-normal Sylow p-subgroup of G. Then $\mathcal{N}_G(P) \ntrianglelefteq G$. Suppose

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that M is a maximal subgroup of G containing $\mathcal{N}_G(P)$. By the Frattini argument $M \ntrianglelefteq G$. Since $\nu(G) = 1$, then P must be a maximal subgroup of G. Since any maximal subgroup of P is normal in G, P is cyclic. Let $q \neq p$ be a prime divisor of $|G|$ and Q be the Sylow q-subgroup of G. Since any subgroup of Q is normal in G and P is maximal, we have $|Q| = q$. Moreover, $\Phi(P) \trianglelefteq G$ implies $[\Phi(P), Q] = 1$ and $p \mid q - 1$. So G is a group as in part (i).

Case 2: G is a nilpotent group.
Suppose that G is not of prime power order. Then $G = A \times B$ where $1 \neq A, B \leqslant G$ and $(|A|, |B|) = 1$. Since $\nu(A \times B) = 1$, we deduce that both $\nu(A)$ and $\nu(B)$ can not be zero simultaneously. Without lose of generality we may assume that $\nu(A) = 1$. If $A_1 \ntrianglelefteq A$, then $A_1 \times 1$ and $A_1 \times B$ are non-conjugate non-normal subgroups of G which implies $\nu(G) \geq 2$, a contradiction. Therefore, G is a p-group, p a prime. Let H be a non-normal subgroup of G. Then H is cyclic, since any maximal subgroup of H is normal in G.

We split the proof of case 2 into the following steps:

Step 1: $|H| = p$, *$Z(G)$ is cyclic and G' is the only central subgroup of order p.*

Assume that $|H| > p$ and let $h \in H$ be of order p. We note that G is not a generalized quaternion group, since otherwise it would have a quotient isomorphic to D_8, which is not possible because $\nu(D_8) = 2$. Hence we may consider a central subgroup $Z = \langle z \rangle$ of order p such that $Z \nleqslant H$. Then we note that $\nu(G/Z) = 0$, i.e., G/Z is a Dedekind group. If p is odd, then G/Z is abelian, which implies $G' = Z$. Since $zh \in Z(G)$, and $zh \notin H$, $G' = \langle zh \rangle$, a contradiction. Therefore, $p = 2$.

Since $HZ \trianglelefteq G$, it follows that $H^t \leqslant HZ$ for all $t \in G$. Let $g \notin \mathcal{N}_G(H)$. Then $\langle g \rangle \trianglelefteq G$, since $g \notin HZ$. Moreover, $|g| > 2$; otherwise $g \in Z(G) \leqslant \mathcal{N}_G(H)$, a contradiction. Now we can consider $Z \leqslant \Omega_1(G)$ such that $Z \nleqslant H$ and $Z \nleqslant \langle g \rangle$. If G/Z were abelian, $Z = [H, g] \leqslant \langle g \rangle$, which is not the case. Then G/Z is a non-abelian Dedekind group and so it has exponent 4. Consequently, $|H| = |g| = 4$ and $g^x = g^{-1}$ where $H = \langle x \rangle$. It follows that $\langle x, g \rangle$ has a quotient isomorphic to D_8, a contradiction. Therefore $|H| = p$.
Let Z_1 and Z_2 be two central subgroup of order p. Since $HZ_i \trianglelefteq G$, for $i = 1, 2$, then $H = HZ_1 \cap HZ_2 \trianglelefteq G$, a contradiction. Thus $Z(G)$ is cyclic. Let $Z \leqslant Z(G)$, $|Z| = p$. Now if G/Z were not abelian, then G would contain a subgroup K such that $K/Z \cong Q_8$, which would imply that $Z \lneqq Z(K)$. Therefore, $K/Z(K) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and then $|K'| = 2$. Since $K \trianglelefteq G$ it would follow that $K' = Z$, a contradiction. Therefore, $G' = Z$ has order p.

Step 2: *$G = MH$ where M is a cyclic maximal subgroup of G and* $[\Phi(M), H] = 1$.

Let $H = \langle h \rangle$. By step 1, $\Phi(G) \leqslant Z(G)$, so there exists a maximal subgroup M of G such that $G = MH$. Since any subgroup of M must be normal in G, $\Omega_1(M) \leqslant Z(G)$, therefore, either M is cyclic, as desired, or M is generalized quaternion. Assume that M in not cyclic. Then it should be $M \cong Q_8$. We note that $G' = Z(M) \leqslant Z(G)$. Also $H^g \leqslant HZ(M)$ for all $g \in G$. Since H is not normal in G, there is $x \in M \backslash Z(M)$ such that $h^x \neq h$. Therefore, $x^h = x^{-1}$. Moreover, $hx \notin HZ(M)$. It follows now that $\langle hx \rangle$ is a normal subgroup of order 2, and consequently, $G' = Z(M) = \langle hx \rangle$, a contradiction.

Now suppose that $M = \langle g \rangle$ and $|g| = p^n$. Since $[g, h] \in M$ is of order p, we may consider $[g, h] = g^{p^{n-1}}$. Therefore, $g^h = g^{1+p^{n-1}}$. If $p = n = 2$ then $g^h = g^{-1}$ and $G \cong D_8$, so $\nu(G) = 2$, a contradiction. Thus G is a group as in part (ii), as desired.

Conversely, let $G = Q \rtimes P$ be a group as in part (i). Then $P \ntrianglelefteq G$. Let H be a non-normal subgroup of G. Since H does not contain Q and is not properly contained in P, then H is conjugate with P. So, $\nu(G) = 1$.
Let $G = \langle g, h \rangle$ be a group as in part (ii). Then $G' \leqslant Z(G) = \langle g^p \rangle$ has order p. Let $x = g^i h^j \in G$. If $p^{n-1} \nmid i$ and $j \geq 0$, then $x^p = g^{pi}[h, g]^{ijp(p-1)/2} \in Z(G)$, thus $G' \leqslant \langle x \rangle$. Otherwise $x^p = 1$ and $x \in \langle g^{p^{n-1}}, h \rangle$. Therefore, $\Omega_1(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Let H be a non-normal subgroup of G. As H does not contain any element of order greater than p, $\exp(H) = p$. Hence, $H \leqslant \Omega_1(G)$ and so $\nu(G) = 1$. $\square$

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