Title:
The Libera operator on Dirichlet spaces

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THE LIBERA OPERATOR ON DIRICHLET SPACES

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(Communicated by Hamid Reza Ebrahimi Vishki)

ABSTRACT. In this paper, we consider the boundedness of the Libera operator on Dirichlet spaces in terms of the Schur test. Moreover, we get its point spectrum and norm.

Keywords: Libera operator, Dirichlet spaces, point spectrum, norm.


1. Introduction

Let \( \mathbb{D} \) be the unit disk in the complex plane \( \mathbb{C} \). Denote by \( \mathcal{D}_\alpha \), \( \alpha \in \mathbb{R} \), the Hilbert space of all analytic functions \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) in \( \mathbb{D} \) with \( f(0) = 0 \) and

\[
\|f\|_{\mathcal{D}_\alpha}^2 = \sum_{n=1}^{\infty} n^{1-\alpha}|a_n|^2 < \infty.
\]

Note that the classical Dirichlet space \( \mathcal{D} \) is obtained for \( \alpha = 0 \). See [1,2,7,9,11] for \( \mathcal{D}_\alpha \) spaces.

The Libera operator \( \mathcal{L} \) on \( \mathcal{D}_\alpha \), \( \alpha < 2 \), is defined by

\[
\mathcal{L} f(z) = \frac{1}{z-1} \int_1^z f(w) \, dw - \int_0^1 f(w) \, dw
\]

for all \( f \in \mathcal{D}_\alpha \). See [4,6,10] for the Libera operator on other spaces. Rhaly [8] gave a series of results about the Libera operator on the classical Dirichlet space \( \mathcal{D} \). In this paper, we consider the Libera operator on \( \mathcal{D}_\alpha \) spaces. By [2], if \( \alpha > 0 \), then there exists a constant \( C \) such that

\[
|f(z)| \leq C\|f\|_{\mathcal{D}_\alpha} \left( \frac{1}{1-|z|} \right)^{\alpha/2}
\]

for all \( f \in \mathcal{D}_\alpha \). Clearly, \( \mathcal{D}_\alpha \subseteq \mathcal{D}_\beta \) when \( \alpha \leq \beta \). Consequently, the Libera operator \( \mathcal{L} \) is well defined on \( \mathcal{D}_\alpha \), \( \alpha < 2 \).
Let $e_n(z) = n^{\frac{\alpha - 1}{2}} z^n$, $n = 1, 2, \ldots$. Then the set $\{e_n\}_{n=1}^\infty$ forms an orthonormal basis for $D_\alpha$. Note that

\[
\mathcal{L}e_n(z) = \frac{1}{z - 1} \int_1^z n^{\frac{\alpha - 1}{2}} w^n dw - \int_0^1 n^{\frac{\alpha - 1}{2}} w^n dw
= \frac{n^{\frac{\alpha - 1}{2}}}{n + 1} (z^n + z^{n-1} + \cdots + z)
= \frac{n^{\frac{\alpha - 1}{2}}}{n + 1} \left( n^{\frac{1-\alpha}{2}} e_n(z) + (n - 1)^{\frac{1-\alpha}{2}} e_{n-1}(z) + \cdots + e_1(z) \right).
\]

Thus, $\mathcal{L}$ has matrix entries

\[
a_{ij} = \langle \mathcal{L} e_j, e_i \rangle = \begin{cases} 0 & i > j \geq 1, \\ \frac{1}{j+1} \left( \frac{i}{j} \right)^{\frac{1-\alpha}{2}} & j \geq i \geq 1. \end{cases}
\]

If $f(z) = \sum_{n=1}^\infty f(n)e_n(z)$, then

\[
\mathcal{L}f(z) = \sum_{n=1}^\infty \sum_{m=n}^\infty \frac{f(m)}{m+1} \left( \frac{n}{m} \right)^{\frac{1-\alpha}{2}} e_n(z).
\]

For $\alpha > -1$, a direct calculation gives that $\|f\|_{D_\alpha}^2$ is comparable with

\[
\int_\mathbb{D} |f'(z)|^2 (1 - |z|^2)^\alpha \, dx dy.
\]

Pavlović [6] investigated the Libera operator on mixed norm spaces. In particular, under the seminorm (1.1) of $D_\alpha$ spaces, he showed that if $\alpha > -1$, then $\mathcal{L}$ is bounded on $D_\alpha$ if and only if $\alpha < 2$. In this paper, by different technique, we give that $\mathcal{L}$ is bounded on $D_\alpha$ if and only if $\alpha < 2$. Particularly, we obtain that $\mathcal{L}$ is also bounded on $D_\alpha$ for $\alpha \leq -1$. Furthermore, the point spectrum and norm of $\mathcal{L}$ on the $D_\alpha$ space are also considered.

2. Main results

Let $T$ be a bounded linear operator on a Hilbert space $H$. Recall that the point spectrum $\sigma_p(T)$ of $T$ is

\[
\sigma_p(T) = \{ \lambda \in \mathbb{C} : \ker(\lambda I - T) \neq 0 \},
\]

where $I$ is the identical operator of $H$. The point spectrum of $\mathcal{L}$ on the Dirichlet space $D$ was obtained in [8]. Using the Schur test, we get the following result.

**Theorem 2.1.** The Libera operator $\mathcal{L}$ is bounded on $D_\alpha$ if and only if $\alpha < 2$. Moreover,

\[
\sigma_p(\mathcal{L}) = \{ \lambda \in \mathbb{C} : |\lambda - \frac{1}{2-\alpha}| < \frac{1}{2-\alpha} \} \cup \left\{ \frac{1}{n} : n = 2, 3, \ldots \right\}.
\]
Proof. Let \( p_i = \frac{i^{1/2}}{i+1}, \ i = 1, 2, \cdots \). Then
\[
\sum_{i=1}^{\infty} a_{ij} p_i = \frac{1}{(j+1)^{1/\alpha}} \sum_{i=1}^{j} \frac{j^{(2-\alpha)/2}}{i+1} \\
\leq \frac{1}{(j+1)^{1/\alpha}} \sum_{i=1}^{j} \int_{i-1}^{i} \max\{t^{-\alpha/2}, (t+1)^{-\alpha/2}\} \ dt \\
\leq \frac{1}{(j+1)^{1/\alpha}} \int_{0}^{j} \max\{t^{-\alpha/2}, (t+1)^{-\alpha/2}\} \ dt.
\]
It follows that
\[
\sum_{i=1}^{\infty} a_{ij} p_i \leq \frac{2}{2-\alpha} p_j, \ 0 \leq \alpha < 2,
\]
and
\[
\sum_{i=1}^{\infty} a_{ij} p_i \leq \frac{2^{2-\frac{\alpha}{2}}}{2-\alpha} p_j, \ \alpha < 0.
\]
On the other hand,
\[
\sum_{j=1}^{\infty} a_{ij} p_j = \sum_{j=1}^{\infty} \frac{1}{j+1} \left( \frac{i}{j} \right)^{1-\alpha} j^{1/2} \\
\leq i^{1-\alpha} \sum_{j=1}^{\infty} \int_{i-1}^{i} \frac{t^{\alpha/2} \max\{t^{\alpha/2}, (t+1/2)^{\alpha/2}\}}{t^2} \ dt \\
\leq i^{1-\alpha} \int_{i}^{\infty} \frac{t^{\alpha/2} \max\{t^{\alpha/2}, (t+1/2)^{\alpha/2}\}}{t^2} \ dt.
\]
Hence,
\[
\sum_{j=1}^{\infty} a_{ij} p_j \leq \frac{2}{2-\alpha} i^{-1/2} \leq \frac{4}{2-\alpha} p_i, \ 0 \leq \alpha < 2,
\]
and
\[
\sum_{j=1}^{\infty} a_{ij} p_j \leq \frac{2^{1-\frac{\alpha}{2}}}{2-\alpha} i^{-1/2} \leq \frac{2^{2-\frac{\alpha}{2}}}{2-\alpha} p_i, \ \alpha < 0.
\]
By the Schur test (see [3, P. 24]), we get that \( \mathcal{L} \) is bounded on \( \mathcal{D}_\alpha \) for \( \alpha < 2 \). For \( \alpha \geq 2 \), let
\[
f_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{2-\alpha} \log(n+1)},
\]
that is,
\[
f_\alpha(z) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n \log(n+1)}} r_\alpha(z).
\]
Then
\[
\|f_\alpha\|_{\mathcal{D}_\alpha}^2 = \sum_{n=1}^{\infty} \frac{1}{n \log^2(n+1)} < \infty.
\]
Note that $\alpha \geq 2$. We deduce that

$$
\|L f\|_{D_\alpha}^2 = \sum_{n=1}^{\infty} \left( \sum_{m=n}^{\infty} \frac{1}{\sqrt{m(m+1)\log(m+1)}} \left( \frac{n}{m} \right)^{\frac{1-\alpha}{2}} \right)^2
\geq \left( \sum_{m=1}^{\infty} \frac{1}{\sqrt{m(m+1)\log(m+1)}} \left( \frac{1}{m} \right)^{\frac{1-\alpha}{2}} \right)^2
\geq \left( \sum_{m=1}^{\infty} \frac{1}{(m+1)\log(m+1)} \right)^2 = +\infty,
$$

which implies that $L$ is not bounded on $D_\alpha$, $\alpha \geq 2$. Thus, $L$ is bounded on $D_\alpha$ if and only if $\alpha < 2$.

We now compute the point spectrum of $L$. If $f(z) = \sum_{n=1}^{\infty} f(n)e_n(z)$, then

$$
Lf(z) = \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \frac{f(m)}{m+1} \left( \frac{n}{m} \right)^{\frac{1-\alpha}{2}} e_n(z).
$$

Thus,

$$
Lf(n) = n^{\frac{1-\alpha}{2}} \sum_{m=n}^{\infty} \frac{f(m)}{(m+1)m^{\frac{1-\alpha}{2}}}, \quad n = 1, 2, \ldots.
$$

Consequently,

$$
n^{\frac{\alpha-1}{2}} Lf(n) - (n+1)^{\frac{\alpha-1}{2}} Lf(n+1) = \frac{f(n)}{(n+1)n^{\frac{1-\alpha}{2}}},
$$

which gives

$$
f(n) = (n+1)Lf(n) - n^{\frac{1-\alpha}{2}} (n+1)^{\frac{\alpha+1}{2}} Lf(n+1).
$$

If $Lf = \lambda f$, then

$$
\lambda n^{\frac{1-\alpha}{2}} (n+1)^{\frac{\alpha+1}{2}} f(n+1) = [\lambda(n+1) - 1] f(n).
$$

If $\lambda = 0$, then $f(n) = 0$ for all $n$. Hence, $0 \notin \sigma_p(L)$. It follows that

$$
f(n+1) = \left( \frac{n+1}{n} \right)^{\frac{1-\alpha}{2}} \left[ 1 - \frac{1}{\lambda(n+1)} \right] f(n).
$$

Thus,

$$
f(n) = n^{\frac{1-\alpha}{2}} \prod_{j=2}^{n} \left( 1 - \frac{1}{\lambda j} \right) f(1), \quad n \geq 2.
$$

We now want to know what nonzero values of $\lambda$ will result in the convergence of $\sum_{n=1}^{\infty} |f(n)|^2$. Clearly, $\left\{ \frac{1}{n} : n = 2, 3, \ldots \right\} \subseteq \sigma_p(L)$. Suppose that $\lambda \notin \left\{ \frac{1}{n} :
\[ n = 2, 3, \cdots \}. \text{ Then} \]

\[ \frac{|f(n)|^2}{|f(n+1)|^2} - 1 = \frac{(n + 1)^{2n^{1-\alpha} + (n + 1)^{3-\alpha}} |\lambda|^2 + (n + 1)^{2-\alpha}(\lambda + \bar{\lambda}) - (n + 1)^{1-\alpha}}{(n + 1)^{3-\alpha}|\lambda|^2 - (n + 1)^{2-\alpha}(\lambda + \bar{\lambda}) + (n + 1)^{1-\alpha}}. \]

Hence,

\[ \lim_{n \to \infty} n \left[ \frac{|f(n)|^2}{|f(n+1)|^2} - 1 \right] = (\alpha - 1)|\lambda|^2 + (\lambda + \bar{\lambda}) \cdot \frac{1}{|\lambda|^2}. \]

By Raabe’s test \[5, \text{Theorem II, P. 396}], \[\sum_{n=1}^{\infty} |f(n)|^2\] converges for \(\frac{\lambda + \bar{\lambda}}{|\lambda|^2} + \alpha - 1 > 1\) and diverges for \(\frac{\lambda + \bar{\lambda}}{|\lambda|^2} + \alpha - 1 < 1\). Namely, the series converges for \(|\lambda - \frac{1}{2-\alpha}| < \frac{1}{2-\alpha}\) and diverges for \(|\lambda - \frac{1}{2-\alpha}| > \frac{1}{2-\alpha}\). If \(|\lambda - \frac{1}{2-\alpha}| = \frac{1}{2-\alpha}\), then

\[ \lim_{n \to \infty} \ln n \left\{ \frac{n \left[ \frac{|f(n)|^2}{|f(n+1)|^2} - 1 \right]}{n(n + 1)^{1-\alpha}} \right\} = 0. \]

By \[5, \text{Theorem III, P. 396}], we obtain that the series \(\sum_{n=1}^{\infty} |f(n)|^2\) diverges if \(|\lambda - \frac{1}{2-\alpha}| = \frac{1}{2-\alpha}\). Therefore,

\[ \sigma_p(\mathcal{L}) = \{ \lambda \in \mathbb{C} : |\lambda - \frac{1}{2-\alpha}| < \frac{1}{2-\alpha} \} \cup \left\{ \frac{1}{n} : n = 2, 3, \cdots \right\}. \]

\[ \square \]

For \(\alpha < 2\), let \(\|\mathcal{L}\|_{D_\alpha}\) be the norm of \(\mathcal{L}\) on \(D_\alpha\) spaces. H. Rhaly \[8] proved that \(\|\mathcal{L}\|_D = 1\). Now we give the following result.

**Theorem 2.2.** Let \(0 \leq \alpha < 2\). Then \(\|\mathcal{L}\|_{D_\alpha} = \frac{2}{2-\alpha}\).
Proof. Let $p_i = i^{-1/2}$, $q_j = \frac{j^{1/2}}{j + 1}$, $i, j = 1, 2, \ldots$. Note that $0 \leq \alpha < 2$. We have

$$
\sum_{i=1}^{\infty} a_{ij} p_i = \frac{1}{(j+1)^{1/2}} \sum_{i=1}^{\infty} i^{-\alpha/2} \\
\leq \frac{1}{(j+1)^{1/2}} \sum_{i=1}^{j} t^{-\alpha/2} dt \\
= \frac{1}{(j+1)^{1/2}} \int_0^1 t^{-\alpha/2} dt \\
= \frac{2}{2 - \alpha} j^{1/2} = \frac{2}{2 - \alpha} q_j.
$$

Similarly,

$$
\sum_{j=1}^{\infty} a_{ij} q_j = \frac{1}{i^{1/2}} \sum_{j=1}^{\infty} j^{1/2} (j+1)^{1/2} \leq \frac{1}{i^{1/2}} \sum_{j=1}^{\infty} \int_j^{j+1} t^{\alpha/2-2} dt \\
= \frac{1}{i^{1/2}} \int_i^{\infty} t^{\alpha/2-2} dt = \frac{2}{2 - \alpha} i^{-1/2} = \frac{2}{2 - \alpha} p_i.
$$

By the Schur test, we get that $\|L\|_{D,2} \leq \frac{2}{2 - \alpha}$. Here we give the details for the completeness. If $f(z) = \sum_{n=1}^{\infty} f(n)e_n(z)$, then

$$
\|Lf\|^2_{D,\alpha} = \sum_{n=1}^{\infty} \left| \sum_{m=1}^{\infty} a_{nm} f(m) \right|^2 q_m \\
\leq \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} |f(m)|^2 \frac{a_{nm}^2}{q_m} \right) \left( \sum_{m=1}^{\infty} a_{nm} q_m \right) \\
\leq \frac{2}{2 - \alpha} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |f(m)|^2 \frac{a_{nm} q_m}{p_n} \\
= \frac{2}{2 - \alpha} \sum_{m=1}^{\infty} |f(m)|^2 \sum_{n=1}^{\infty} a_{nm} p_n \\
\leq \left( \frac{2}{2 - \alpha} \right)^2 \sum_{m=1}^{\infty} |f(m)|^2 = \left( \frac{2}{2 - \alpha} \right)^2 \|f\|^2_{\mathbb{D},\alpha}.
$$

Hence $\|L\|_{D,\alpha} \leq \frac{2}{2 - \alpha}$. Since $\|L\|_{D,\alpha} \geq |\lambda|$ for $\lambda \in \sigma_p(L)$, we get that $\|L\|_{D,\alpha} \geq \frac{2}{2 - \alpha}$. The proof is complete. \hfill \Box

Remark 2.3. Observe that $\|L\|_{\mathbb{D},\alpha} \geq \max\{1/2, \frac{2}{2 - \alpha}\}$. If $\alpha < -2$, then Theorem 2.2 is not true.
Acknowledgments

The authors would like to thank the referee for his/her helpful comments. G. Bao is supported in part by NSF of China (No. 11371234). J. Yang is supported by Innovation Program of Shanghai Municipal Education Commission (No. 13YZ090) and NSF of China (No. 11501357).

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