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# THE LIBERA OPERATOR ON DIRICHLET SPACES

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Abstract. In this paper, we consider the boundedness of the Libera operator on Dirichlet spaces in terms of the Schur test. Moreover, we get its point spectrum and norm.
**Keywords:** Libera operator, Dirichlet spaces, point spectrum, norm.
**MSC(2010):** Primary: 30H10; Secondary: 47A30.

## 1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$. Denote by $\mathcal{D}_\alpha$, $\alpha \in \mathbb{R}$, the Hilbert space of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{D}$ with $f(0) = 0$ and

$$\|f\|_{\mathcal{D}_\alpha}^2 = \sum_{n=1}^{\infty} n^{1-\alpha} |a_n|^2 < \infty.$$

Note that the classical Dirichlet space $\mathcal{D}$ is obtained for $\alpha = 0$. See [1,2,7,9,11] for $\mathcal{D}_\alpha$ spaces.

The Libera operator $\mathcal{L}$ on $\mathcal{D}_\alpha$, $\alpha < 2$, is defined by

$$\mathcal{L}f(z) = \frac{1}{z-1} \int_1^z f(w)dw - \int_0^1 f(w)dw$$

for all $f \in \mathcal{D}_\alpha$. See [4,6,10] for the Libera operator on other spaces. Rhaly [8] gave a series of results about the Libera operator on the classical Dirichlet space $\mathcal{D}$. In this paper, we consider the Libera operator on $\mathcal{D}_\alpha$ spaces. By [2], if $\alpha > 0$, then there exists a constant $C$ such that

$$|f(z)| \le C\|f\|_{\mathcal{D}_\alpha} \left(\frac{1}{1-|z|}\right)^{\alpha/2}$$

for all $f \in \mathcal{D}_\alpha$. Clearly, $\mathcal{D}_\alpha \subseteq \mathcal{D}_\beta$ when $\alpha \le \beta$. Consequently, the Libera operator $\mathcal{L}$ is well defined on $\mathcal{D}_\alpha$, $\alpha < 2$.

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Let $e_n(z) = n^{\frac{\alpha-1}{2}} z^n$, $n = 1, \quad 2, \cdots$. Then the set $\{e_n\}_{n=1}^{\infty}$ forms an orthonormal basis for $\mathcal{D}_\alpha$. Note that

$$\begin{aligned}\mathcal{L}e_n(z) &= \frac{1}{z-1}\int_1^z n^{\frac{\alpha-1}{2}} w^n dw - \int_0^1 n^{\frac{\alpha-1}{2}} w^n dw \\ &= \frac{n^{\frac{\alpha-1}{2}}}{n+1}(z^n + z^{n-1} + \cdots + z) \\ &= \frac{n^{\frac{\alpha-1}{2}}}{n+1}\left(n^{\frac{1-\alpha}{2}} e_n(z) + (n-1)^{\frac{1-\alpha}{2}} e_{n-1}(z) + \cdots + e_1(z)\right).\end{aligned}$$

Thus, $\mathcal{L}$ has matrix entries

$$a_{ij} = \langle \mathcal{L}e_j, e_i \rangle = \begin{cases} 0 & i > j \geq 1, \\ \frac{1}{j+1}\left(\frac{i}{j}\right)^{\frac{1-\alpha}{2}} & j \geq i \geq 1. \end{cases}$$

If $f(z) = \sum_{n=1}^{\infty} f(n)e_n(z)$, then

$$\mathcal{L}f(z) = \sum_{n=1}^{\infty}\sum_{m=n}^{\infty} \frac{f(m)}{m+1}\left(\frac{n}{m}\right)^{\frac{1-\alpha}{2}} e_n(z).$$

For $\alpha > -1$, a direct calculation gives that $\|f\|^2_{\mathcal{D}_\alpha}$ is comparable with

$$\int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^\alpha dxdy. \tag{1.1}$$

Pavlović [6] investigated the Libera operator on mixed norm spaces. In particular, under the seminorm (1.1) of $\mathcal{D}_\alpha$ spaces, he showed that if $\alpha > -1$, then $\mathcal{L}$ is bounded on $\mathcal{D}_\alpha$ if and only if $\alpha < 2$. In this paper, by different technique, we give that $\mathcal{L}$ is bounded on $\mathcal{D}_\alpha$ if and only if $\alpha < 2$. Particularly, we obtain that $\mathcal{L}$ is also bounded on $\mathcal{D}_\alpha$ for $\alpha \leq -1$. Furthermore, the point spectrum and norm of $\mathcal{L}$ on the $\mathcal{D}_\alpha$ space are also considered.

## 2. Main results

Let $T$ be a bounded linear operator on a Hilbert space $H$. Recall that the point spectrum $\sigma_p(T)$ of $T$ is

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \ker(\lambda I - T) \neq 0\},$$

where $I$ is the identical operator of $H$. The point spectrum of $\mathcal{L}$ on the Dirichlet space $\mathcal{D}$ was obtained in [8]. Using the Schur test, we get the following result.

**Theorem 2.1.** *The Libera operator $\mathcal{L}$ is bounded on $\mathcal{D}_\alpha$ if and only if $\alpha < 2$. Moreover,*

$$\sigma_p(\mathcal{L}) = \{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2-\alpha}| < \frac{1}{2-\alpha}\} \cup \{\frac{1}{n} : n = 2, 3, \cdots\}.$$

*Proof.* Let $p_i = \frac{i^{1/2}}{i+1}$, $i = 1, \ 2, \cdots$. Then

$$\begin{aligned}\sum_{i=1}^{\infty} a_{ij} p_i &= \frac{1}{(j+1) j^{\frac{1-\alpha}{2}}} \sum_{i=1}^{j} \frac{i^{(2-\alpha)/2}}{i+1} \\ &\leq \frac{1}{(j+1) j^{\frac{1-\alpha}{2}}} \sum_{i=1}^{j} \int_{i-1}^{i} \max\{t^{-\alpha/2}, (t+1)^{-\alpha/2}\} dt \\ &\leq \frac{1}{(j+1) j^{\frac{1-\alpha}{2}}} \int_{0}^{j} \max\{t^{-\alpha/2}, (t+1)^{-\alpha/2}\} dt.\end{aligned}$$

It follows that

$$\sum_{i=1}^{\infty} a_{ij} p_i \leq \frac{2}{2-\alpha} p_j, \ 0 \leq \alpha < 2,$$

and

$$\sum_{i=1}^{\infty} a_{ij} p_i \leq \frac{2^{2-\frac{\alpha}{2}}}{2-\alpha} p_j, \ \alpha < 0.$$

On the other hand,

$$\begin{aligned}\sum_{j=1}^{\infty} a_{ij} p_j &= \sum_{j=i}^{\infty} \frac{1}{j+1} \left(\frac{i}{j}\right)^{\frac{1-\alpha}{2}} \frac{j^{1/2}}{j+1} \\ &\leq i^{\frac{1-\alpha}{2}} \sum_{j=i}^{\infty} \int_{j}^{j+1} \frac{\max\{t^{\alpha/2}, (t/2)^{\alpha/2}\}}{t^2} dt \\ &\leq i^{\frac{1-\alpha}{2}} \int_{i}^{\infty} \frac{\max\{t^{\alpha/2}, (t/2)^{\alpha/2}\}}{t^2} dt.\end{aligned}$$

Hence,

$$\sum_{j=1}^{\infty} a_{ij} p_j \leq \frac{2}{2-\alpha} i^{-1/2} \leq \frac{4}{2-\alpha} p_i, \ 0 \leq \alpha < 2,$$

and

$$\sum_{j=1}^{\infty} a_{ij} p_j \leq \frac{2^{1-\frac{\alpha}{2}}}{2-\alpha} i^{-1/2} \leq \frac{2^{2-\frac{\alpha}{2}}}{2-\alpha} p_i, \ \alpha < 0.$$

By the Schur test (see [3, P. 24]), we get that $\mathcal{L}$ is bounded on $\mathcal{D}_\alpha$ for $\alpha < 2$.

For $\alpha \geq 2$, let

$$f_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{\frac{2-\alpha}{2}} \log(n+1)},$$

that is,

$$f_\alpha(z) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n} \log(n+1)} e_n(z).$$

Then

$$\|f_\alpha\|_{\mathcal{D}_\alpha}^2 = \sum_{n=1}^{\infty} \frac{1}{n \log^2(n+1)} < \infty.$$

Note that $\alpha \geq 2$. We deduce that

$$\begin{aligned}
\|\mathcal{L}f_\alpha\|^2_{\mathcal{D}_\alpha} &= \sum_{n=1}^{\infty}\left|\sum_{m=n}^{\infty}\frac{1}{\sqrt{m}(m+1)\log(m+1)}\left(\frac{n}{m}\right)^{\frac{1-\alpha}{2}}\right|^2 \\
&\geq \left|\sum_{m=1}^{\infty}\frac{1}{\sqrt{m}(m+1)\log(m+1)}\left(\frac{1}{m}\right)^{\frac{1-\alpha}{2}}\right|^2 \\
&\geq \left|\sum_{m=1}^{\infty}\frac{1}{(m+1)\log(m+1)}\right|^2 = +\infty,
\end{aligned}$$

which implies that $\mathcal{L}$ is not bounded on $\mathcal{D}_\alpha$, $\alpha \geq 2$. Thus, $\mathcal{L}$ is bounded on $\mathcal{D}_\alpha$ if and only if $\alpha < 2$.

We now compute the point spectrum of $\mathcal{L}$. If $f(z) = \sum_{n=1}^{\infty} f(n)e_n(z)$, then

$$\mathcal{L}f(z) = \sum_{n=1}^{\infty}\sum_{m=n}^{\infty}\frac{f(m)}{m+1}\left(\frac{n}{m}\right)^{\frac{1-\alpha}{2}} e_n(z).$$

Thus,

$$\mathcal{L}f(n) = n^{\frac{1-\alpha}{2}}\sum_{m=n}^{\infty}\frac{f(m)}{(m+1)m^{\frac{1-\alpha}{2}}}, \quad n = 1, \quad 2, \cdots.$$

Consequently,

$$n^{\frac{\alpha-1}{2}}\mathcal{L}f(n) - (n+1)^{\frac{\alpha-1}{2}}\mathcal{L}f(n+1) = \frac{f(n)}{(n+1)n^{\frac{1-\alpha}{2}}},$$

which gives

$$f(n) = (n+1)\mathcal{L}f(n) - n^{\frac{1-\alpha}{2}}(n+1)^{\frac{\alpha+1}{2}}\mathcal{L}f(n+1).$$

If $\mathcal{L}f = \lambda f$, then

$$\lambda n^{\frac{1-\alpha}{2}}(n+1)^{\frac{\alpha+1}{2}}f(n+1) = [\lambda(n+1) - 1]f(n).$$

If $\lambda = 0$, then $f(n) = 0$ for all $n$. Hence, $0 \notin \sigma_p(\mathcal{L})$. It follows that

$$f(n+1) = \left(\frac{n+1}{n}\right)^{\frac{1-\alpha}{2}}\left[1 - \frac{1}{\lambda(n+1)}\right]f(n).$$

Thus,

$$f(n) = n^{\frac{1-\alpha}{2}}\left[\prod_{j=2}^{n}\left(1 - \frac{1}{\lambda j}\right)\right]f(1), \quad n \geq 2.$$

We now want to know what nonzero values of $\lambda$ will result in the convergence of $\sum_{n=1}^{\infty}|f(n)|^2$. Clearly, $\{\frac{1}{n} : n = 2, 3, \cdots\} \subseteq \sigma_p(\mathcal{L})$. Suppose that $\lambda \notin \{\frac{1}{n} :$

$n = 2, 3, \cdots\}$. Then

$$\begin{aligned} & \frac{|f(n)|^2}{|f(n+1)|^2} - 1 \\ = & \frac{\left((n+1)^2 n^{1-\alpha} - (n+1)^{3-\alpha}\right)|\lambda|^2 + (n+1)^{2-\alpha}(\lambda + \overline{\lambda}) - (n+1)^{1-\alpha}}{(n+1)^{3-\alpha}|\lambda|^2 - (n+1)^{2-\alpha}(\lambda + \overline{\lambda}) + (n+1)^{1-\alpha}}. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} n\left[\frac{|f(n)|^2}{|f(n+1)|^2} - 1\right] = \frac{(\alpha-1)|\lambda|^2 + (\lambda+\overline{\lambda})}{|\lambda|^2}.$$

By Raabe's test [5, Theorem II, P. 396], $\sum_{n=1}^{\infty}|f(n)|^2$ converges for $\frac{\lambda+\overline{\lambda}}{|\lambda|^2} + \alpha - 1 > 1$ and diverges for $\frac{\lambda+\overline{\lambda}}{|\lambda|^2} + \alpha - 1 < 1$. Namely, the series converges for $|\lambda - \frac{1}{2-\alpha}| < \frac{1}{2-\alpha}$ and diverges for $|\lambda - \frac{1}{2-\alpha}| > \frac{1}{2-\alpha}$. If $|\lambda - \frac{1}{2-\alpha}| = \frac{1}{2-\alpha}$, then

$$\begin{aligned} & \lim_{n\to\infty} \ln n\big\{n\big[\frac{|f(n)|^2}{|f(n+1)|^2} - 1\big] - 1\big\} \\ = & \lim_{n\to\infty} \frac{\ln n}{n}\Big\{\frac{n\left[(n+1)^2 n^{2-\alpha} - (n+1)^{4-\alpha} + (2-\alpha)(n+1)^{3-\alpha}\right]|\lambda|^2}{(n+1)^{2-\alpha}(n+\alpha-1)|\lambda|^2 + (n+1)^{1-\alpha}} \\ & - \frac{n(n+1)^{2-\alpha}}{(n+1)^{2-\alpha}(n+\alpha-1)|\lambda|^2 + (n+1)^{1-\alpha}}\Big\} \\ = & \ 0 \cdot \frac{\frac{\alpha^2-3\alpha+2}{2}|\lambda|^2 - 1}{|\lambda|^2} = 0. \end{aligned}$$

By [5, Theorem III, P. 396], we obtain that the series $\sum_{n=1}^{\infty}|f(n)|^2$ diverges if $|\lambda - \frac{1}{2-\alpha}| = \frac{1}{2-\alpha}$. Therefore,

$$\sigma_p(\mathcal{L}) = \{\lambda \in \mathbb{C} : |\lambda - \frac{1}{2-\alpha}| < \frac{1}{2-\alpha}\} \cup \{\frac{1}{n} : n = 2, 3, \cdots\}.$$

□

For $\alpha < 2$, let $\|\mathcal{L}\|_{\mathcal{D}_\alpha}$ be the norm of $\mathcal{L}$ on $\mathcal{D}_\alpha$ spaces. H. Rhaly [8] proved that $\|\mathcal{L}\|_{\mathcal{D}} = 1$. Now we give the following result.

**Theorem 2.2.** *Let* $0 \le \alpha < 2$. *Then* $\|\mathcal{L}\|_{\mathcal{D}_\alpha} = \frac{2}{2-\alpha}$.

*Proof.* Let $p_i = i^{-1/2}$, $q_j = \frac{j^{1/2}}{j+1}$, $i,\ j = 1, \quad 2, \cdots$. Note that $0 \le \alpha < 2$. We have

$$\begin{aligned}\sum_{i=1}^{\infty} a_{ij} p_i &= \frac{1}{(j+1) j^{\frac{1-\alpha}{2}}} \sum_{i=1}^{j} i^{-\alpha/2} \\ &\le \frac{1}{(j+1) j^{\frac{1-\alpha}{2}}} \sum_{i=1}^{j} \int_{i-1}^{i} t^{-\alpha/2} dt \\ &= \frac{1}{(j+1) j^{\frac{1-\alpha}{2}}} \int_{0}^{j} t^{-\alpha/2} dt \\ &= \frac{2}{2-\alpha} \frac{j^{1/2}}{j+1} = \frac{2}{2-\alpha} q_j.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{j=1}^{\infty} a_{ij} q_j &= i^{\frac{1-\alpha}{2}} \sum_{j=i}^{\infty} \frac{j^{\alpha/2}}{(j+1)^2} \le i^{\frac{1-\alpha}{2}} \sum_{j=i}^{\infty} \int_{j}^{j+1} t^{\alpha/2-2} dt \\ &= i^{\frac{1-\alpha}{2}} \int_{i}^{\infty} t^{\alpha/2-2} dt = \frac{2}{2-\alpha} i^{-1/2} = \frac{2}{2-\alpha} p_i.\end{aligned}$$

By the Schur test, we get that $\|\mathcal{L}\|_{\mathcal{D}_\alpha} \le \frac{2}{2-\alpha}$. Here we give the details for the completeness. If $f(z) = \sum_{n=1}^{\infty} f(n) e_n(z)$, then

$$\begin{aligned}\|\mathcal{L}f\|_{\mathcal{D}_\alpha}^2 &= \sum_{n=1}^{\infty} \left| \sum_{m=1}^{\infty} f(m) a_{nm} \right|^2 \\ &\le \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} |f(m)|^2 \frac{a_{nm}}{q_m} \right) \left( \sum_{m=1}^{\infty} a_{nm} q_m \right) \\ &\le \frac{2}{2-\alpha} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |f(m)|^2 \frac{a_{nm}}{q_m} p_n \\ &= \frac{2}{2-\alpha} \sum_{m=1}^{\infty} \frac{|f(m)|^2}{q_m} \sum_{n=1}^{\infty} a_{nm} p_n \\ &\le \left( \frac{2}{2-\alpha} \right)^2 \sum_{m=1}^{\infty} |f(m)|^2 = \left( \frac{2}{2-\alpha} \right)^2 \|f\|_{\mathcal{D}_\alpha}^2.\end{aligned}$$

Hence $\|\mathcal{L}\|_{\mathcal{D}_\alpha} \le \frac{2}{2-\alpha}$. Since $\|\mathcal{L}\|_{\mathcal{D}_\alpha} \ge |\lambda|$ for $\lambda \in \sigma_p(\mathcal{L})$, we get that $\|\mathcal{L}\|_{\mathcal{D}_\alpha} \ge \frac{2}{2-\alpha}$. The proof is complete. □

**Remark 2.3.** *Observe that* $\|\mathcal{L}\|_{\mathcal{D}_\alpha} \ge \max\{1/2, \frac{2}{2-\alpha}\}$. *If* $\alpha < -2$, *then Theorem 2.2 is not true.*

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