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ON THE MODIFIED ITERATIVE METHODS FOR M-MATRIX LINEAR SYSTEMS

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Abstract. This paper deals with scrutinizing the convergence properties of iterative methods to solve linear system of equations. Recently, several types of the preconditioners have been applied for ameliorating the rate of convergence of the Accelerated Overrelaxation (AOR) method. In this paper, we study the applicability of a general class of the preconditioned iterative methods under certain conditions. More precisely, it is demonstrated that the preconditioned Mixed-Type Splitting (MTS) iterative methods can surpass the preconditioned AOR iterative methods for an entirely general class of preconditioners handled by Wang and Song [J. Comput. Appl. Math. 226 (2009), no. 1, 114–124]. Finally some numerical results are elaborated which confirm the validity of the established results.

Keywords: Linear system, iterative method,mixed-Type splitting, preconditioner, convergence rate.
MSC(2010): Primary: 65F10.

1. Introduction

Consider the linear system

$$Ax = b, \tag{1.1}$$

where the nonsingular matrix $A \in \mathbb{R}^{n\times n}$ and the right-hand side vector $b \in \mathbb{R}^n$ are given and x is the unknown vector to be determined. Without loss of generality, we may assume that the elements of main diagonal of A are nonzero.

For a given matrix $A \in \mathbb{R}^{n\times n}$, the decomposition $A = M - N$ is called a splitting if M and N belong to $\mathbb{R}^{n\times n}$ and M is nonsingular. For an arbitrary given splitting $A = M - N$, a basic stationary iterative method for solving $Ax = b$ has the subsequent form:

$$x^{(k+1)} = \mathcal{V}x^{(k)} + M^{-1}b, \quad k = 0, 1, 2, \ldots, \tag{1.2}$$

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where the initial vector $x^{(0)}$ is given and $\mathcal{V} = M^{-1}N$ is called the iteration matrix. It is well-known that the convergence analysis of the iterative method (1.2) relies on the spectral radius of the iteration matrix $\mathcal{V}$, i.e., the smaller spectral radius is the faster the convergence is; for further details see [24].

In this paper we work with special kinds of matrices which are introduced in the following two definitions.

Definition 1.1. (Berman and Plemmons [3]). The matrix $A \in \mathbb{R}^{n\times n}$ is called a Z-matrix if $a_{ij} \leq 0$ for $i, j = 1, 2, 3, \ldots, n$ $(i \neq j)$. A Z-matrix with positive diagonal elements is named an L-matrix.

Definition 1.2. (Berman and Plemmons [3]). Let A be an L-matrix. Then the matrix A is said to be an M-matrix if A is nonsingular and $A^{-1} \geq 0$.

The next definition expounds different types of the splittings exploited in this work.

Definition 1.3. (Woznicki [27]). The splitting $A = M - N$ is called

(1) a *regular* splitting of A if $M^{-1} \geq 0$ and $N \geq 0$,
(2) a *nonnegative* splitting of A if $M^{-1} \geq 0$, $M^{-1}N \geq 0$ and $NM^{-1} \geq 0$,
(3) a *weak nonnegative* splitting of A if $M^{-1} \geq 0$ and either $M^{-1}N \geq 0$ (the first type) or $NM^{-1} \geq 0$ (the second type),
(4) a *convergent* splitting of A if $\rho(M^{-1}N) < 1$.

Hitherto, several kinds of the preconditioners have been examined to improve the speed of convergence of the iterative methods. In fact, there is a growing interest to study about the performance of the preconditioners to ameliorate the speed of convergence of both stationary and nonstationary iterative methods in the literature; for instance see [2,5,18,19,22,23,25,28,29,31] and the references therein.

For example, Moghadam and Beik [19] have applied the preconditioner

$$P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{a_{n1}}{a_{11}} & 0 & \cdots & 1 \end{pmatrix}, \tag{1.3}$$

to accelerate the speed of convergence of the MTS method. It has been proved that the preconditioned MTS method associated with the preconditioner (1.3) can outperform the preconditioned AOR method for solving M-matrix linear systems.

Note 1.4. In the sequel, we assume that the main diagonal elements of the coefficient matrix A are all equal to one.

In [5], the next two preconditioners have been introduced

$$P_1 = I + S_1, \quad P_2 = I + S_2,$$

where

$$S_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -(a_{21} + \alpha_2) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(a_{n1} + \alpha_n) & 0 & \cdots & 0 \end{pmatrix},$$

and

$$S_2 = \begin{pmatrix} 0 & \cdots & 0 & -(a_{1n} + \beta_1) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -(a_{n-1n} + \beta_{n-1}) \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The authors have improved the speed of convergence of the AOR method when A is an L-matrix. The inspiration of the authors was proposing a modified version of the AOR method under weaker conditions than usually assumed in the literature, i.e., $a_{1i}a_{i1} < 1$ $(i = 2, \ldots, n)$ which have been also assumed in [18] when the preconditioner (1.3) applied for improving the rate of convergence of the Jacobi and Gauss-Seidel iterative methods. Nevertheless, the improvement has been reached only in the circumstance that the AOR splitting is a convergent splitting for an L-matrix. Whereas, Moghadam and Beik [19] have shown that the AOR splitting is a regular splitting for an L-matrix under certain conditions for parameters which have been also assumed in [5]. Hence, the convergence of the AOR splitting for an L-matrix implies that its inverse is nonnegative, i.e., the matrix is in fact an M-matrix. Consequently, in [5], the modified AOR methods are elaborated for an M-matrix. In the current paper, it reveals that if A is an M-matrix then the matrix A gratifies the conditions $a_{1i}a_{i1} < 1$ for $i = 2, \ldots, n$. Meanwhile, this fact can be also seen in the authors' utilized numerical examples; see [5] for more details.

In [25], the next general preconditioner have been handled

$$P = \begin{pmatrix} 1 & \cdots & -\alpha_{1n-1}a_{1n-1} & -\alpha_{1n}a_{1n} \\ -\alpha_{21}a_{21} & \ddots & -\alpha_{2n-1}a_{2n-1} & -\alpha_{2n}a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -\alpha_{n1}a_{n1} & \cdots & -\alpha_{nn-1}a_{nn-1} & 1 \end{pmatrix}, \tag{1.4}$$

where $p_{ii} = 1$ and $0 \leq \alpha_{ij} \leq 1$ for $i, j = 1, \ldots, n$. As seen, the preconditioner (1.4) is quite general and incorporates many of the previously studied preconditioners such as (1.3) and those used in [6–11,15–17,20,25,30]. More precisely, Wang and Song [25] have improved the rate of convergence of the AOR method by applying the preconditioner (1.4) and the numerical experiments illustrate that the best case occurs when $\alpha_{ij} = 1$ for $i, j = 1 \ldots, n$ $(i \neq j)$. However, the results have been established under certain conditions which imply that the (preconditioned) AOR method is a regular splitting. In this case, we show that the preconditioned MTS is superior than the preconditioned AOR method by choosing appropriate auxiliary matrices.

1.1. **A brief survey on the MTS method.** In [13,14], the mixed-type splitting (MTS) method has been introduced for solving the linear system $Ax = b$, where A is a nonsingular, large, sparse and nonsymmetric matrix and $A^T + A$ is a symmetric positive definite (SPD) matrix. Consider the decomposition $A = D - L - U$ for the matrix A where D, L and U denote the diagonal, strictly lower triangular and strictly upper triangular matrices, respectively. The MTS iterative method to solve $Ax = b$ is given as follows.

<u>The MTS method :</u>

$$(D + D_1 + L_1 - L)x^{(k+1)} = (D_1 + L_1 + U)x^{(k)} + b, \quad k = 0, 1, 2, \ldots.$$

The iteration matrix of the MTS method is expounded by

$$T = (D + D_1 + L_1 - L)^{-1}(D_1 + L_1 + U),$$

where D_1 is an available auxiliary nonnegative diagonal matrix and L_1 is a given auxiliary strictly lower triangular matrix such that $0 \le L_1 \le L$.

It is well-known that the MTS method incorporates the Gauss-Seidel (GS) method by setting $D_1 = L_1 = 0$ and the AOR method by the following choices of D_1 and L_1.

<u>The AOR method :</u>

$$\begin{aligned} D_1 &= \frac{1}{\omega}(1-\omega)D, \quad L_1 = \frac{1}{\omega}(\omega - r)L, \\ (D - rL)x^{(k+1)} &= ((1-\omega)D + (\omega - r)L + \omega U)x^{(k)} + \omega b, \quad k = 0, 1, 2, \ldots. \end{aligned}$$

The iteration matrix of the AOR method is determined by

$$T_{r,\omega} = (D - rL)^{-1}[(1-\omega)D + (\omega - r)L + \omega U],$$

where ω and r are real parameters such that $0 \le r \le \omega < 1$ and $\omega \ne 0$.

Cheng et al. [4] have considered the linear system $Ax = b$ where A is a Z-matrix and propounded a new class of the MTS method which is similar to the MTS approach presented in [13,14,26]. The method utilizes auxiliary matrices and contains SOR and AOR iterative methods as special cases. It has been proved that if the SOR (AOR) iterative method is convergent, then by suitable selections of auxiliary matrices, the MTS method converges faster than the SOR (AOR) iterative method. In [19], Moghadam and Beik have ameliorated the rate of convergence of the MTS method [4] by applying the preconditioner introduced by Milaszewicz [18]. Moreover, under the same assumptions considered in [4], it has been demonstrated that if the SOR (AOR) method is convergent for solving $Ax = b$ where A is a Z-matrix, then the coefficient matrix is in fact an M-matrix. In addition, it has revealed that the (preconditioned) GS iterative method converges faster than the (preconditioned) SOR and (preconditioned) AOR iterative methods.

1.2. **Motivations and highlight points.** In [5], Dehghan and Hajarian have presented a modified AOR iterative method ($0 < r \le \omega \le 1$) by exploiting two new preconditioners for solving linear system $Ax = b$ where A is an L-matrix. In [18], Milaszewicz has assumed that the matrix A is an L-matrix which satisfies the following properties

$$a_{i,i+1}a_{i+1,i} > 0 \quad \text{and} \quad 0 < a_{1i}a_{i1} < 1 \quad \text{for} \quad i = 2, 3, \ldots, n, \tag{1.5}$$

and improved the convergence speed of the Jacobi and Gauss-Seidel iterative methods. Dehghan and Hajarian have proved that if A is an irreducible L-matrix, their offered preconditioners can improve the rate of convergence the AOR method under weaker conditions than those given in (1.5). More precisely, the authors try to omit the restrictions $a_{1i}a_{i1} < 1$ and $a_{i,i+1}a_{i+1,i} > 0$ for $i = 2, 3, \ldots, n$. We would like to comment here that the authors have established the improvements of the AOR method under the assumption that the AOR splitting is convergent. Thence, it is not difficult to verify the fact that the modified AOR methods have been elaborated for M-matrices. Recently, Saberi-Najafi et al. [22] have demonstrated that if A is an irreducible M-matrix then without setting the restriction that $0 < a_{1i}a_{i1} < 1$ and $a_{i,i+1}a_{i+1,i} > 0$ for $i = 2, 3, \ldots, n$, the Milaszewicz's preconditioner outperforms those proposed by Dehghan and Hajarian.

Considering the pointed works cited in the previous paragraph, we have motivated to answer this question that

- *Does there exist an M-matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ such that $a_{1i}a_{i1} \ge 1$ for some i ($2 \le i \le n$)?*

In this paper we show that the answer of the above question is negative. That is if A is an M-matrix then $a_{1i}a_{i1} < 1$ for $i = 2, 3, \ldots, n$. This fact demonstrates that by setting the assumption that A is an M-matrix (or A is an L-matrix and the AOR splitting is convergent), Saberi-Najafi et al. (Dehghan and Hajarian) do not omit the restriction $a_{1i}a_{i1} < 1$ for $i = 2, 3, \ldots, n$.

As pointed out, Moghadam and Beik [19] have shown that the (preconditioned) MTS method can outperform the (preconditioned) AOR method for solving M-matrix linear systems when $0 \le r \le \omega \le 1$ and $\omega \ne 0$. More precisely, the authors have investigated the preconditioner offered in [18]. Meanwhile, a general class of preconditioner is considered for improving the rate of convergence of the AOR iterative method ($0 \le r \le \omega \le 1$ and $\omega \ne 0$) in [25]. The preconditioner mentioned by Wang and Song [25] includes that given by Milaszewicz. This inspirits us to investigate that

- *whether the preconditioned MTS method is superior than the preconditioned AOR method for the general class of preconditioners examined in* [25].

In this paper, it turns out that under mild conditions the preconditioned MTS method works better than the preconditioned AOR method when the Wang and Song's handled preconditioner is utilized.

The rest of this paper is organized as follows. In Section 2, we state some required concepts including symbols, definitions, properties, theorems, and etc. Some useful theoretical results are presented in Section 3. Afterward, we give a brief survey on some comparison theorems which have been elaborated in [4,19] as the first part of Section 4. The rest of the fourth section is devoted to investigating about the application of the preconditioned MTS method with a general preconditioner applied by Wang and Song [25]. In Section 5, we report some numerical experiments which confirm the validity of the theoretical results established throughout the current work. Finally, a brief conclusion is a subject of Section 6.

2. **Preliminaries**

In what follows, we recall some definitions and results which are utilized during the paper. For a square matrix A, the spectral radius of A is denoted by $\rho(A)$. For a given matrix $U \in \mathbb{R}^{n\times m}$, we say $U \geq 0$ $(U > 0)$ when all entries of U are nonnegative (positive). For two $n \times m$ real matrices U and V, the notation $U \geq V$ $(U > V)$ means that $U - V \geq 0$ $(U - V > 0)$.

Before ending this section, some theoretical results and facts are recollected which are utilized for proving our main results.

Lemma 2.1. *Let A be a Z-matrix. Then, A is an M-matrix if and only if there is a positive vector x such that $Ax > 0$.*

Proof. See [28]. □

Definition 2.2. (Varga [24]). A matrix A is said to be reducible if there is a permutation matrix P such that PAP^T is a block upper triangular matrix. Otherwise, it is irreducible.

In what follows, the notation $\mathcal{G}(A)$ denotes the directed graph of matrix A. The next lemma supplies another way to verify that whether a matrix is irreducible or not.

Lemma 2.3. *A matrix A is irreducible if $\mathcal{G}(A)$ is strongly connected.*

Proof. See [24]. □

Theorem 2.4. *Let $A = M - N$ be a regular splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is nonsingular and A^{-1} is nonnegative.*

Proof. See [24]. □

Now we recall the following theorem which is an essential tool for analyzing the speed of convergence of the (preconditioned) iterative methods through this work.

Theorem 2.5. *Let $A = M_1 - N_1 = M_2 - N_2$ be two convergent weak nonnegative splittings of A where $A^{-1} \geq 0$, if $M_1^{-1} \geq M_2^{-1}$ then*

$$\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2).$$

In particular, if $A^{-1} > 0$ and $M_1^{-1} > M_2^{-1}$ then

$$\rho(M_1^{-1}N_1) < \rho(M_2^{-1}N_2).$$

Proof. See [27]. □

Theorem 2.6. *Let A be a Z-matrix. Moreover, suppose that $A = M - N$ is a weak nonnegative splitting of the first type. Then $\rho(M^{-1}N) < 1$ if and only if A is an M-matrix.*

Proof. See [2]. □

3. Fundamental theoretical results

In this section, we present some useful theoretical results. It is not difficult to establish these facts and their proofs are left to the conscious reader. It is demonstrated that some of the restrictions, assumed in the literature while preconditioners are applied for improving the convergence rate of the iterative methods, are satisfied when A is an M-matrix; see for instance [12, 18].

Proposition 3.1. *Let $A \in \mathbb{R}^{n\times n}$ be an M-matrix. Then each of its principle submatrices is an M-matrix.*

Remark 3.2. Let A be an M-matrix. The $(n-1)\times(n-1)$ matrix $A(k)$ obtained from A by removing its kth row and column is an M-matrix where $1 \leq k \leq n$.

By Proposition 3.1 and Remark 3.2, we may establish the next proposition.

Proposition 3.3. *Let $A \in \mathbb{R}^{n\times n}$ be an M-matrix whose all of its main diagonal elements are equal to one. Then*

$$a_{1i}a_{i1} < 1, \qquad i = 2, \ldots, n. \tag{3.1}$$

Remark 3.4. In [12], Lei et al have presented some theoretical results and supposed that $A = I - L - U$ is an M-matrix with $0 < a_{ii+1}a_{i+1i} < 1$ for $i = 1, \ldots, n-1$; see for instance Lemma 3.1, Theorem 3.2, and etc. It is not difficult to see that if A is assumed to be an M matrix then $a_{ii+1}a_{i+1i} < 1$

for $i = 1, \dots, n-1$. As a matter of fact, in view of Proposition 3.3, if A is an M-matrix then the following principle submatrix of A is an M-matrix,

$$A_2 = \begin{pmatrix} 1 & a_{ii+1} \\ a_{i+1i} & 1 \end{pmatrix}, \qquad i = 1, \dots, n-1,$$

and hence by Proposition 3.3, we deduce that $a_{ii+1}a_{i+1,i} < 1$. We would like to point here that in this work we do not aim to analyze the iterative methods proposed in [12].

The following proposition reveals that under certain sufficient conditions, the iteration matrix of the MTS method associated with an irreducible matrix is irreducible. As the proposition can be established with a simmer manner exploited in [Beik and Shams [2], Proposition 3.1], we omit its proof.

Proposition 3.5. *Let $A = D - L - U$ be an irreducible L-matrix where D is a diagonal matrix, L and U are respectively lower and upper triangular matrices. Suppose that D_1 is nonsingular, $0 \le D_1 \le D$ and $0 \le L_1 \le L$. Then the iteration matrix of the MTS method is irreducible.*

In the following, for an M-matrix, it is seen that the spectral radius of the iteration matrix of the AOR iterative method is a decreasing function with respect to the parameters r and ω in the case that $0 \le r \le \omega \le 1$ and $\omega \ne 0$.

Theorem 3.6. *Let A be an M-matrix. Suppose that $r \ge 0$ and $0 \ne \omega_1 \le \omega_2 \le 1$ with $\omega_i \ge r$ for $i = 1, 2$. Presume that $\rho(T_{r,\omega_1})$ and $\rho(T_{r,\omega_2})$ denote the spectral radius of the AOR method associated with (r, ω_1) and (r, ω_2), respectively. Then $\rho(T_{r,\omega_2}) \le \rho(T_{r,\omega_1})$.*

Theorem 3.7. *Let A be an M-matrix. Suppose that $0 < \omega \le 1$ and $0 \le r_1 \le r_2$ with $\omega \ge r_i$ for $i = 1, 2$. Suppose that $\rho(T_{r_1,\omega})$ and $\rho(T_{r_2,\omega})$ stand for the spectral radius of the AOR method associated with (r_1, ω) and (r_2, ω), respectively. Then $\rho(T_{r_2,\omega}) \le \rho(T_{r_1,\omega})$.*

4. **Comparison results on preconditioned MTS methods**

The present section consists of two main parts. In the first subsection, we recall some comparison results between MTS and AOR iterative methods. In the second part of this section, in order to improve the rate of convergence of the MTS iterative method, we apply the general precoditioner examined by Wang and Song [25].

4.1. **Comparison results.**
Throughout this subsection, we give an overview on some comparison results established in [4] and [19].

Theorem 4.1. *If A is an M-matrix, $D_1 \ge 0$ and $0 \le L_1 \le L$. Then the MTS method is a convergent regular splitting.*

Proof. See [19]. □

Corollary 4.2. *Let A be an M-matrix and $0 \leq r \leq \omega < 1$, $\omega \neq 0$. Then the AOR splitting for the matrix A is convergent.*

In the next two theorems, the authors have supposed that the matrix A is a Z-matrix with positive diagonal elements, in fact A is assumed to be an L-matrix. It has been seen that the speed of convergence of the MTS iterative method is faster than the speed of convergence of the AOR method in the case that the AOR is a convergent method; for more details see [4].

Theorem 4.3. *Let A be a Z-matrix with positive diagonal elements and*

$$0 \leq D_1 \leq (\frac{1}{\omega} - 1)D, \; 0 \leq L_1 \leq (1 - \frac{r}{\omega})L.$$

Moreover assume that the matrices T and $T_{r,\omega}$ are the mixed-type splitting and AOR iteration matrices, respectively, where $0 < r < \omega < 1$. If T and $T_{r,\omega}$ are irreducible, then

(1) $\rho(T) < \rho(T_{r,\omega})$ *if* $\rho(T_{r,\omega}) < 1$,
(2) $\rho(T) = \rho(T_{r,\omega})$ *if* $\rho(T_{r,\omega}) = 1$,
(3) $\rho(T) > \rho(T_{r,\omega})$ *if* $\rho(T_{r,\omega}) > 1$.

In the following proposition, Moghadam and Beik [19] have shown that if A is an L-matrix, then MTS method is convergent iff A is an M-matrix.

Proposition 4.4. *Let A be a Z-matrix with positive diagonal elements. Then the MTS is a convergent splitting for the matrix A if and only if A is an M-matrix.*

Remark 4.5. Note that if the MTS (AOR) is a convergent splitting for a given L-matrix A, then the matrix A is an M-matrix. Hence, we only focus on M-matrix linear systems.

Theorem 4.6. *Let A be an M-matrix, $D_1 \geq 0$ and $0 \leq L_1 \leq L$. Suppose that T_G and T are the MTS and GS iteration matrices, respectively. If T_G and T are irreducible, then $\rho(T_G) < \rho(T)$.*

Proof. See [19] □

Theorem 4.7. *Let A be an M-matrix. Suppose that T_G and $T_{r,\omega}$ are the GS and AOR iteration matrices, respectively, where $0 \leq r \leq \omega < 1$, $\omega \neq 0$. If T_G and $T_{r,\omega}$ are irreducible, then*

$$\rho(T_G) < \rho(T_{r,\omega}).$$

Proof. See [19] □

4.2. **Preconditioned MTS methods in a general form.** In [25], the authors have considered the general preconditioner (1.4) and improved the rate of the convergence of the AOR iterative method. Moreover, it has been revealed that the preconditioner $P = I + L + U$ is the best preconditioner between the preconditioners belong to the general form (1.4). In the current subsection we demonstrate that with the aid of appropriate auxiliary matrices, preconditioned MTS iterative method outperforms MTS and (preconditioned) AOR methods.

In view of Lemma 2.1, it is not difficult to see that if A is an M-matrix and PA is a Z-matrix, then PA is an M-matrix where P is a nonnegative and nonsingular matrix.

Note 4.8. Let A be a Z-matrix whose diagonal elements are all equal to one; i.e., $A = I - L - U$. Suppose that P is defined by (1.4), straightforward computations show that $\bar{A} = PA$ is an M-matrix.

Let us consider the subsequent preconditioned linear system

$$\bar{A}x = Pb, \tag{4.1}$$

where $\bar{A} = PA$ and P is defined by (1.4). Assume that $\bar{A} = \bar{D} - \bar{L} - \bar{U}$ where $\bar{D}$ is a diagonal matrix, $\bar{L}$ and $\bar{U}$ are respectively strictly lower and strictly upper triangular matrices. The preconditioned MTS iterative method is specified by

$$(\bar{D} + \bar{D}_1 + \bar{L}_1 - \bar{L})x^{(k+1)} = (\bar{D}_1 + \bar{L}_1 + \bar{U})x^{(k)} + Pb \qquad k = 0, 1, \ldots,$$

where $x^{(0)}$ is given and the auxiliary matrices $\bar{D}_1$ and $\bar{L}_1$ have the following form

$$\bar{D}_1 = \xi\bar{D} \quad \text{and} \quad \bar{L}_1 = \zeta\bar{L}, \tag{4.2}$$

in which $\xi \in [0, 1]$ and $\zeta \in [0, 1]$ are given parameters.

As known the preconditioned MTS (AOR, GS) iterative method is derived by considering the MTS (AOR, GS) iterative method for solving the preconditioned linear system $\bar{A}x = Pb$. As PA is a Z-matrix, hence the preconditioned matrix $\bar{A}$ is an M-matrix when A is assumed to be an M-matrix. As a result the preconditioned MTS (AOR, GS) method is convergent; see Proposition 4.4.

Note that the MTS method includes the preconditioned GS method for $\xi = \zeta = 0$. Moreover, if we set $\xi = \frac{1}{\omega}(1-\omega)$, $\zeta = \frac{1}{\omega}(\omega - r)$ for $0 \leq r \leq \omega \leq 1$ and $\omega \geq 0.5$, then the preconditioned MTS method reduces to the preconditioned AOR method which has been mentioned in [25]. In view of Theorem 3.6, it is not difficult to verify that the preconditioned AOR method for $\omega \geq 0.5$ converges faster than the preconditioned AOR method for $0 < \omega < 0.5$. Hence, in what follows, we only need to mention the preconditioned AOR method for $0.5 \leq \omega \leq 1$ which is a special case of the preconditioned MTS method.

Theorem 4.9. *Let A be an M-matrix and $\bar{A} = PA$ where P is defined by* (1.4)*. Suppose that the auxiliary matrices $\bar{D}_1$ and $\bar{L}_1$ satisfy* (4.2)*. Assume* *that $\rho(\bar{T}_{r,\omega})$ and $\rho(\bar{T})$ represent the iteration matrices of the preconditioned AOR and preconditioned MTS methods, respectively. If*

$$0 \le \bar{D}_1 \le (\frac{1}{\omega} - 1)\bar{D} \quad \textit{and} \quad 0 \le \bar{L}_1 \le (1 - \frac{r}{\omega})\bar{L}.$$

where $0 \le r \le \omega \le 1$ and $0.5 \le \omega \le 1$, then $\rho(\bar{T}) \le \rho(\bar{T}_{r,\omega})$.

Proof. As A is an M-matrix, it is not difficult to verify that $\bar{A}$ is an M-matrix. Therefore, Proposition 4.4 implies that $\rho(\bar{T}_{r,\omega}) < 1$ and $\rho(\bar{T}) < 1$. Presume that $\bar{A} = \bar{M} - \bar{N}$ and $\bar{A} = \bar{M}_{r,\omega} - \bar{N}_{r,\omega}$ denote the MTS and AOR splittings for $\bar{A}$, respectively. Straightforward computations show that

$$\bar{M} - \bar{M}_{r,\omega} = \bar{D}_1 - (\frac{1}{\omega} - 1)\bar{D} + \bar{L}_1 - (1 - \frac{r}{\omega})\bar{L}.$$

Thus, we deuce that $\bar{M} \le \bar{M}_{r,\omega}$. As $\bar{M}^{-1}$ and $\bar{M}_{r,\omega}^{-1}$ are nonnegative matrices, we may conclude that $\bar{M}^{-1} \ge \bar{M}_{r,\omega}^{-1}$. Now the result follows immediately from Theorem 2.5. □

Exploiting a quite similar strategy utilized in the proof of Theorem 4.9, the next proposition can be established. The proposition turns out that when A is an M-matrix and a nonnegative preconditioner $P = [p_{ij}]$ is applied with $p_{ii} = 1$ and $p_{ij} \ge 0$ for $i, j = 1, \ldots, n$ $(i \ne j)$ such that PA is a Z-matrix, the preconditioned GS method outperforms both preconditioned AOR and preconditioned MTS methods. That is the best convergence speed in the (preconditioned) AOR method occurs when $r = \omega = 1$ for $1 \le r \le \omega \le 1$ and $\omega \ne 0$. This fact has been also seen in the numerical experiments part of [22, Table 2].

Proposition 4.10. *Let A be an M-matrix. Suppose that the auxiliary matrices $\bar{D}_1$ and $\bar{L}_1$ satisfy (4.2). Presume that $\rho(\bar{T})$ and $\rho(\bar{T}_G)$ stand for the iteration matrices of the preconditioned MTS and preconditioned GS methods, then $\rho(\bar{T}_G) \le \rho(\bar{T})$.*

Let us consider the preconditioner (1.4), the ensuing theorem demonstrates that the preconditioned MTS (AOR, GS) converges faster than the MTS (AOR, GS) method.

Theorem 4.11. *Let A be an M-matrix. Presume that T and $\bar{T}$ stand for the iteration matrices of the MTS and preconditioned MTS iterative methods. Suppose that $D_1 = \xi D$, $L_1 = \zeta L$, $\bar{D}_1 = \xi\bar{D}$ and $\bar{L}_1 = \zeta\bar{L}$ for $\xi \in [0, 1]$ and $\zeta \in [0, 1]$. Then $\rho(\bar{T}) \le \rho(T) < 1$.*

Proof. The matrix A is an M-matrix which implies that $\rho(T) < 1$ and $\rho(\bar{T}) < 1$. Now we show that $\rho(\bar{T}) \le \rho(T)$. Suppose that

$$\bar{A} = \bar{M} - \bar{N},$$

where

$$\bar{M} = \bar{D} + \bar{D}_1 + \bar{L}_1 - \bar{L}, \qquad \bar{N} = \bar{D}_1 + \bar{L}_1 + \bar{U}.$$

We handel the auxiliary matrices in the next forms

$$D_1 = \xi D, \quad \bar{D}_1 = \xi \bar{D}, \quad L_1 = \zeta L \quad \text{and} \quad \bar{L}_1 = \zeta \bar{L},$$

with $0 \le \xi, \zeta \le 1$. It is not difficult to see that $\bar{D} \le D$, hence $\bar{D}_1 \le D_1$. As $\bar{A}$ is an M-matrix, it is concluded that $\bar{A} = \bar{M} - \bar{N}$ is a regular splitting. Consider the following splitting for the matrix A,

$$M_1 = P^{-1}\bar{M}, \quad N_1 = P^{-1}\bar{N}.$$

Evidently, $A = M_1 - N_1$ is a weak nonnegative splitting of the first kind. Hence Lemma 2.6 implies that $\rho(M_1^{-1}N_1) < 1$. By some straightforward computations, we have $L \le \bar{L}$. Therefore, it can be seen that

$$\begin{aligned} L_1 - L - (\bar{L}_1 - \bar{L}) &= \zeta L - L - \zeta \bar{L} + \bar{L} \\ &= \zeta(L - \bar{L}) + (\bar{L} - L) \\ &= (\zeta - 1)(L - \bar{L}) \ge 0. \end{aligned}$$

Thence, we may conclude that

$$\bar{L}_1 - \bar{L} \le L_1 - L.$$

The above inequality together with the facts that $\bar{D} \le D$ and $\bar{D}_1 \le D_1$ imply that

$$\bar{M} - M = (\bar{D} - D) + (\bar{D}_1 - D_1) + (\bar{L}_1 - \bar{L}) - (L_1 - L) \le 0,$$

i.e., $\bar{M} \le M$. Now, it can be seen that $M^{-1} \le \bar{M}^{-1}$. The result follows from Theorem 2.5 and the subsequent relations

$$M^{-1} \le \bar{M}^{-1} \le \bar{M}^{-1}P = M_1^{-1},$$

and

$$0 \le \bar{M}^{-1}\bar{N} = M_1^{-1}N_1.$$

□

Remark 4.12. We would like to comment here that Theorem 4.11 is true for any preconditioner $P = [p_{ij}]_{n\times n}$ where $p_{ii} = 1$ and $p_{ij} \ge 0$ for $i \ne j$ and $i, j = 1, \dots, n$ such that PA is a Z-matrix; e.g., the preconditioners mentioned in [5].

As seen the preconditioner $P = I + L + U$ is the best between the preconditioners incorporated in the form (1.4); for more details see [25]. Hence, in what follows, we focus on the application of the preconditioner $P = I + L + U$. Considering Theorem 2.5, it is interesting to know that when $A^{-1} > 0$. It is well-known that the inverse of an irreducible M-matrix is a positive matrix. The next theorem provides a mild condition under which the irreducibility of A implies that $\bar{A} = (I + L + U)A$ is irreducible. The proof of the theorem follows from straightforward computations, hence we discard the proof.

n	ω	r	AOR	MTS	GS
100	0.5	0.4	0.9749	0.9619	0.9206
900	0.8	0.6	0.9949	0.9932	0.9898
2500	0.9	0.7	0.9974	0.9969	0.9962

TABLE 1. Comparison results between spectral radii of the iteration matrices for Example 5.1.

Theorem 4.13. *Assume that $A = [a_{ij}] = I - L - U$ is an irreducible Z-matrix where L and U are strictly lower and strictly upper triangular matrices respectively. Moreover, suppose that there exists $\tau \neq i, j$ such that $a_{i\tau}a_{\tau j} \neq 0$ whenever $a_{ij} \neq 0$ where $i \neq j$. Then $\bar{A} = (I + L + U)A$ is irreducible.*

Remark 4.14. Suppose that the same hypophyses assumed in Theorem 4.13 satisfy. With a similar manner exploited in the proof of Proposition 3.5, we may conclude that the iteration matrices of the preconditioned AOR, preconditioned MTS and preconditioned GS methods are irreducible.

5. **Numerical experiments**

In order to illustrate the validity of the results established in this work, we report some numerical experiments in this section. All numerical procedures were computed in MATHEMATICA 6.

In the subsequent two examples, the right sides b are chosen such that $b = Ae$ where $e = (1, \ldots, 1)^T$. The initial guess is taken to be zero. It is not difficult to verify that the considered coefficient matrices are M-matrix. For simplicity, we convert all of the main diagonal elements of A to one.

In all of the following simulations, the auxiliary matrices are selected such that $D_1 = 0.5(\frac{1}{\omega} - 1)I$, $L_1 = 0.5(1 - \frac{r}{\omega})L$, $\bar{D}_1 = 0.5(\frac{1}{\omega} - 1)I$ and $\bar{L}_1 = 0.5(1 - \frac{r}{\omega})\bar{L}$.

Example 5.1. As a symmetric example, we mention the Poisson equation in two dimensions, i.e.

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega = (0,1) \times (0,1), \\ u &= 0 \quad \text{on} \quad \Gamma := \partial\Omega. \end{aligned}$$

A finite difference discretisation with mesh-width $h = \frac{1}{N+1}$ leads to a linear system $Ax = b$ where $A \in \mathbb{R}^{n \times n}$ and $n = N^2$. It can be seen that the matrix A has the following form $A = I_N \otimes B + T \otimes I_N$ where $B = \text{trid}\,(-1/4, 1, -1/4)$ and $T = \text{trid}\,(-1/4, 0, -1/4)$ are $N \times N$ matrices.

Example 5.2. As a non-symmetric example, let us consider the two dimensional convection–diffusion equation

$$-(u_{xx} + u_{yy}) + 2exp(x + y)(xu_x + yu_y) = f(x, y),$$

n	ω	r	PAOR	PMTS	PGS
100	0.5	0.4	0.9357	0.9033	0.8020
900	0.8	0.6	0.9864	0.9819	0.9729
2500	0.9	0.7	0.9930	0.9918	0.9899

TABLE 2. Comparison results between spectral radii of the iteration matrices for Example 5.1.

n	ω	r	AOR	MTS	GS
100	0.5	0.4	0.9786	0.9675	0.9322
900	0.8	0.6	0.9950	0.9937	0.9913
2500	0.9	0.7	0.9978	0.9973	0.9968

TABLE 3. Comparison results between spectral radii of the iteration matrices for Example 5.2.

on $\Omega = (0,1) \times (0,1)$, with the homogeneous Dirichlet boundary conditions. By discretizatig the above equation with the central difference scheme on a uniform grid with $N \times N$ interior nodes ($n = N^2$), we can derive the system of linear equations $Ax = b$ with the five diagonal coefficient matrix; see [1].

In Tables 1 and 2 (3 and 4), we report the comparison results between the spectral radii of the AOR, MTS and GS iterative methods and their preconditioned versions where the preconditioner $P = I + L + U$ is applied. The results confirm the validity of our established results. It is seen that the (preconditioned) MTS method surpasses the (preconditioned) AOR method. As demonstrated theoretically, the (preconditioned) GS iterative method is superior than both (preconditioned) AOR and (preconditioned) MTS methods. For more details, we utilize the AOR, MTS, GS, preconditioned AOR (PAOR), preconditioned MTS (PMTS) and preconditioned GS (PGS) methods for solving linear system $Ax = b$ and stop the iterations as soon as

$$\frac{\|b - Ax(k)\|}{\|b\|} < \epsilon, \tag{5.1}$$

where $x(k)$ denotes the kth approximate solution and $\epsilon = 10^{-5}$. The corresponding results are reported in Table 5 which demonstrate the validity of the established results.

Let us compare the application of the preconditioners (1.3) and (1.4) for improving the speed of convergence of the GMRES(10) [21]. In Tables 5 and 6, we present the required number of iterations and CPU-times(s) of the GMRES(10) and its preconditioned version (PGMRES(10)) methods when the stopping criterion (5.1) is exploited with $\epsilon = 10^{-6}$.

n	ω	r	PAOR	PMTS	PGS
100	0.5	0.4	0.9446	0.9166	0.8286
900	0.8	0.6	0.9868	0.9832	0.9769
2500	0.9	0.7	0.9940	0.9930	0.9914

TABLE 4. Comparison results between spectral radii of the iteration matrices for Example 5.2.

Example	GS	MTS	AOR	PGS	PMTS	PAOR
Example 5.1	920	1124	1328	357	436	516
	(0.69)	(0.73)	(1.04)	(0.24)	(0.33)	(0.38)
Example 5.2	1074	1319	1564	415	510	605
	(0.83)	(1.03)	(1.24)	(0.34)	(0.40)	(0.48)

TABLE 5. Number of iterations (CPU-time in second) for $n = 1024$ ($N = 32$) with $r = 0.7$ and $\omega = 0.9$.

Example	PGMRES(10) $P = I + L + U$	PGMRES(10) [18, 22]	GMRES(10)
Example 5.1	15 (0.062)	21 (0.156)	20 (0.125)
Example 5.2	15 (0.063)	39 (0.172)	42 (0.203)

TABLE 6. Number of iterations (CPU-time in second) for $n = 2500$ ($N = 50$).

6. **Conclusion**

Recently, Dehghan and Hajarian [J. Vib. Control, 20 (2014), no. 5, 661–669] have offered two types of preconditioners to improve the rate of convergence of the AOR method for solving linear system $Ax = b$ under the restriction that the method is convergent. The authors have aimed to propose modified AOR methods under weaker condition than those assumed by Milaszewicz [Linear Algebra Appl. 93 (1987) 161–170] when A is an L-matrix.

Lately, Saberi-Najafi et al. [Mediterr. J. Math. DOI: 10.1007/s00009-014-0412-3] have demonstrated that if A is an irreducible M-matrix, the Milaszewicz's preconditioner can be exploited under mild conditions and it also performs better than preconditioners introduced by Dehghan and Hajarian for ameliorating the rate of convergence of the AOR method.

We have demonstrated that some of the restrictions assumed by the Milaszewicz hold if A is assumed to be an M-matrix (or the AOR method is convergent for the L-matrix A). Hence these restrictions have not been relaxed really. To improve the rate of convergence of the Mixed-Type Splitting (MTS) iterative method, we have examined the application of a quite general preconditioner applied by Wang and Song [J. Comput. Appl. Math. 226 (2009), no.

1, 114–124]. In this work, it has been proved that the (preconditioned) MTS works better than the (preconditioned) AOR method by choosing appropriate auxiliary matrices. Numerical experiments have illustrated the validity of the theoretical results established through the paper.

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