ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Special Issue of the

Bulletin of the Iranian Mathematical Society

in Honor of Professor Heydar Radjavi's 80th Birthday

Vol. 41 (2015), No. 7, pp. 29–45

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 7, pp. 29–45
Online ISSN: 1735-8515

HIGHER NUMERICAL RANGES OF MATRIX POLYNOMIALS

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(Communicated by Peter Šemrl)

Dedicated to Professor Heydar Radjavi on his 80th birthday

Abstract. Let $P(\lambda)$ be an n-square complex matrix polynomial, and $1 \leq k \leq n$ be a positive integer. In this paper, some algebraic and geometrical properties of the k-numerical range of $P(\lambda)$ are investigated. In particular, the relationship between the k-numerical range of $P(\lambda)$ and the k-numerical range of its companion linearization is stated. Moreover, the k-numerical range of the basic A-factor block circulant matrix, which is the block companion matrix of the matrix polynomial $P(\lambda) = \lambda^m I_n - A$, is studied.

Keywords: k-Numerical range, matrix polynomial, companion linearization, basic A-factor block circulant matrix.
MSC(2010): Primary: 15A60; Secondary: 15A18, 47A56.

1. Introduction and preliminaries

Let $\mathbb{M}_{n\times m}$ be the vector space of all $n \times m$ complex matrices. For the case $n = m$, $\mathbb{M}_{n\times n}$ is denoted by $\mathbb{M}_n$; namely, the algebra of all $n \times n$ complex matrices. Throughout the paper, k, m and n are considered as positive integers, and $k \leq n$. Moreover, I_k denotes the $k \times k$ identity matrix. The set of all $n \times k$ isometry matrices is denoted by $\mathcal{X}_{n\times k}$,; i.e., $\mathcal{X}_{n\times k} = \{X \in \mathbb{M}_{n\times k} : X^*X = I_k\}$. Also, the group of $n \times n$ unitary matrices is denoted by $\mathcal{U}_n$; namely, $\mathcal{U}_n = \{U \in \mathbb{M}_n \ : \ U^*U = I_n\} = \mathcal{X}_{n\times n}$. The notion of the k-numerical range of $A \in \mathbb{M}_n$, which was first introduced by P. R. Halmos [10], is defined and denoted by

$$W_k(A) = \{\frac{1}{k}\mathrm{tr}\,(\mathrm{X^*AX}) \ : \ \mathrm{X} \in \mathcal{X}_{\mathrm{n\times k}}\},$$

Article electronically published on December 31, 2015.
Received: 15 October 2014, Accepted: 11 January 2015.
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where tr(.) denotes the trace. The sets $W_k(A)$, where $k \in \{1, 2, \ldots, n\}$, are generally called *higher numerical ranges* of A. When $k = 1$, we have the classical numerical range $W_1(A) = W(A) := \{x^*Ax \ : \ x \in \mathbb{C}^n, \ x^*x = 1\}$, which has been studied extensively; see for example [9] and [12, Chapter 1]. Motivation of our study comes from finite-dimensional quantum systems. In quantum physics, e.g., see [8], quantum states are represented by density matrices, i.e., positive semidefinite matrices with trace one. If a quantum state $D \in \mathbb{M}_n$ has rank one, i.e., $D = xx^*$ for some $x \in \mathbb{C}^n$ with $x^*x = 1$, then D is called a pure quantum state; otherwise, D is said to be a mixed quantum state, which can be written as a convex combination of pure quantum states. So, for $A \in \mathbb{M}_n$, we have $W(A) = \{\mathrm{tr(AD)} \ : \ \mathrm{D} \in \mathbb{M}_\mathrm{n} \text{ is a pure quantum state}\}$. Also, by the fact that the convex hull of the set $\{\frac{1}{k}P \ : P \in \mathbb{M}_n, \ P^2 = P = P^*, \ tr(P) = k\}$ equals to the set $\mathcal{S}_k$ of density matrices $D \in \mathbb{M}_n$ satisfying $\frac{1}{k}I_n - D$ is positive semidefinite, we have

$$\begin{aligned} W_k(A) &= \{\frac{1}{k}\mathrm{tr(AP)} \ : \ \mathrm{P} \in \mathbb{M}_\mathrm{n}, \ \mathrm{P}^2 = \mathrm{P} = \mathrm{P}^*, \ \mathrm{tr(P)} = \mathrm{k}\} \\ &= \{tr(AD) \ : \ D \in \mathcal{S}_k\}. \end{aligned}$$

Let $A \in \mathbb{M}_n$ have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ counting multiplicities. The set of all k-averages of eigenvalues of A is denoted by $\sigma^{(k)}(A)$; namely,

$$\sigma^{(k)}(A) = \{\frac{1}{k}\left(\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k}\right) \ : \ 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}.$$

Notice that if $k = 1$, then $\sigma^{(1)}(A) = \sigma(A)$, i.e., the spectrum of A. Next, we list some properties of the k-numerical range of matrices which will be useful in our discussion. For more details, see [6, 10, 14, 15, 18].

Proposition 1.1. *Let $A \in \mathbb{M}_n$. Then the following assertions are true:*
(i) $W_k(A)$ *is a compact and convex set in* $\mathbb{C}$*;*
(ii) $\mathrm{conv}\left(\sigma^{(\mathrm{k})}(\mathrm{A})\right) \subseteq \mathrm{W_k(A)}$*, where* $\mathrm{conv(S)}$ *denotes the convex hull of a set* $S \subseteq \mathbb{C}$*. The equality holds if* A *is normal;*
(iii) $\{\frac{1}{n}\mathrm{tr(A)}\} = \mathrm{W_n(A)} \subseteq \mathrm{W_{n-1}(A)} \subseteq \cdots \subseteq \mathrm{W_2(A)} \subseteq \mathrm{W_1(A)} = \mathrm{W(A)}$*;*
(iv) If $V \in \mathcal{X}_{n\times s}$*, where* $k \leq s \leq n$*, then* $W_k(V^*AV) \subseteq W_k(A)$*. The equality holds if* $s = n$*, i.e.,* $W_k(U^*AU) = W_k(A)$*, where* $U \in \mathcal{U}_n$*;*
(v) For any $\alpha, \beta \in \mathbb{C}$*,* $W_k(\alpha A + \beta I_n) = \alpha W_k(A) + \beta$*, and for the case* $k < n$*,* $W_k(A) = \{\alpha\}$ *if and only if* $A = \alpha I_n$*;*
(vi) $W_k(A^*) = \overline{W_k(A)}$*;*
(vii) For the case $k < n$*,* $W_k(A) \subseteq \mathbb{R}$ *if and only if* A *is Hermitian.*

At the end of this section, we give some information about matrix polynomials. Notice that matrix polynomials arise in many applications and their spectral analysis is very important when studying linear systems of ordinary differential equations with constant coefficients; e.g., see [7]. Suppose that

$$P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0 \tag{1.1}$$

is a matrix polynomial, where $A_i \in \mathbb{M}_n$ $(i = 0, 1, \ldots, m)$, $A_m \neq 0$ and λ is a complex variable. The numbers m and n are referred as the *degree* and the *order* of $P(\lambda)$, respectively. The matrix polynomial $P(\lambda)$, as in (1.1), is called a *monic matrix polynomial* if $A_m = I_n$. It is said to be a *selfadjoint* matrix polynomial if all the coefficients A_i are Hermitian matrices. A scalar $\lambda_0 \in \mathbb{C}$ is an *eigenvalue* of $P(\lambda)$ if the system $P(\lambda_0)x = 0$ has a nonzero solution $x_0 \in \mathbb{C}^n$. This solution x_0 is known as an *eigenvector* of $P(\lambda)$ corresponding to λ_0, and the set of all eigenvalues of $P(\lambda)$ is said to be the *spectrum* of $P(\lambda)$; namely,

$$\sigma[P(\lambda)] = \{\mu \in \mathbb{C} \ : \ \det(P(\mu)) = 0\}.$$

The *(classical) numerical range* of $P(\lambda)$ is defined and denoted by

$$W[P(\lambda)] := \{\mu \in \mathbb{C} \ : \ x^* P(\mu) x = 0 \text{ for some nonzero} x \in \mathbb{C}^n\},$$

which is closed and contains $\sigma[P(\lambda)]$. The numerical range of matrix polynomials plays an important role in the study of overdamped vibration systems with finite number of degrees of freedom, and it is also related to the stability theory; e.g., see [13] and its references. Notice that the notion of $W[P(\lambda)]$ is a generalization of the classical numerical range of a matrix $A \in \mathbb{M}_n$; namely, $W[\lambda I_n - A] = W(A)$.

Let $C \in \mathbb{M}_n$ and $P(\lambda)$ be a matrix polynomial as in (1.1). The *C-numerical range* and the *C-spectrum* of $P(\lambda)$ are, respectively, defined and denoted, see [1], by

$$W_C[P(\lambda)] = \{\mu \in \mathbb{C} \ : \ \mathrm{tr}(\mathrm{C}\mathrm{U}^*\mathrm{P}(\mu)\mathrm{U}) = 0 \text{ for some } \mathrm{U} \in \mathcal{U}_\mathrm{n}\}, \tag{1.2}$$

and

$$\sigma_C[P(\lambda)] = \{\mu \in \mathbb{C} \ : \ \sum_{j=1}^{n} \gamma_j \alpha_{i_j}^{(\mu)} = 0 \ \ \textit{for some permutation} \ (i_1, \ldots, i_n) \ \textit{of} \ \{1, 2, \ldots, n\}\}, \tag{1.3}$$

where $\gamma_1, \ldots, \gamma_n$ are the eigenvalues of C, and for $\mu \in \mathbb{C}$, $\alpha_1^{(\mu)}, \ldots, \alpha_n^{(\mu)}$ are the eigenvalues of the matrix $P(\mu) \in M_n$. Denote by $E_{ij} \in \mathbb{M}_n$, where $i, j \in \{1, 2, \ldots, n\}$, the matrix whose (i, j)-entry is equal to one and all the others are equal to zero. For the case $C = E_{11} \in \mathbb{M}_n$, we have $W_{E_{11}}[P(\lambda)] = W[P(\lambda)]$ and $\sigma_{E_{11}}[P(\lambda)] = \sigma[P(\lambda)]$. So, $W_C[P(\lambda)]$ is a generalization of the numerical range of $P(\lambda)$. In the last few years, the generalization of the numerical range of matrix polynomials has attracted much attention and many interesting results have been obtained; e.g., see [1–3, 19] and [20]. In this paper, we continue the study of the C-numerical range of matrix polynomials for the case $C = \frac{1}{k}E_{11} + \frac{1}{k}E_{22} + \cdots + \frac{1}{k}E_{kk} \in \mathbb{M}_n$. For this, in Section 2, we introduce the notion of the k-numerical range of matrix polynomials as a spacial case of the C-numerical range of matrix polynomials, and we state all results from [1] which hold for this set. In Section 3, we study the relationship between the k-numerical range of a matrix polynomial and the k-numerical range of its companion linearization. In this section, the emphasis is on the study algebraic and geometrical properties of the k-numerical range of the basic A-factor block circulant matrix, which is the block companion matrix of the matrix polynomial $P(\lambda) = \lambda^m I_n - A$. In Section 4, we study the number of connected components, the isolated points and the boundedness of the k-numerical range of matrix polynomials.

2. k-numerical range of matrix polynomials

The our aim of this section is to introduce the notion of k-numerical range of matrix polynomials and also is to state all results from [1] which are hold for this notion. For this, let $P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$ be a matrix polynomial as in (1.1). By setting $C = \frac{1}{k}(E_{11}+E_{22}+\cdots+E_{kk}) \in \mathbb{M}_n$ in (1.2), we denote $W_C[P(\lambda)]$ by $W_k[P(\lambda)]$, and we call this set as the *k-numerical range* of $P(\lambda)$; namely,

$$W_k[P(\lambda)] = \{\mu \in \mathbb{C} \ : \ \mathrm{tr(X^*P(\mu)X) = 0 \text{ for some } X \in \mathcal{X}_{n\times k}}\}. \tag{2.1}$$

Also, in this case, we denote the C-spectrum of $P(\lambda)$, $\sigma_C[P(\lambda)]$ as in (1.3), by $\sigma^{(k)}[P(\lambda)]$; namely,

$$\sigma^{(k)}[P(\lambda)] = \left\{\mu \in \mathbb{C} \ : \ 0 \in \sigma^{(k)}(P(\mu))\right\}. \tag{2.2}$$

We also define the *joint k-numerical range* of $P(\lambda)$ as the joint k-numerical range of its coefficients, i.e.,

$$\begin{aligned} JW_k[P(\lambda)] &= W_k(A_0, A_1, \ldots, A_m) \\ &:= \left\{\left(\frac{1}{k}\mathrm{tr(X^*A_0X)}, \ldots, \frac{1}{\mathrm{k}}\mathrm{tr(X^*A_mX)}\right) \ : \ X \in \mathcal{X}_{n\times k}\right\}. \end{aligned}$$

It is clear that:

$$\begin{aligned} W_k[P(\lambda)] &= \{\mu \in \mathbb{C} \ : \ 0 \in W_k(P(\mu))\} \\ &= \{\mu \in \mathbb{C} \ : \ a_m\mu^m + \cdots + a_1\mu + a_0 = 0, \\ &\qquad\qquad (a_0, a_1, \ldots, a_m) \in W_k(A_0, A_1, \ldots, A_m)\}. \end{aligned}$$

Moreover, if $P(\lambda) = \lambda I_n - A$, where $A \in \mathbb{M}_n$, then $W_k[P(\lambda)] = W_k(A)$ and $\sigma^{(k)}[P(\lambda)] = \sigma^{(k)}(A)$. The sets $W_k[P(\lambda)]$, where $k \in \{1, 2, \ldots, n\}$, are generally called the *higher numerical ranges* of $P(\lambda)$. Now we are ready to state all results from [1] which hold for the k-numerical range of matrix polynomials. Recall that these results follow from this fact that the matrix C in [1] equals to $\frac{1}{k}(E_{11} + E_{22} + \cdots + E_{kk}) \in \mathbb{M}_n$.

In the following theorem, we state some basic properties.

Theorem 2.1. *[1, Theorem 2.3] Let $P(\lambda)$ be a matrix polynomial as in (1.1). Then the following assertions are true:*

(i) $W_k[P(\lambda)]$ *is a closed set in* $\mathbb{C}$ *which contains* $\sigma^{(k)}[P(\lambda)]$*;*
(ii) $W_k[P(\lambda+\alpha)] = W_k[P(\lambda)] - \alpha$*, where* $\alpha \in \mathbb{C}$*;*
(iii) $W_k[\alpha P(\lambda)] = W_k[P(\lambda)]$*, where* $\alpha \in \mathbb{C}$ *is nonzero;*
(iv) If $Q(\lambda) = \lambda^m P(\lambda^{-1}) := A_0\lambda^m + A_1\lambda^{m-1} + \cdots + A_{m-1}\lambda + A_m$*, then*

$$W_k[Q(\lambda)] \setminus \{0\} = \{\frac{1}{\mu} \ : \ \mu \in W_k[P(\lambda)], \ \mu \neq 0\};$$

(v) If all the powers of λ *in* $P(\lambda)$ *are even (or all of them are odd), then* $W_k[P(\lambda)]$ *is symmetric with respect to the origin;*
(vi) If $P(\lambda)$ *is a selfadjoint matrix polynomial, or if all* $A_0, A_1, \ldots, A_m$ *are real matrices, then* $W_k[P(\lambda)]$ *is symmetric with respect to the real axis.*

In the following theorem, some geometrical properties are stated.

Theorem 2.2. *Let* $P(\lambda)$ *be a matrix polynomial as in (1.1). Then the following assertions are true:*
(i) [1, Theorem 2.4] If $0 \notin W_k(A_m)$*, then* $W_k[P(\lambda)]$ *is bounded;*
(ii) [1, Theorem 2.7] If $\mu \in \partial W_k[P(\lambda)]$*, then* $0 \in \partial W_k(P(\mu))$*;*
(iii) [1, Theorem 3.3]

$$W_k[P(\lambda)] = \{\mu \in \mathbb{C} : a_m\mu^m + \cdots + a_1\mu + a_0 = 0, \\ (a_0, a_1, \ldots, a_m) \in \operatorname{conv}(W_k(A_0, A_1, \ldots, A_m))\}$$

(iv) [1, Theorem 3.1(ii)] $W_k[P(\lambda)] = \bigcup W_k[D(\lambda)]$*, where the union is taken over all diagonal matrix polynomials* $D(\lambda)$ *of degree* m *and order* n *such that* $JW_k[D(\lambda)] \subseteq JW_k[P(\lambda)]$*;*
(v) [1, Corollary 3.2] If $(0, 0, \ldots, 0) \in JW_k[P(\lambda)]$*, then* $W_k[P(\lambda)] = \mathbb{C}$*;*
(vi) [1, Theorem 3.4] If $a_m\mu^m + \cdots + a_1\mu + a_0 = 0$*, where* $\mu \in \mathbb{C}$ *and* $(a_0, a_1, \ldots, a_m) \in \operatorname{Int}(JW_k[P(\lambda)])$*, then* $\mu \in \operatorname{Int}(W_k[P(\lambda)])$*. Here,* $\operatorname{Int}(S)$ *denotes the set of all interior points of* $S \subseteq \mathbb{C}$*.*

For the final result from [1], we recall that the *k-numerical radius* of $A \in \mathbb{M}_n$ is

$$r_k(A) = \max_{z \in W_k(A)} |z|.$$

Theorem 2.3. *Let* $P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$*, as in (1.1), be a monic matrix polynomial (i.e.,* $A_m = I_n$*). Then:*
(i) [1, Theorem 2.9] $W_k[P(\lambda)] \subseteq \{z \in \mathbb{C} \ : \ p \leq |z| \leq 1 + q\}$*, where*

$$p = \frac{\mathrm{dist}\,(0, \mathrm{W_k(A_0)})}{\mathrm{dist}\,(0, \mathrm{W_k(A_0)}) + \max_{\mathrm{j=1,2,\ldots,m}} \mathrm{r_k(A_j)}} \ \textit{and}\ q = \max_{j=0,1,\ldots,m-1} r_k(A_j);$$

(ii) [1, Theorem 2.10] If $\mu \notin W_k[P(\lambda)]$*, then*

$$W_k[P(\lambda)] \bigcap \{z \in \mathbb{C} \ : |\ z - \mu \ | < \rho_\mu\} = \emptyset,$$

where $\rho_\mu = \frac{\mathrm{dist}(0, \mathrm{W_k(P(\mu))})}{\mathrm{dist}(0, \mathrm{W_k(P(\mu))}) + \frac{1}{\mathrm{k}} \max_{\mathrm{j}=1,2,\ldots,\mathrm{m}} \sum_{\mathrm{i}=1}^{\mathrm{k}} \mathrm{s_i}(\frac{1}{\mathrm{j!}}\mathrm{P^{(j)}}(\mu))}$,
in which, for a matrix $X \in \mathbb{M}_n$, $s_1(X) \geq s_2(X) \geq \cdots \geq s_n(X)$ *are the sigular values of* X.

3. k-Numerical range of basic A-factor block circulant matrices

Consider a matrix polynomial $P(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0$ as in (1.1), in which $m \geq 2$. The *companion linearization* of $P(\lambda)$ is defined, e.g., see [7], as the following linear pencil $L(\lambda)$ of order mn:

$$L(\lambda) = \begin{pmatrix} I_n & 0 & 0 & \cdots & 0 \\ 0 & I_n & 0 & \cdots & 0 \\ \vdots & \ldots & \ddots & \ldots & \vdots \\ 0 & \cdots & 0 & I_n & 0 \\ 0 & 0 & \cdots & 0 & A_m \end{pmatrix} \lambda - \begin{pmatrix} 0 & I_n & 0 & 0 & \cdots & 0 \\ 0 & 0 & I_n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & I_n \\ -A_0 & -A_1 & \cdots & \cdots & \cdots & -A_{m-1} \end{pmatrix}. \tag{3.1}$$

By [7, page 186], there are unimodular matrix polynomials $E(\lambda)$ and $F(\lambda)$ of order mn such that $E(\lambda)L(\lambda)F(\lambda) = \begin{pmatrix} P(\lambda) & 0 \\ 0 & I_{n(m-1)} \end{pmatrix}$. So, every eigenvalue of $P(\lambda)$ is an eigenvalue of $L(\lambda)$ with the same multiplicity, and vice versa. Hence, for any positive integer $1 \leq k \leq mn$,

$$\sigma^{(k)}[P(\lambda)] = \sigma^{(k)}[L(\lambda)].$$

Proposition 3.1. *Let* $P(\lambda)$*, as in (1.1), be a matrix polynomial such that all the powers of* λ *are even (or all of them are odd). Moreover, let* $L(\lambda)$*, as in (3.1), be the companion linearization of* $P(\lambda)$*, and* $1 \leq k \leq mn$ *be a positive integer. Then* $W_k[L(\lambda)]$ *is symmetric with respect to the origin.*

Proof. Without loss of generality, we assume that all the powers of λ are even. Now, let $\mu \in W_k[L(\lambda)]$ be given. Then there exists a

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \ldots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mk} \end{pmatrix} \in \mathcal{X}_{mn\times k},$$

where $x_{ij} \in \mathbb{C}^n$, such that $\mathrm{tr}(\mathrm{X}^*\mathrm{L}(\mu)\mathrm{X}) = 0$. By setting

$$Y = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1k} \\ y_{21} & y_{22} & \cdots & y_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mk} \end{pmatrix} \in \mathbb{M}_{mn\times k},$$

where $y_{ij} = \begin{cases} x_{ij} & \text{for } i \text{ odd} \\ -x_{ij} & \text{for } i \text{ even} \end{cases}$ we have $Y^*Y = X^*X = I_k$ and

$$\mathrm{tr}\,(\mathrm{Y}^*\mathrm{L}(-\mu)\mathrm{Y}) = -\mathrm{tr}\,(\mathrm{X}^*\mathrm{L}(\mu)\mathrm{X}) = 0.$$

So, $-\mu \in W_k[L(\lambda)]$, and hence the proof is complete. □

In the following theorem, which is a generalization of [17, Proposition 2.4], we state the relationship between the k-numerical range of $P(\lambda)$ and the k-numerical range of its companion linearization $L(\lambda)$.

Theorem 3.2. *Let $1 \le k \le n$ be a positive integer, and $P(\lambda)$, as in (1.1), be a matrix polynomial with the companion linearization $L(\lambda)$ as in (3.1). Then*

$$W_k[P(\lambda)] \cup \{0\} \subseteq W_k[L(\lambda)].$$

Proof. For any $\mu \in \mathbb{C}$ and $X \in \mathcal{X}_{n\times k}$, we consider the following matrix:

$$Y = \frac{1}{\sqrt{1 + |\mu|^2 + |\mu|^4 + \cdots + |\mu|^{2m-2}}} \begin{pmatrix} I_n \\ \mu I_n \\ \vdots \\ \mu^{m-1} I_n \end{pmatrix} X \in \mathbb{M}_{mn\times k}.$$

Then we have $Y^*Y = X^*X = I_k$, and

$$Y^*L(\mu)Y = \frac{\overline{\mu^{m-1}}}{1+ \mid \mu \mid^2 + \cdots + \mid \mu \mid^{2m-2}} X^*P(\mu)X. \qquad (*)$$

Now, let $\mu \in W_k[P(\lambda)] \cup \{0\}$ be given. We will show that $\mu \in W_k[L(\lambda)]$. If $\mu = 0$, then by selecting any $X \in \mathcal{X}_{n\times k}$, the relation $(*)$ shows that $Y \in \mathcal{X}_{mn\times k}$ and $\mathrm{tr}\,(\mathrm{Y}^*\mathrm{L}(0)\mathrm{Y}) = 0$. So, $0 \in W_k[L(\lambda)]$.

If $\mu \in W_k[P(\lambda)]$, then by (2.1), there exists a $X \in \mathcal{X}_{n\times k}$ such that $\mathrm{tr}\,(\mathrm{X}^*\mathrm{P}(\mu)\mathrm{X}) = 0$. Therefore, the relation $(*)$ shows that $Y \in \mathcal{X}_{mn\times k}$ and $\mathrm{tr}\,(\mathrm{Y}^*\mathrm{L}(\mu)\mathrm{Y}) = 0$. Hence, $\mu \in W_k[L(\lambda)]$. □

Corollary 3.3. *If $W_k[L(\lambda)]$ is bounded, then $W_k[P(\lambda)]$ is also bounded.*

The converse statement in Corollary 3.3 is not true, as is illustrated in the following example.

Example 3.4. Let $P(\lambda) = I_2\lambda^2 - I_2\lambda$. So, by (2.1), we have $W_2[P(\lambda)] = \{0, 1\}$, which is bounded. The companion linearization of $P(\lambda)$ is $L(\lambda) = S_1\lambda - S_0$, where $S_1 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \in M_4$ and $S_0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \in M_4$. By setting $X = \begin{pmatrix} e_1 & e_3 \end{pmatrix} \in M_{4\times 2}$, where $e_i \in \mathbb{C}^4$ is the ith standard vector, we have $X \in \mathcal{X}_{4\times 2}$ and $\mathrm{tr}(X^*S_1X) = 0 = \mathrm{tr}(X^*S_0X)$. So, $(0, 0) \in JW_2[L(\lambda)]$, and hence, by Theorem 2.2(v), $W_2[L(\lambda)] = \mathbb{C}$, which is unbounded.

For the remainder of this section, we study some algebraic and geometrical properties of the k-numerical range of the companion linearization of the matrix polynomial $P(\lambda) = \lambda^m I_n - A$, where $m \geq 2$ and $A \in M_n$. By (3.1), the companion linearization of $P(\lambda)$ is $L(\lambda) = \lambda I_{mn} - \Pi_A$, where

$$\Pi_A = \begin{pmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_n \\ A & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathbb{M}_{mn}. \tag{3.2}$$

The matrix Π_A, as in (3.2), is called the *basic A-factor block circulant matrix*. These matrices have important applications in vibration analysis and differential equations. For more information, see [4, 5] and their references.

The following theorem shows that $W_k(\Pi_A)$ is invariant under some rotations.

Theorem 3.5. *Let $A \in M_n$, $1 \leq k \leq mn$ be a positive integer, and ω be an mth root of unity (i.e., $\omega^m = 1$). Then*

$$\omega W_k(\Pi_A) = W_k(\Pi_A).$$

Consequently, if m is even, then $W_k(\Pi_A)$ is symmetric with respect to the origin.

Proof. Since $\omega^m = 1$, there exists a $\theta \in \mathbb{R}$ such that $\omega = e^{i\theta}$. Let $\mu \in W_k(\Pi_A)$ be arbitrary. Then, there exists a $X \in \mathcal{X}_{mn\times k}$ such that $\mu = \frac{1}{k}\mathrm{tr}\,(X^*\Pi_A X)$. Consider $Y = U_\theta X$, where

$$U_\theta = \mathrm{diag}\left(1, e^{i(m-1)\theta}, \ldots, e^{i\theta}\right) \otimes I_n.$$

So, we have $Y^*Y = X^*X = I_k$, and $Y^*\Pi_A Y = e^{-i\theta}X^*\Pi_A X$. Therefore, $e^{-i\theta}\mu = e^{-i\theta}\frac{1}{k}\mathrm{tr}\,(X^*\Pi_A X) = \frac{1}{k}\mathrm{tr}\,(Y^*\Pi_A Y) \in W_k(\Pi_A)$, and hence, $W_k(\Pi_A) \subseteq e^{i\theta}W_k(\Pi_A)$. By changing θ by $-\theta$, we see that $e^{i\theta}W_k(\Pi_A) \subseteq W_k(\Pi_A)$. So, the set equality holds.

If m is even, then by setting $\omega = -1$ in the first case, we have that $\mu \in W_k(\Pi_A)$ if and only if $-\mu \in W_k(\Pi_A)$. So, the second assertion also holds, and hence, the proof is complete. □

Using Theorem 3.2, we state the following result. Recall that for any set $S \subseteq \mathbb{C}$, $\sqrt[m]{S} := \{\mu \in \mathbb{C} \ : \ \mu^m \in S\}$.

Theorem 3.6. *Let $1 \leq k \leq n$ be a positive integer, $A \in \mathbb{M}_n$ and Π_A be the basic A-factor block circulant matrix as in (3.2). Then*

$$\operatorname{conv}\left(\sqrt[m]{W_k(A)}\right) = \operatorname{conv}\left(\sqrt[m]{W_k(A)} \cup \{0\}\right) \subseteq \mathrm{W_k(\Pi_A)}.$$

Proof. Consider the matrix polynomial $P(\lambda) = \lambda^m I_n - A$. The companion linearization of $P(\lambda)$ is $L(\lambda) = \lambda I_{mn} - \Pi_A$. So, by Theorem 3.2, we have:

$$\begin{aligned}
\sqrt[m]{W_k(A)} \cup \{0\} &= W_k[P(\lambda)] \cup \{0\} \\
&\subseteq W_k[L(\lambda)] \\
&= W_k(\Pi_A).
\end{aligned}$$

Now, since $m \geq 2$, $\operatorname{conv}\left(\sqrt[m]{W_k(A)} \cup \{0\}\right) = \operatorname{conv}\left(\sqrt[m]{W_k(A)}\right)$, and hence, the result follows from the above inclusion and the fact that $W_k(\Pi_A)$ is convex (Proposition 1.1(i)). $\square$

The set equality in Theorem 3.6 does not hold in general, which is illustrated in the following example. We use a Matlab program from Li, which is available at *http : //people.wm.edu/ ∼ cklixx/mathlib.html*, for plotting all shapes in this section.

Example 3.7. Let $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{M}_2$, $k = 2$ *and* $m = 3$. We have $W_2(A) = \{\frac{1}{2}\mathrm{tr(A)}\} = \{0\}$ and so, $\operatorname{conv}\left(\sqrt[3]{W_2(A)}\right) = \{0\}$. Since A is unitary, by [2, Theorem 3.3], Π_A is also a unitary matrix. Hence by Proposition 1.1(ii), we have $W_2(\Pi_A) = \operatorname{conv}\left(\sigma^{(2)}(\Pi_\mathrm{A})\right)$. By setting $P(\lambda) = \lambda^3 I_2 - A$, the companion linearization of $P(\lambda)$ is $L(\lambda) = \lambda I_6 - \Pi_A$, and hence, we have $\sigma(\Pi_A) = \sigma[L(\lambda)] = \sigma[P(\lambda)] = \sqrt[3]{\sigma(A)} = \{1, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}, -1, e^{i\frac{\pi}{3}}, e^{i\frac{5\pi}{3}}\}$. So, $\sigma^{(2)}(\Pi_A) = \{0, \pm\frac{1}{2}(1 + e^{i\frac{\pi}{3}}), \pm\frac{1}{2}(1 + e^{i\frac{2\pi}{3}}), \pm\frac{1}{2}(e^{i\frac{\pi}{3}} + e^{i\frac{2\pi}{3}}), \pm\frac{1}{2}(e^{i\frac{\pi}{3}} + e^{i\frac{5\pi}{3}}), \pm\frac{1}{2}(-1 + e^{i\frac{\pi}{3}}), \pm\frac{1}{2}(-1 + e^{i\frac{2\pi}{3}})\}$. Hence,

$$\begin{aligned}
W_2(\Pi_A) &= \operatorname{conv}\left(\sigma^{(2)}(\mathrm{A})\right) \\
&= \operatorname{conv}\left(\left\{\pm\frac{1}{2}(1 + \mathrm{e}^{\mathrm{i}\frac{\pi}{3}}), \pm\frac{1}{2}(\mathrm{e}^{\mathrm{i}\frac{\pi}{3}} + \mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}), \pm\frac{1}{2}(-1 + \mathrm{e}^{\mathrm{i}\frac{2\pi}{3}})\right\}\right) \\
&\neq \{0\},
\end{aligned}$$

which is shown in Figure 1.

In the following example, we characterize the k-numerical range of Π_A, for the case $A = I_n$.

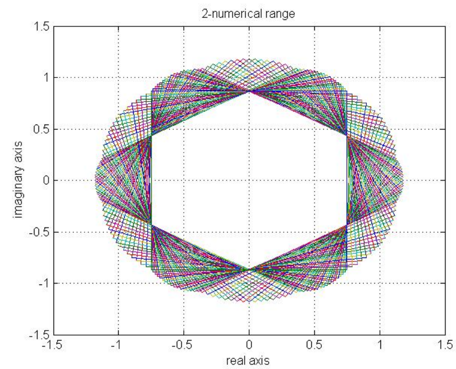

FIGURE 1. $W_2(\Pi_A)$

Example 3.8. Let $m \geq 2$ be a positive integer, and $\Pi_{I_n} \in \mathbb{M}_{mn}$ be the matrix as in (3.2). It is clear that the eigenvalues of Π_{I_n}, counting multiplicities, are

$$\underbrace{1,\ldots,1}_{n-times},\underbrace{\omega,\ldots,\omega}_{n-times},\underbrace{\omega^2,\ldots,\omega^2}_{n-times},\ldots,\underbrace{\omega^{m-1},\ldots,\omega^{m-1}}_{n-times},$$

where $\omega = e^{i\frac{2\pi}{m}}$. So, $\sigma^{(k)}(\Pi_{I_n})$ equals to all points of the following form:

$$\frac{1}{k}(r_0 + r_1\omega + r_2\omega^2 + \cdots + r_{m-1}\omega_{m-1}), \tag{3.3}$$

where $0 \leq r_0, r_1, \ldots, r_{m-1} \leq k$ are positive integers and $r_0+r_1+\cdots+r_{m-1} = k$. Since Π_{I_n} is unitary, by Proposition 1.1(ii), we have

$$W_k(\Pi_{I_n}) = \mathrm{conv}(\sigma^{(\mathrm{k})}(\Pi_{\mathrm{I_n}})).$$

Now, we consider the following cases:
case 1: If $1 \leq k \leq n$, then $\{1, \omega, \omega^2, \ldots, \omega^{m-1}\} \subseteq \sigma^{(k)}(\Pi_{I_n})$ and so,

$$W_k(\Pi_{I_n}) = \mathrm{conv}(\sigma^{(\mathrm{k})}(\Pi_{\mathrm{I_n}})) = \mathrm{conv}(\{1, \omega, \ldots, \omega^{\mathrm{m}-1}\}).$$

case 2: If $k = tn + l$, where $1 \leq t \leq m$ and $0 \leq l \leq n - 1$ are integer numbers, then by considering all the points of the form

$$p_\alpha = \frac{1}{k}\left(n\omega^{\alpha_1} + n\omega^{\alpha_2} + \cdots + n\omega^{\alpha_t} + l\omega^{\alpha_{t+1}}\right),$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{t+1})$ is a $(t+1)$-permutation of $\{0, 1, \ldots, n-1\}$, we have

$$\mathrm{conv}\left(\sigma^{(\mathrm{k})}(\Pi_{\mathrm{I}})\right) = \mathrm{conv}\left(\{\, p_\alpha \ : \ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{t+1}) \text{ is a } (t+1) - \text{permutation of } \{0, 1, \ldots, n-1\}\}\right).$$

For example, if $m=4$ and $n=2$, we see, as in Figure 2, that

$$W_1(\Pi_{I_2}) = W_2(\Pi_{I_2}) = \mathrm{conv}(\{1, \mathrm{i}, -1, -\mathrm{i}\}),$$

$$W_3(\Pi_{I_2}) = \mathrm{conv}\left(\left\{\frac{2+\mathrm{i}}{3}, \frac{2i+1}{3}, \frac{2i-1}{3}, \frac{i-2}{3}, \frac{-2i-1}{3}, \frac{-i-2}{3}, \frac{2-i}{3}, \frac{1-2i}{3}\right\}\right),$$

$$W_4(\Pi_{I_2}) = \mathrm{conv}\left(\left\{\frac{1+\mathrm{i}}{2}, \frac{\mathrm{i}_1}{2}, \frac{-1-\mathrm{i}}{2}, \frac{1-\mathrm{i}}{2}\right\}\right),$$

$$W_5(\Pi_{I_2}) = \mathrm{conv}\left(\left\{\frac{1+2\mathrm{i}}{5}, \frac{2+\mathrm{i}}{5}, \frac{\mathrm{i}}{5}, \frac{-\mathrm{i}}{5} = \frac{2-\mathrm{i}}{5}, \frac{1-2\mathrm{i}}{5}, \frac{-1+2\mathrm{i}}{5}, \frac{-2+i}{5}, \frac{-1-2i}{5}, \frac{-2-i}{5}\right\}\right)$$

$$= \mathrm{conv}\left(\left\{\frac{1+2\mathrm{i}}{5}, \frac{2+\mathrm{i}}{5}, \frac{2-\mathrm{i}}{5}, \frac{1-2\mathrm{i}}{5}, \frac{-1+2\mathrm{i}}{5}, \frac{-2+\mathrm{i}}{5}, \frac{-1-2i}{5}, \frac{-2-i}{5}\right\}\right),$$

$$W_6(\Pi_{I_2}) = \mathrm{conv}\left(\left\{\frac{\mathrm{i}}{3}, \frac{1}{3}, \frac{-\mathrm{i}}{3}, \frac{-1}{3}\right\}\right),$$

$$W_7(\Pi_{I_2}) = \mathrm{conv}\left(\left\{\frac{\mathrm{i}}{7}, \frac{1}{7}, \frac{-\mathrm{i}}{7}, \frac{-1}{7}\right\}\right),$$

and

$$W_8(\Pi_{I_2}) = \{\frac{1}{8}\mathrm{tr}(\Pi_{\mathrm{I}_2})\} = \{0\}.$$

At the end of this section, we find a circular disk which contains $W_k(\Pi_A)$. Then, using this disk, we obtain an upper bound for $r_k(\Pi_A)$ and we show that this bound is sharp.

Theorem 3.9. *Let $1 \le k \le mn$ be a positive integer, $A \in \mathbb{M}_n$ and Π_A be the basic A-factor block circulant matrix as in (3.2). Then*

$$W_k(\Pi_A) \subseteq \{\mu \in \mathbb{C} : |\mu| \le 1 + \|A - I_n\|\},$$

where $\|.\|$ is the spectral matrix norm (i.e., the matrix norm subordinate to the Euclidian vector norm).

Proof. Let $\mu \in W_k(\Pi_A)$ be given. Then there exists a $X \in \mathcal{X}_{mn\times k}$ such that $\mu = \frac{1}{k}\mathrm{tr}\,(\mathrm{X}^*\Pi_\mathrm{A}\mathrm{X})$. By setting

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mk} \end{pmatrix} \in \mathcal{X}_{mn\times k},$$

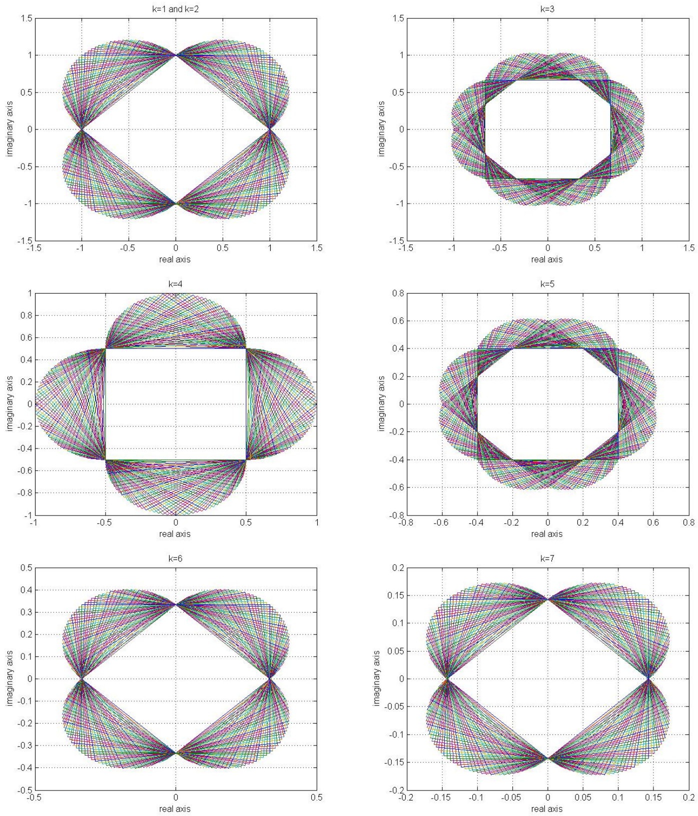

FIGURE 2. $W_k(\Pi_{I_2}),\ \ for\ k = 1, 2, 3, 4, 5, 6, 7$

where $x_{ij} \in \mathbb{C}^n$, we have:

$$\begin{aligned}\mu = \frac{1}{k}\operatorname{tr}\left(\mathrm{X}^*\Pi_{\mathrm{A}}\mathrm{X}\right) &= \frac{1}{k}\left(\sum_{j=1}^{k}\sum_{i=1}^{m-1} x_{ij}^* x_{(i+1)j} + \sum_{j=1}^{k} x_{mj}^* A x_{1j}\right)\\ &= \frac{1}{k}\left(\sum_{j=1}^{k}\sum_{i=1}^{m} x_{ij}^* x_{(i+1)j} + \sum_{j=1}^{k} x_{mj}^* (A - I_n) x_{1j}\right),\end{aligned}$$

where $x_{(m+1)j} := x_{1j}$ for all $1 \leq j \leq k$. Since $X^*X = I_k$,

$$|\ \mu\ | \leqslant \frac{1}{k}\left(\sum_{j=1}^{k}\sum_{i=1}^{m} \parallel x_{ij}^* \parallel\parallel x_{(i+1)j} \parallel + \sum_{j=1}^{k} \parallel x_{mj}^* \parallel\parallel A - I_n \parallel\parallel x_{1j} \parallel\right)$$

$$\leqslant \frac{1}{k}\left(\sum_{j=1}^{k}\sum_{i=1}^{m}\frac{1}{2}(\|x_{ij}\|^2+\|x_{(i+1)j}\|^2)+k \parallel A-I_n \parallel\right)$$
$$= 1+\parallel A-I_n \parallel,$$

and hence the result holds. □

Corollary 3.10. *Let $1 \leq k \leq mn$ be a positive integer, $A \in \mathbb{M}_n$ and Π_A be the basic A-factor block circulant matrix as in (3.2). Then*

$$r_k(\Pi_A) \leq 1+\parallel A-I_n \parallel .$$

Moreover, for the case $1 \leq k \leq n$, the estimate is sharp.

Proof. The first assertion follows directly from Theorem 3.9.
To show, for the case $1 \leq k \leq n$, that the estimate is sharp, we consider $A = I_n$. Then, by Example 3.8, we have:

$$W_k(\Pi_A) = \mathrm{conv}\left(\sqrt[m]{W_k(I_n)}\right) = \mathrm{conv}\left(\{1,\omega,\omega^2,\ldots,\omega^{\mathrm{m}-1}\}\right),$$

where $\omega = e^{\frac{2\pi}{m}i}$. So, $r_k(\Pi_A) = 1 = 1+\parallel A-I_n \parallel$, and hence, the proof is complete. □

4. **Additional results**

In this section, we are going to continue the study of the k-numerical range of matrix polynomials. By (2.1) and Proposition 1.1((*iii*) and (*iv*)), we have the following result.

Proposition 4.1. *Let $P(\lambda)$ be a matrix polynomial as in (1.1). Then:*
(i) $\{\mu \in \mathbb{C} \ : \ \mathrm{tr}(\mathrm{A_m})\mu^{\mathrm{m}}+\cdots+\mathrm{tr}(\mathrm{A_1})\mu+\mathrm{tr}(\mathrm{A_0}) = 0\} = W_n[P(\lambda)]$
$$\subseteq W_{n-1}[P(\lambda)] \subseteq \cdots \subseteq W_1[P(\lambda)] = W[P(\lambda)];$$
(ii) If $V \in \mathcal{X}_{n\times s}$, where $k \leq s \leq n$, then $W_k[V^*P(\lambda)V] \subseteq W_k[P(\lambda)]$. The equality holds if $s = n$, i.e., $W_k[U^*P(\lambda)U] = W_k[P(\lambda)]$, where $U \in \mathcal{U}_n$.

It is known, e.g., see [13, Example 1], that $W_1[P(\lambda)]$ is not necessarily connected. Now, we are going to study the number of connected components of $W_k[P(\lambda)]$. For this, we need the following lemma.

Lemma 4.2. *The set $\mathcal{X}_{n\times k}$ is a path-connected set in $\mathbb{M}_{n\times k}$.*

Proof. Let $X, Y \in \mathcal{X}_{n\times k}$ be given. Then there exists a unitary matrix $U \in \mathcal{U}_n$ such that $Y = UX$. Since $\mathcal{U}_n$ is path-connected [11, Lemma in p. 266], there exists a continuous curve $f : [0,1] \longrightarrow \mathcal{U}_n$ such that $f(0) = I_n$ and $f(1) = U$. We know that the function $g : \mathcal{U}_n \longrightarrow \mathcal{X}_{n\times k}$ with $g(V) = VX$ is continuous, and so, the function $\varphi := g \circ f : [0,1] \longrightarrow \mathcal{X}_{n\times k}$ is a continuous curve such that $\varphi(0) = X$ and $\varphi(1) = Y$. Hence, the result holds. $\square$

Theorem 4.3. *Let $P(\lambda)$ be a matrix polynomial as in (1.1). If $0 \notin W_k(A_m)$, then $W_k[P(\lambda)]$ has no more than m connected components.*

Proof. Let l be the minimum of the number of distinct roots of equations $\mathrm{tr}(X^*P(\lambda)X) = 0$ over all $X \in \mathcal{X}_{n\times k}$. Since $0 \notin W_k(A_m)$, the integer number l belongs to $\{1, 2, \dots, m\}$. Moreover, there exists a $X_0 \in \mathcal{X}_{n\times k}$ such that the equation $\mathrm{tr}(x_0^*P(\lambda)X_0) = 0$ has solution $\lambda_1, \lambda_2, \dots, \lambda_m$, in which exactly l roots are distinct. Now, let $X \in \mathcal{X}_{n\times k}$ be arbitrary. By Lemma 4.2, there exists a continuous curve $s : [0,1] \longrightarrow \mathcal{X}_{n\times k}$ such that $s(0) = X_0$ and $s(1) = X$. Since $0 \notin W_k(A_m)$, $\mathrm{tr}(s(t)^*A_m s(t)) \neq 0$ for all $t \in [0,1]$. So, the solutions $\lambda_1(t), \lambda_2(t), \dots, \lambda_m(t)$ of the equation $\mathrm{tr}(s(t)^*P(\lambda)s(t)) = 0$ are continuous function of t. Thus the zeros of equation $\mathrm{tr}(X^*P(\lambda)X) = \mathrm{tr}(s(1)^*P(\lambda)s(1)) = 0$, are connected to those of the equation $\mathrm{tr}(X_0^*P(\lambda)X_0) = \mathrm{tr}(s(0)^*P(\lambda)s(0)) = 0$ by continuous curves in $W_k[P(\lambda)]$, and hence, the zeros of the equation $\mathrm{tr}(X^*P(\lambda)X) = 0$ must lie in the connected components containing the zeros of the equation $\mathrm{tr}(X_0^*P(\lambda)X_0) = 0$. So, $W_k[P(\lambda)]$ has no more than l connected components. Hence, the result holds. $\square$

In the following theorem, which is a direct extension of [16, Theorem 2.1 and its Corollary], we study the isolated points of the k-numerical range of matrix polynomials.

Theorem 4.4. *Let $k < n$ and $P(\lambda)$, as in (1.1), be a matrix polynomial such that $0 \notin W_k(A_m)$. If $\lambda_1, \lambda_2, \dots, \lambda_s$ are isolated points of $W_k[P(\lambda)]$, then*
(i) $P(\lambda_j) = 0$ for $j = 1, 2, \dots, s$;
(ii) $P(\lambda) = (\lambda - \lambda_1)^{t_1}(\lambda - \lambda_2)^{t_2} \cdots (\lambda - \lambda_s)^{t_s} P_0(\lambda)$, where

$$W_k[P_0(\lambda)] = W_k[P(\lambda)] \setminus \{\lambda_1, \lambda_2, \dots, \lambda_s\}.$$

Conversely, by the factorization in (ii) for $P(\lambda)$, the scalars $\lambda_1, \dots, \lambda_s$ are isolated points of $W_k[P(\lambda)]$.

Proof. Without lost of generality, we assume that $s = 1$. We know that $\lambda_1 \in W_k[P(\lambda)]$. Hence there exists a $X \in \mathcal{X}_{n\times k}$ such that $\mathrm{tr}(X^*P(\lambda_1)X) = 0$. Since $0 \notin W_k(A_m)$, in the same manner as in the proof of Theorem 4.3, for every $Y \in \mathcal{X}_{n\times k}$, the roots of the equation $\mathrm{tr}(Y^*P(\lambda)Y) = 0$ are connected to those of the equation $\mathrm{tr}(X^*P(\lambda)X) = 0$ by continuous curves in $W_k[P(\lambda)]$. Now, by the fact that λ_1 is an isolated point of $W_k[P(\lambda)]$, we have $\mathrm{tr}(Y^*P(\lambda_1)Y) = 0$. Since $Y \in \mathcal{X}_{n\times k}$ is arbitrary, Proposition 1.1(v) shows that $P(\lambda_1) = 0$. So, we have $P(\lambda) = (\lambda - \lambda_1)P_1(\lambda)$, where $P_1(\lambda)$ is an $n \times n$ matrix polynomial of degree $m-1$. If $\lambda_1 \in W_k[P_1(\lambda)]$, then λ_1 is an isolated point of $W_k[P_1(\lambda)]$, and hence by the first case, there exists a matrix polynomial $P_2(\lambda)$ of degree $m-2$

such that $P_1(\lambda) = (\lambda - \lambda_1)P_2(\lambda)$. In this way, we can find a positive integer t_1 such that

$$P(\lambda) = (\lambda - \lambda_1)^{t_1} P_0(\lambda), \tag{4.1}$$

where $P_0(\lambda)$ is an $n \times n$ matrix polynomial of degree $m - t_1$ and $\lambda_1 \notin W_k[P_0(\lambda)]$. Hence, the result in (ii) also holds. Conversely, if the factorization in (4.1) holds and $\lambda_1 \notin W_k[P_0(\lambda)]$, then $P(\lambda_1) = 0$ and λ_1 is an isolated point of $W_k[P(\lambda)] = W_k[P_0(\lambda)] \cup \{\lambda_1\}$. So, the proof is complete. □

The result in Theorem 4.4 dose not hold for the case $k = n$, as is illustrated in the following example.

Example 4.5. Consider the following quadratic matrix polynomial:

$$P(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \lambda + \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Then, by Theorem 4.1(i), $W_2[P(\lambda)] = \{0\}$. So, 0 is an isolated point of $W_2[P(\lambda)]$. But $P(0) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \neq 0$.

At the end of this section, we study the boundedness of the k-numerical range of matrix polynomials. We recall, e.g., see Theorem 2.2(i), that if $0 \notin W_k(A_m)$, then $W_k[P(\lambda)]$ is bounded. For the converse, we state the following proposition.

Proposition 4.6. *Let $P(\lambda)$, as in (1.1), be a matrix polynomial with the reversal $Q(\lambda) := \lambda^m P(\lambda^{-1})$. Then $W_k[P(\lambda)]$ is unbounded if and only if $0 \in W_k(A_m)$ and 0 is not an isolated point of $W_k[Q(\lambda)]$.*

Proof. For the implication ($\Leftarrow$), since $0 \in W_k(A_m)$, Relation (2.1) implies that $0 \in W_k[Q(\lambda)]$. Moreover, since 0 is not an isolated point of $W_k[Q(\lambda)]$, there exists a sequence $\{\mu_t\}_{t=1}^{\infty} \subseteq W_k[Q(\lambda)] \setminus \{0\}$ such that converges to 0. So, by Theorem 2.1(iv), the sequence $\{\frac{1}{\mu_t}\}_{t=1}^{\infty}$, which is unbounded, lies in $W_k[P(\lambda)]$, and hence, $W_k[P(\lambda)]$ is unbounded. Using Theorem 2.2(i) and the same manner as in the proof of ($\Leftarrow$), the proof of ($\Rightarrow$) is easy to verify. So, the proof is complete. □

In Proposition 4.6, we proved that if $0 \in W_k(A_m)$ and 0 is not an isolated point of $W_k[\lambda^m P(\lambda^{-1})]$, then $W_k[P(\lambda)]$ is unbounded. But, we think that the condition "0 is not an isolated point of $W_k[\lambda^m P(\lambda^{-1})]$" can be removed. Hence, we state the following conjecture.

Conjecture 4.7. Let $k < n$, and $P(\lambda)$ be a matrix polynomial as in (1.1). If $0 \in W_k(A_m)$, then $W_k[P(\lambda)]$ is unbounded.

The following example shows that the result in Conjecture 4.7 does not hold for the case $k = n$.

Example 4.8. Let $P(\lambda) = A_m\lambda^m + \cdots + A_1\lambda + A_0$, as in (1.1), be a matrix polynomial such that $\mathrm{tr(A_m)} = 0$ and $\mathrm{tr(A_i)} \neq 0$ for some $i \in \{0, 1, \ldots, m-1\}$. Thus, $0 \in \{0\} = W_n(A_m)$. Moreover, by Proposition 4.1(i), we have

$$W_n[P(\lambda)] = \left\{\mu \in \mathbb{C} \ : \ \mathrm{tr(A_{m-1})}\mu^{\mathrm{m-1}} + \cdots + \mathrm{tr(A_1)}\mu + \mathrm{tr(A_0)} = 0\right\},$$

which has at most $m-1$ elements, and hence is bounded. So, the result in Conjecture 4.7 does not hold for the case $k = n$.

Acknowledgement

The authors are very grateful to the anonymous referees for helpful comments and useful suggestions, especially for Proposition 4.6 and its proof, which greatly improved the presentation of the original manuscript.

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