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SELF-COMMUTATORS OF COMPOSITION OPERATORS WITH MONOMIAL SYMBOLS ON THE BERGMAN SPACE

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(Communicated by Bamdad Yahaghi)

Abstract. Let $\varphi(z) = z^m, z \in \mathbb{U}$, for some positive integer m, and C_φ be the composition operator on the Bergman space $\mathcal{A}^2$ induced by φ. In this article, we completely determine the point spectrum, spectrum, essential spectrum, and essential norm of the operators $C_\varphi^* C_\varphi, C_\varphi C_\varphi^*$ as well as self-commutator and anti-self-commutators of C_φ. We also find the eigenfunctions of these operators.

Keywords: Bergman space, composition operator, essential spectrum, essential norm, self-commutator.
MSC(2010): Primary 47B33; Secondary 47A10, 47B47.

1. Introduction

Let φ be a holomorphic self-map of the unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$. The function φ induces the *composition operator* C_φ, defined on the space of holomorphic functions on $\mathbb{U}$ by $C_\varphi f = f \circ \varphi$. The restriction of C_φ to various Banach spaces of holomorphic functions on $\mathbb{U}$ has been an active subject of research for more than three decades and it will continue to be for decades to come (see [11], [12] and [6]).

Let $\varphi(z) = z^m, z \in \mathbb{U}$, for some positive integer m, and $C_\varphi : \mathcal{A}^2 \longrightarrow \mathcal{A}^2$ be the composition operator on the Bergman space $\mathcal{A}^2$ induced by φ. The main aim here is to find the spectrum, point spectrum, essential spectrum and essential norm of $C_\varphi^* C_\varphi, C_\varphi C_\varphi^*$ as well as self-commutator $[C_\varphi^*, C_\varphi] = C_\varphi^* C_\varphi - C_\varphi C_\varphi^*$ and anti-self-commutator $\{C_\varphi^*, C_\varphi\} = C_\varphi^* C_\varphi + C_\varphi C_\varphi^*$, for composition operators C_φ on the Bergman space.

In [3], by using Cowen's formula for the adjoint of C_φ on $H^2(\mathbb{U})$, Bourdon and MacCluer completely determined the spectrum, essential spectrum

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and point spectrum for self-commutators of automorphic composition operators acting on the Hardy space of unit disk. In [1], the first author extended these results from the Hardy space to the Dirichlet space. In [2], the authors obtained similar results for composition operators with monomial symbols on the Dirichlet space.

The other problem which is important to the study of composition operators is finding the relationships between the properties of the symbol φ and essential normality of the composition operator C_φ. Recall that an operator T on a Hilbert space $\mathcal{H}$ is called *essentially normal* if its image in the Calkin algebra is normal or equivalently if the *self-commutator* $[T^*, T] = T^*T - TT^*$ is compact on $\mathcal{H}$.

In [8], MacCluer and Pons determined which composition operators with automorphism symbol are essentially normal on $A^2(B_N)$ and $H^2(B_N)$ for $N \geq 1$. They showed that the only essential normal automorphic composition operators are actually normal. This fact was first shown in the setting $H^2(\mathbb{U})$ by N. Zorboska in [13]. Related results and some historical remarks can be found in [3, 9, 13] and [8].

In [4], G. A. Chacón and G. R. Chacón considered composition operators C_φ acting on the Dirichlet space $\mathcal{D}$, where φ is a linear-fractional self-map of the unit disk $\mathbb{U}$. By using the E. Gallardo and A. Montes' adjoint formula given in [7], they showed that the essentially normal linear fractional composition operators on $\mathcal{D}$ are precisely those whose symbol is not a hyperbolic non-automorphism with a boundary fixed point. They also obtained conditions for the linear fractional symbols φ and ψ of the unit disk for which $C_\psi^* C_\varphi$ or $C_\varphi C_\psi^*$ is compact.

In the next section, after giving some background material, we present a formula for the adjoint of C_φ on $\mathcal{A}^2$, when φ is an arbitrary monomial symbol $\varphi(z) = z^m$. In Section 3, we completely determine the point spectrum, spectrum, and essential spectrum of $C_\varphi^* C_\varphi$ and $C_\varphi C_\varphi^*$. Finally, in Section 4 we determine the same for $[C_\varphi^*, C_\varphi]$ and $\{C_\varphi^*, C_\varphi\}$.

2. Preliminaries

Throughout this paper, $\mathcal{B}(\mathcal{H})$ denotes the set of all bounded operators on a Hilbert space $\mathcal{H}$, and $\mathcal{B}_0(\mathcal{H})$ denotes the closed ideal of all compact operators in $\mathcal{B}(\mathcal{H})$. The quotient Banach algebra $\mathcal{B}(\mathcal{H})/\mathcal{B}_0(\mathcal{H})$ is known as the *Calkin algebra*. For an operator $T \in \mathcal{B}(\mathcal{H})$, the essential norm of T is defined by

$$\|T\|_e := \inf \{\|T + K\| : K \in \mathcal{B}_0(\mathcal{H})\},$$

and the *essential spectrum* $\sigma_e(T)$ is defined as the spectrum of the image $\widetilde{T} = T + \mathcal{B}_0(\mathcal{H})$ of T in the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{B}_0(\mathcal{H})$. It is well known that the essential spectrum of a normal operator consists of all points in the spectrum of the operator except the isolated eigenvalues of finite multiplicity (see [5]).

The Bergman space $\mathcal{A}^2$ is the set of all analytic functions in the disk that are square integrable with respect to $dA(z) = \pi^{-1} r dr d\theta$, the normalized Lebesgue area measure on $\mathbb{U}$. Equivalently an analytic function f is in $\mathcal{A}^2$ if $\sum_{n=0}^{\infty} \frac{|\hat{f}(n)|^2}{n+1} < \infty$, where $\hat{f}(n)$ denotes the nth Taylor coefficients of f. The inner product in this space is given by

$$\langle f, g\rangle_{\mathcal{A}^2} = \int_{\mathbb{U}} f(z)\overline{g(z)}dA(z).$$

The inner product of two functions $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ and $g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n$ in $\mathcal{A}^2$ may also be computed by

$$\langle f, g\rangle_{\mathcal{A}^2} = \sum_{n=0}^{\infty} \frac{\hat{f}(n)\overline{\hat{g}(n)}}{n+1}.$$

The *reproducing kernel* in $\mathcal{A}^2$ for the point w in the disk is given by

$$K_w(z) = \frac{1}{(1-\overline{w}z)^2}, \quad z \in \mathbb{U}.$$

M. J. Martín and D. Vukotić in [10] expressed and proved some formulas for the adjoint of C_φ on the Hardy space, when φ is finite Blaschke product that is also a rational self-map of the unit disk $\mathbb{U}$. By using the same arguments as in [10] for the Hardy space, one can prove the following theorem for the Bergman space case.

Theorem 2.1. *Let $\varphi(z) = z^m$. For an arbitrary point $w = re^{i\theta}$ in $\mathbb{U}$, the adjoint of C_φ (viewed as an operator on the Bergman space $\mathcal{A}^2$) is given by the formula*

$$C_\varphi^* f(w) = \sum_{n=0}^{\infty} (n+1) \frac{f^{(mn)}(0)}{(mn+1)!} w^n. \tag{2.1}$$

For the case $m = 1$, we have $C_\varphi = C_\varphi^* = I$. Hence any results relating to $[C_\varphi^*, C_\varphi]$ and $\{C_\varphi^*, C_\varphi\}$ in this case would be trivial. So we assume throughout the paper that $m \geq 2$.

3. Spectrum of $C_\varphi^* C_\varphi$ and $C_\varphi C_\varphi^*$

Let $\varphi(z) = z^m$. In this section we are going to find the point spectrum, spectrum, essential spectrum, and the eigenfunctions of the operators $C_\varphi^* C_\varphi$ and $C_\varphi C_\varphi^*$.

Theorem 3.1. *Let* $\varphi(z) = z^m$. *If* $m \geq 2$, *then*

$$\sigma_p(C_\varphi C_\varphi^*) = \{0,1\} \bigcup \left\{ \frac{n+1}{mn+1} | n \in \mathbb{N} \right\},$$

$$\sigma(C_\varphi C_\varphi^*) = \sigma_p(C_\varphi C_\varphi^*) \bigcup \left\{ \frac{1}{m} \right\},$$

$$\sigma_e(C_\varphi C_\varphi^*) = \left\{ 0, \frac{1}{m} \right\},$$

$$\sigma_p(C_\varphi^* C_\varphi) = \sigma_p(C_\varphi C_\varphi^*) \setminus \{0\},$$

$$\sigma(C_\varphi^* C_\varphi) = \sigma(C_\varphi C_\varphi^*) \setminus \{0\},$$

and

$$\sigma_e(C_\varphi^* C_\varphi) = \left\{ \frac{1}{m} \right\}.$$

Proof. Since any points in the spectrum of a normal operator which is not in the essential spectrum is an isolated eigenvalue of finite multiplicity, we first find the eigenvalues of the operator $C_\varphi C_\varphi^*$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of the operator $C_\varphi C_\varphi^*$ with a corresponding eigenfunction $f \in \mathcal{A}^2$. Then $C_\varphi C_\varphi^* f = \lambda f$. By using formula (2.1) for $C_\varphi C_\varphi^*$, we have

$$\sum_{n=0}^{\infty} \frac{(n+1)}{mn+1} \frac{f^{(mn)}(0)}{(mn)!} w^{mn} = \lambda \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} w^k, \quad w \in \mathbb{U}. \tag{3.1}$$

By putting $w = 0$, it follows that $f(0) = \lambda f(0)$. If $f(0) \neq 0$, then $\lambda = 1$. Thus for the case $\lambda = 1$, the function $f \equiv f(0)$ is a nonzero function in $\mathcal{A}^2$ that satisfies the equation and hence $\lambda = 1$ is an eigenvalue of the operator $C_\varphi C_\varphi^*$. If $f(0) = 0$, we have

$$\sum_{n=1}^{\infty} \frac{(n+1)}{mn+1} \frac{f^{(mn)}(0)}{(mn)!} w^{mn} = \lambda \sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{k!} w^k, \quad w \in \mathbb{U}.$$

So we conclude that

$$\lambda \frac{f^{(k)}(0)}{k!} = 0 \quad , k \neq mn \ , \ n \in \mathbb{N},$$

and

$$\left(\lambda - \frac{n+1}{mn+1} \right) \frac{f^{(mn)}(0)}{(mn)!} = 0 \quad , n \in \mathbb{N}.$$

If $\lambda \notin \{0\} \bigcup \left\{ \frac{n+1}{mn+1} | n \in \mathbb{N} \right\}$, then the above equations imply $f = 0$. For a given natural number n and $\lambda = \frac{n+1}{mn+1}$, the function $f_n(z) = z^{mn}$ is a non-zero function in $\mathcal{A}^2$ that satisfies equation (3.1), and hence $\lambda = \frac{n+1}{mn+1}$ is an eigenvalue of the operator $C_\varphi C_\varphi^*$. If $\lambda = 0$, for each natural number k which is not a multiplie of m, the function $f(z) = z^k$ is a non-zero function in $\mathcal{A}^2$ that satisfies equation (3.1). Hence $\lambda = 0$ is an eigenvalue of the operator $C_\varphi C_\varphi^*$ with infinite multiplicity. So $0 \in \sigma_e(C_\varphi C_\varphi^*)$, and

$$\sigma_p(C_\varphi C_\varphi^*) = \{0,1\} \bigcup \left\{ \frac{n+1}{mn+1} | n \in \mathbb{N} \right\}.$$

Now we show that

$$\sigma(C_\varphi C_\varphi^*) = \sigma_p(C_\varphi C_\varphi^*) \bigcup \left\{ \frac{1}{m} \right\}.$$

Since $C_\varphi C_\varphi^*$ is a self-adjoint operator, we conclude that $\sigma(C_\varphi C_\varphi^*) \subseteq \mathbb{R}$. First assume that $\frac{1}{m} \neq \lambda \in \mathbb{R}$ and λ is not in $\sigma_p(C_\varphi C_\varphi^*)$. So we have two cases:

Case 1: $\lambda < \frac{1}{m}$ or $\lambda > \frac{2}{m+1}$; in the case $\lambda < \frac{1}{m}$, we show that $\mathrm{ran}(\mathrm{C}_\varphi \mathrm{C}_\varphi^* - \lambda) = \mathcal{A}^2$. Let $g(w) = \sum_{n=0}^{\infty} a_n w^n$ be a function in $\mathcal{A}^2$. By a simple computation, we conclude that the function f defined by $f(w) = \sum_{n=0}^{\infty} c_n w^n$, where

$$c_k = \begin{cases} \frac{-a_{mn}}{\lambda - \frac{n+1}{mn+1}}, & k = mn \text{ for } n \in \mathbb{N} \cup \{0\} \\ \frac{-a_k}{\lambda}, & k \in \mathbb{N} \setminus \{mn | n \in \mathbb{N}\} \end{cases}$$

is convergent in the unit disk $\mathbb{U}$. Furthermore, since

$$\sum_{k=0}^{\infty} \frac{|c_k|^2}{1+k} = \frac{|a_0|^2}{|1-\lambda|^2} + \sum_{k \in \mathbb{N} \setminus \{mn|n \in \mathbb{N}\}} \frac{|\frac{-a_k}{\lambda}|^2}{1+k} + \sum_{n=1}^{\infty} \frac{|a_{mn}|^2}{mn+1} \frac{1}{|\frac{n+1}{mn+1} - \lambda|^2},$$

from

$$\sum_{k \in \mathbb{N} \setminus \{mn|n \in \mathbb{N}\}} \frac{|\frac{-a_k}{\lambda}|^2}{1+k} \leq \frac{1}{|\lambda|^2} \sum_{k=0}^{\infty} \frac{|a_k|^2}{1+k} < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{|a_{mn}|^2}{mn+1} \frac{1}{|\frac{n+1}{mn+1} - \lambda|^2} \leq \frac{1}{(\frac{1}{m} - \lambda)^2} \sum_{k=1}^{\infty} \frac{|a_k|^2}{1+k} < \infty,$$

it follows that $f \in \mathcal{A}^2$. Also for each $w \in \mathbb{U}$, $(C_\varphi C_\varphi^* - \lambda) f(w) = g(w)$. So $\mathrm{ran}(\mathrm{C}_\varphi \mathrm{C}_\varphi^* - \lambda) = \mathcal{A}^2$. For the case $\lambda > \frac{2}{m+1}$, by a similar proof as above we conclude that $\mathrm{ran}(\mathrm{C}_\varphi \mathrm{C}_\varphi^* - \lambda) = \mathcal{A}^2$.

Case 2: λ is between two consecutive terms of the sequence $\left\{ \frac{n+1}{mn+1} \right\}$; let

$$\frac{(n_0+1)+1}{m(n_0+1)+1} < \lambda < \frac{n_0+1}{mn_0+1},$$

for some n_0. In the case $n \geq n_0 + 1$, we have $\lambda > \frac{n+1}{mn+1}$ and by putting

$$B_0 = \max \left\{ \frac{1}{|\frac{2}{m+1} - \lambda|^2}, \ldots, \frac{1}{|\lambda - \frac{n_0+2}{m(n_0+1)+1}|^2}, \frac{1}{|\lambda - \frac{n_0+1}{mn_0+1}|^2} \right\},$$

for each $n \in \mathbb{N}$ we have
$$\frac{1}{|\frac{n+1}{mn+1} - \lambda|^2} \leq B_0.$$
By using this inequality and a similar proof as in Case 1 above, we conclude that $\mathrm{ran}(\mathrm{C}_\varphi \mathrm{C}_\varphi^* - \lambda) = \mathcal{A}^2$.

Now let $\lambda = \frac{1}{m}$, and define $g \in \mathcal{A}^2$ by
$$g(w) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{mn+1}} w^{mn}.$$
Suppose that there exists $f(w) = \sum_{k=0}^{\infty} c_k w^k$ such that $(C_\varphi C_\varphi^* - \frac{1}{m})f = g$. The last equality implies
$$c_k = \begin{cases} \frac{m}{m-1}\sqrt{mk+1} & k = mn \text{ for } n \in \mathbb{N}, \\ 0 & k \in \mathbb{N} \setminus \{mn | n \in \mathbb{N}\} \cup \{0\}. \end{cases}$$
Since $\sum_{k=0}^{\infty} \frac{|c_k|^2}{k+1} = \infty$, we conclude that f is not in $\mathcal{A}^2$. So we have
$$\sigma(C_\varphi C_\varphi^*) = \left\{\frac{1}{m}\right\} \cup \sigma_p(C_\varphi C_\varphi^*).$$
Since the essential spectrum of a normal operator consists of all accumulation points of the spectrum plus all isolated eigenvalues of infinite multiplicity, we conclude that
$$\sigma_e(C_\varphi C_\varphi^*) = \left\{0, \frac{1}{m}\right\}.$$

Now we do the same for the operator $C_\varphi^* C_\varphi$. In this case, we have
$$(C_\varphi^* C_\varphi f)(w) = f(0) + \sum_{n=1}^{\infty} \frac{n+1}{mn+1} \; \frac{(f \circ \varphi)^{(mn)}(0)}{(mn)!} \; w^n, \quad w \in U.$$
If $\lambda = 0$, and $(C_\varphi^* C_\varphi) f(w) = \lambda f(w)$, then
$$f(0) + \sum_{n=1}^{\infty} \frac{n+1}{mn+1} \frac{f^{(n)}(0)}{n!} \; w^n = 0,$$
and so $f = 0$. Therefore 0 is not an eigenvalue of $C_\varphi^* C_\varphi$, and so
$$\sigma_p(C_\varphi^* C_\varphi) = \{1\} \bigcup \left\{\frac{n+1}{mn+1} | n \in \mathbb{N}\right\}.$$
Now we show that $\mathrm{ran}(\mathrm{C}_\varphi^* \mathrm{C}_\varphi - 0) = \mathcal{A}^2$. For given $g(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{A}^2$, we define $f(w) = \sum_{n=0}^{\infty} b_n w^n$, where $b_0 = a_0$ and for each natural number n, $b_n = a_n \frac{mn+1}{n+1}$. Hence $f \in \mathcal{A}^2$, and $(C_\varphi^* C_\varphi f)(w) = g(w)$, for each $w \in U$. So 0 is not in the spectrum of $C_\varphi^* C_\varphi$, and from
$$\sigma(C_\varphi C_\varphi^*) \cup \{0\} = \sigma(C_\varphi^* C_\varphi) \cup \{0\}$$

we conclude that

$$\sigma(C_\varphi^* C_\varphi) = \sigma(C_\varphi C_\varphi^*) \setminus \{0\}\,.$$

For the essential spectrum of $C_\varphi^* C_\varphi$, since

$$\sigma(C_\varphi C_\varphi^* + \mathcal{B}_0(\mathcal{A}^2)) \cup \{0\} = \sigma(C_\varphi^* C_\varphi + \mathcal{B}_0(\mathcal{A}^2)) \cup \{0\}\,,$$

it follows that

$$\sigma_e(C_\varphi^* C_\varphi) = \left\{\frac{1}{m}\right\}.$$

□

4. The spectrum of $[C_\varphi^*, C_\varphi]$ and $\left\{C_\varphi^*, C_\varphi\right\}$

In this section, we shall find the point spectrum, spectrum, essential spectrum, and eigenfunctions of the operators $[C_\varphi^*, C_\varphi] = C_\varphi^* C_\varphi - C_\varphi C_\varphi^*$ and $\left\{C_\varphi^*, C_\varphi\right\} = C_\varphi^* C_\varphi + C_\varphi C_\varphi^*$.

Theorem 4.1. *Let $\varphi(z) = z^m$. Then for $m \geq 2$*

$$\begin{aligned}\sigma_p([C_\varphi^*, C_\varphi]) \quad = \quad & \left\{\frac{mn+1}{m^2n+1} - \frac{n+1}{mn+1} | n \in \mathbb{N}\right\} \\ & \bigcup \left\{\frac{-(n+1)}{mn+1} | n \in \mathbb{N} \setminus \{mk | k \in \mathbb{N}\}\right\} \cup \{0\}\,,\end{aligned}$$

$$\sigma([C_\varphi^*, C_\varphi]) = \sigma_p([C_\varphi^*, C_\varphi]) \bigcup \left\{-\frac{1}{m}\right\}$$

and

$$\sigma_e([C_\varphi^*, C_\varphi]) = \left\{0, -\frac{1}{m}\right\}.$$

Proof. Let $T = C_\varphi^* C_\varphi - C_\varphi C_\varphi^*$. Then for each $f \in \mathcal{A}^2$ and $w \in \mathbb{U}$,

$$\begin{aligned}(Tf)(w) \quad = \quad & \sum_{n=1}^{\infty} \left(\frac{mn+1}{m^2n+1} - \frac{n+1}{mn+1}\right) \frac{f^{(mn)}(0)}{(mn)!} w^{mn} \\ & + \sum_{n \in \mathbb{N} \setminus \{mk | k \in \mathbb{N}\}} \frac{-(n+1)}{mn+1} \frac{f^{(n)}(0)}{n!} w^n.\end{aligned} \tag{4.1}$$

Since all points in the spectrum of a normal operator which are not in the essential spectrum are isolated eigenvalues of finite multiplicity, we first find the eigenvalues.

Now let $\lambda \in \mathbb{C}$ be an eigenvalue of the operator T with a corresponding eigenfunction $f \in \mathcal{A}^2$. Then $Tf = \lambda f$ and for each $w \in \mathbb{U}$,

$$(4.2)\quad \lambda\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}w^k = \sum_{n=1}^{\infty}\left(\frac{mn+1}{m^2n+1}-\frac{n+1}{mn+1}\right)\frac{f^{(mn)}(0)}{(mn)!}w^{mn} + \sum_{n\in\mathbb{N}\setminus\{mk|k\in\mathbb{N}\}}\frac{-(n+1)}{mn+1}\frac{f^{(n)}(0)}{n!}w^n.$$

So $\lambda f(0)=0$. If $f(0)\neq 0$ we conclude that $\lambda=0$. For the case $\lambda=0$, the function $f\equiv f(0)$ is a non-zero function in $\mathcal{A}^2$ that satisfies equation (4.2), and hence $\lambda=0$ is an eigenvalue of the operator T. If $f(0)=0$, we have

$$\left(\frac{mn+1}{m^2n+1}-\frac{n+1}{mn+1}-\lambda\right)\frac{f^{(mn)}(0)}{(mn)!}=0\quad ,n\in\mathbb{N},$$

and

$$\left(\frac{-(n+1)}{mn+1}-\lambda\right)\frac{f^{(n)}(0)}{n!}=0\quad ,n\in\mathbb{N}\setminus\{mk|k\in\mathbb{N}\}.$$

For each natural number n, we put $\lambda_n=\frac{mn+1}{m^2n+1}-\frac{n+1}{mn+1}$. For the case $\lambda_n=\frac{mn+1}{m^2n+1}-\frac{n+1}{mn+1}$, the function z^{mn} is a non-zero function in $\mathcal{A}^2$ that satisfies equation (4.2), and hence $\lambda_n=\frac{mn+1}{m^2n+1}-\frac{n+1}{mn+1}$ is an eigenvalue of the operator T. For the case $\lambda'_n=-\frac{n+1}{mn+1}$, when $n\in\mathbb{N}\setminus\{mk|k\in\mathbb{N}\}$ the function z^n is a non-zero function in $\mathcal{A}^2$ that satisfies equation (4.2), and hence $\lambda'_n=-\frac{n+1}{mn+1}$ is an eigenvalue of the operator T. Now if $\lambda\neq 0$, and for each natural number n, $\lambda\neq\lambda_n$, and for each $n\in\mathbb{N}\setminus\{mk|k\in\mathbb{N}\}$, $\lambda\neq\lambda'_n$, and if $([C_\varphi^*,C_\varphi]f)(w)=\lambda f(w)$, we conclude that $f\equiv 0$. Hence

$$\sigma_p([C_\varphi^*,C_\varphi] = \left\{\frac{mn+1}{m^2n+1}-\frac{n+1}{mn+1}|n\in\mathbb{N}\right\} \bigcup\left\{\frac{-(n+1)}{mn+1}|n\in\mathbb{N}\setminus\{mk|k\in\mathbb{N}\}\right\}\bigcup\{0\}.$$

Now we show that

$$\sigma([C_\varphi^*,C_\varphi])=\sigma_p([C_\varphi^*,C_\varphi])\bigcup\left\{-\frac{1}{m}\right\}.$$

Since $[C_\varphi^*,C_\varphi]$ is a self-adjoint operator, we conclude that $\sigma([C_\varphi^*,C_\varphi])\subseteq\mathbb{R}$. For this case we assume that $\lambda\in\mathbb{R}$, $\lambda\neq\frac{-1}{m}$, and $\lambda\notin\sigma_p([C_\varphi^*,C_\varphi])$, and we show that $\mathrm{ran}([\mathrm{C}_\varphi^*,\mathrm{C}_\varphi]-\lambda)=\mathcal{A}^2$. For a given function $g(w)=\sum_{n=0}^{\infty}b_nw^n$ in $\mathcal{A}^2$, we define $f\in\mathcal{A}^2$ by $f(w)=\sum_{n=0}^{\infty}a_nw^n$ such that

$$b_k=\begin{cases}\dfrac{-a_k}{\frac{k+1}{mk+1}+\lambda} & k\in\mathbb{N}\setminus\{mn|n\in\mathbb{N}\},\\[2ex] \dfrac{-a_{mn}}{\frac{n(m-1)^2}{(m^2n+1)(mn+1)}+\lambda} & k=mn \text{ for } n\in\mathbb{N}\cup\{0\}.\end{cases}$$

By a simple computation we conclude that $f \in \mathcal{A}^2$ and $(([C_\varphi^*, C_\varphi] - \lambda)f)(w) = g(w)$. Now we assume that $\lambda = -\frac{1}{m}$. We define $g(w) = \sum_{n=0}^{\infty} a_n w^n$ on $\mathbb{U}$ such that

$$a_k = \begin{cases} \frac{1}{\sqrt{mk+1}} & k \in \mathbb{N} \setminus \{mn | n \in \mathbb{N}\}, \\ 0 & k = mn \text{ for } n \in \mathbb{N} \cup \{0\}. \end{cases}$$

Since $\lim_{n \longrightarrow \infty} \sqrt[n]{\frac{1}{\sqrt{mn+1}}} = 1$, we have $\limsup_{n \longrightarrow \infty} \sqrt[n]{|a_n|} = 1$. So $\sum_{n=0}^{\infty} a_n w^n$ is convergent on unit disk. Since

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} = \sum_{n \in \mathbb{N} \setminus \{mk | k \in \ \mathbb{N}\}} \frac{1}{(\sqrt{mn+1})^2} \ \frac{1}{n+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2},$$

we conclude that $\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < \infty$, and hence $g \in \mathcal{A}^2$. Now we suppose that there exists $f \in \mathcal{A}^2$ such that, for each $w \in U$,

$$(([C_\varphi^*, C_\varphi] + \frac{1}{m})f)(w) = g(w).$$

If $f(w) = \sum_{n=0}^{\infty} b_n w^n$, then

$$b_k = \begin{cases} \frac{-m}{m-1} \ \frac{mk+1}{\sqrt{mk+1}} & k \in \mathbb{N} \setminus \{mn | n \in \mathbb{N}\}, \\ 0 & k = mn \text{ for } n \in \mathbb{N} \cup \{0\}. \end{cases}$$

So we conclude that f is not in $\mathcal{A}^2$, and hence

$$\sigma([C_\varphi^*, C_\varphi]) = \sigma_p([C_\varphi^*, C_\varphi]) \bigcup \left\{ -\frac{1}{m} \right\}.$$

Since all points in the spectrum of a normal operator which are not in the essential spectrum are isolated eigenvalues of finite multiplicity, we conclude that

$$\sigma_e([C_\varphi^*, C_\varphi]) = \left\{ 0, -\frac{1}{m} \right\}.$$

□

In the next theorem, we do the same for the operator $\left\{C_\varphi^*, C_\varphi\right\}$.

Theorem 4.2. *Let $\varphi(z) = z^m$, and for each $n \in \mathbb{N}$, $\lambda_n = \frac{mn+1}{m^2 n+1} + \frac{n+1}{mn+1}$, and for each $n \in \mathbb{N} \setminus \{mk | k \in \mathbb{N}\}$, $\lambda'_n = \frac{n+1}{mn+1}$. Then for $m \geq 2$,*

$$\sigma_p(\left\{C_\varphi^*, C_\varphi\right\}) = \{2\} \cup \{\lambda_n | n \in \mathbb{N}\} \cup \{\lambda'_n | n \in \mathbb{N} \setminus \{mk | k \in \mathbb{N}\}\},$$

$$\sigma(\left\{C_\varphi^*, C_\varphi\right\}) = \sigma_p(\left\{C_\varphi^*, C_\varphi\right\}) \bigcup \left\{ \frac{1}{m}, \frac{2}{m} \right\},$$

and

$$\sigma_e(\left\{C_\varphi^*, C_\varphi\right\}) = \left\{ \frac{1}{m}, \frac{2}{m} \right\}.$$

Proof. Let $S = C_\varphi^* C_\varphi + C_\varphi C_\varphi^*$. Then for each $f \in \mathcal{A}^2$ and $w \in \mathbb{U}$,

$$(Sf)(w) = 2f(0) \quad + \quad \sum_{n=1}^{\infty} \left(\frac{mn+1}{m^2n+1} + \frac{n+1}{mn+1} \right) \frac{f^{(mn)}(0)}{(mn)!} w^{mn}$$
$$+ \sum_{n \in \mathbb{N} \setminus \{mk | k \in \mathbb{N}\}} \frac{n+1}{mn+1} \frac{f^{(n)}(0)}{n!} w^n.$$

Let $\lambda \in \mathbb{C}$ be an eigenvalue of the operator S with a corresponding eigenfunction $f \in \mathcal{A}^2$. Then $Sf = \lambda f$. So we have

$$(Sf)(w) \quad = \quad \lambda f(0) + \sum_{n=1}^{\infty} \lambda \frac{f^{(n)}(0)}{n!} w^n, \quad w \in \mathbb{U}. \tag{4.3}$$

Now we define:

$$\begin{cases} \lambda_n = \frac{mn+1}{m^2n+1} + \frac{n+1}{mn+1}, & n \in \mathbb{N}, \\ \lambda'_n = \frac{n+1}{mn+1}, & n \in \mathbb{N} \setminus \{mk | k \in \mathbb{N}\}. \end{cases}$$

By using formula (4.3), if $f(0) \neq 0$, for the case $\lambda = 2$ the function $f \equiv f(0)$, is a non-zero function in $\mathcal{A}^2$ that satisfies equation (4.3), and hence $\lambda = 2$ is an eigenvalue of the operator S. If $f(0) = 0$, we have

$$\begin{cases} (\lambda_n - \lambda) \frac{f^{(mn)}(0)}{(mn)!} = 0 \ , & n \in \mathbb{N}, \\ (\lambda'_n - \lambda) \frac{f^{(n)}(0)}{n!} = 0 \ , & n \in \mathbb{N} \setminus \{mk | k \in \mathbb{N}\}. \end{cases}$$

For each natural number n, λ_n is an eigenvalue of the operator S with corresponding eigenfunction $z^{nm} \in \mathcal{A}^2$, and for each $n \in \mathbb{N} \setminus \{mk | k \in \mathbb{N}\}$, λ'_n is an eigenvalue of the operator S with corresponding eigenfunction $z^n \in \mathcal{A}^2$. So we conclude that

$$\sigma_p(\{C_\varphi^*, C_\varphi\}) = \{2\} \cup \{\lambda_n | n \in \mathbb{N}\} \cup \{\lambda'_n | n \in \mathbb{N} \setminus \{mk | k \in \mathbb{N}\}\}.$$

Assume that, $\lambda \in \mathbb{R}$, $\lambda \notin \sigma_p(S)$, $\lambda \neq \frac{1}{m}$ and $\lambda \neq \frac{2}{m}$. Then we obtain that $\mathrm{ran}(\mathrm{S} - \lambda) = \mathcal{A}^2$. Now we show that $\frac{1}{m} \in \sigma(S)$. For this case we define $g(w) = \sum_{n=0}^{\infty} a_n w^n$ with

$$a_k = \begin{cases} \frac{1}{\sqrt{mk+1}} & k \in \mathbb{N} \setminus \{mn | n \in \mathbb{N}\}, \\ 0 & k = mn \text{ for } n \in \mathbb{N} \cup \{0\}. \end{cases}$$

Then we conclude that $\sum_{n=0}^{\infty} a_n w^n$ is convergent on U. So the function g defined by $g(w) = \sum_{n=0}^{\infty} a_n w^n$ is in $\mathcal{A}^2$. If we assume that there exists an $f \in \mathcal{A}^2$, such that $(S - \frac{1}{m})f = g$, then we conclude that $f(w) = \sum_{n=0}^{\infty} b_n w^n$ with

$$b_k = \begin{cases} \frac{m}{m-1} \sqrt{mk+1} & k \in \mathbb{N} \setminus \{mn | n \in \mathbb{N}\}, \\ 0 & k = mn \text{ for } n \in \mathbb{N} \cup \{0\}. \end{cases}$$

Since $\sum_{n=0}^{\infty} \frac{|b_n|^2}{n+1} = \infty$, we conclude that $\mathrm{ran}(\mathrm{S} - \frac{1}{\mathrm{m}}) \neq \mathcal{A}^2$, and hence $\frac{1}{m} \in \sigma(S)$. Now we define $G(w) = \sum_{n=0}^{\infty} c_n w^n$ with

$$c_k = \begin{cases} \sqrt{\lambda_k - \frac{2}{m}} & k = mn \text{ for } n \in \mathbb{N}, \\ 0 & k \in \mathbb{N} \cup \{0\} \setminus \{mn | n \in \mathbb{N}\}. \end{cases}$$

By a similar proof as above, we conclude that there does not exist $F \in \mathcal{A}^2$ such that $(S - \frac{2}{m})F = G$. So $\frac{2}{m} \in \sigma(S)$. Then if $m \geq 2$, we have

$$\sigma(\{C_\varphi^*, C_\varphi\}) = \sigma_p(\{C_\varphi^*, C_\varphi\}) \bigcup \left\{\frac{1}{m}, \frac{2}{m}\right\}.$$

Since the essential spectrum of a normal operator consists of all points in the spectrum of the operator except the isolated eigenvalues of finite multiplicity, we conclude that

$$\sigma_e(\{C_\varphi^*, C_\varphi\}) = \left\{\frac{1}{m}, \frac{2}{m}\right\}.$$

□

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