Title:
Linear maps preserving or strongly preserving majorization on matrices

Author(s):
F. Khalooei
LINEAR MAPS PRESERVING OR STRONGLY PRESERVING MAJORIZATION ON MATRICES

F. KHALOOEI

(Communicated by Bamdad Yahaghi)

Dedicated to Professor Heydar Radjavi on his 80th birthday

Abstract. For $A, B \in M_{nm}$, we say that $A$ is left matrix majorized (resp. left matrix submajorized) by $B$ and write $A \preceq \ell B$ (resp. $A \preceq \ell s B$), if $A = RB$ for some $n \times n$ row stochastic (resp. row substochastic) matrix $R$. Moreover, we define the relation $\sim_{\ell s}$ on $M_{nm}$ as follows: $A \sim_{\ell s} B$ if $A \preceq_{\ell s} B \preceq_{\ell s} A$. This paper characterizes all linear preservers and all linear strong preservers of $\preceq_{\ell s}$ and $\sim_{\ell s}$ from $M_{nm}$ to $M_{nm}$.

Keywords: Linear preserver, row substochastic matrix, matrix majorization.


1. Introduction

Throughout the paper, the notation $M_{nm}$ is used for the space of all $n \times m$ real matrices. We also write $M_{nn} = M_n$ and $M_{n1} = R^n$. $I_n$ is the $n \times n$ identity matrix and $P(n)$ will denote all $n \times n$ permutation matrices. An $n \times m$ matrix $R = [r_{ij}]$ is called row stochastic (resp. row substochastic) if for all $i, j, r_{ij} \geq 0$ and $\sum_{k=1}^{n} r_{ik}$ is equal (resp. at most equal) to 1. For $A, B \in M_{nm}$, we say that $A$ is left matrix majorized (resp. left matrix submajorized) by $B$ and write $A \preceq \ell B$ (resp. $A \preceq \ell s B$) if $A = RB$ for some $n \times n$ row stochastic (resp. row substochastic) matrix $R$. For a given relation $\preceq$, we write $A \sim B$ if $A \preceq B \preceq A$.

A linear operator $T : M_{nm} \to M_{nm}$ is said to be a linear preserver of $\preceq$ if $A \preceq B$ implies that $T(A) \preceq T(B)$ for all $A, B \in M_{nm}$. It is a strong preserver of $\preceq$ when $A \sim B$ if and only if $T(A) \sim T(B)$.

A.M. Hasani and M. Radjabalipour [7] characterized the structure of all linear operators $T : M_{nm} \to M_{nm}$ preserving $\sim_{\ell}$. In particular, they proved that if $T : M_n \to M_n$ strongly preserves $\sim_{\ell}$, then there exists a permutation...
matrix \( P \in \mathcal{P}(n) \) and an invertible matrix \( L \in M_n \) such that \( T(X) = PXL \) for all \( X \in M_n \).

A. Armandnejad and A. Salemi [2] characterized the structure of all linear preservers of \( \prec_\ell \) on complex matrices. Also, M. Radjabalipour and P. Torabian [14] characterized all preservers of \( \prec_\ell \) on \( \mathbb{R}^n \) which are not necessarily linear.

For more information about left matrix majorization and the previous work on this subject we also refer to [3, 5, 8, 9, 10] and [13]. The structure of linear operators that preserve other types of majorization have been derived by Ando [1], Beasley, Lee and Y.H. Lee [4], Dahl [6], and Li and E. Poon [11]. Marshall and Olkin’s text [12] is a standard general reference for majorization.

The present paper is organized as follows. In Section 2 we derive necessary and sufficient conditions for a linear operator \( T \) from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) to preserve \( \prec_{\ell_2} \). In particular, we prove that the structure of linear preservers of \( \prec_\ell \), \( \prec_{\ell_2} \) and \( \prec_{\ell_\infty} \) are the same for \( n \geq 3 \). In Section 3 we characterize a general linear preserver \( T \) from \( M_{nm} \) to \( M_{nm} \). In particular, we give necessary and sufficient conditions for a linear operator \( T : M_{nm} \to M_{nm} \) to strongly preserve \( \prec_{\ell_\infty} \).

We note that necessary and sufficient conditions for \( T : \mathbb{R}^n \to \mathbb{R}^n \) to be a linear preserver of \( \prec_\ell \) have been derived before and the following theorems are known.

**Theorem 1.1.** [7, Theorem 2.3] Let \( n \geq 3 \). Then \( T : \mathbb{R}^n \to \mathbb{R}^n \) is a linear preserver of \( \prec_\ell \) if and only if \( T \) has the form \( T(X) = aPX \), for all \( X \in \mathbb{R}^n \), for some some \( a \in \mathbb{R} \) and some \( P \in \mathcal{P}(n) \).

**Theorem 1.2.** [7, Theorem 2.3] Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be a linear operator. Then, \( T \) is a linear preserver of \( \prec_\ell \) if and only if \( T \) has the form \( T(X) = (aI + bP)X \) for all \( X \in \mathbb{R}^2 \), where \( P \) is a \( 2 \times 2 \) permutation matrix not equal \( I_2 \), and \( ab \leq 0 \).

The following theorem states necessary and sufficient conditions for a linear operator \( T : M_{nm} \to M_{nm} \) to be a linear preserver of \( \prec_\ell \).

**Theorem 1.3.** [7, Theorem 3.1] Let \( T : M_{nm} \to M_{nm} \) be a linear operator. Then \( T \) preserves \( \prec_\ell \) if and only if \( T(X) = (aI + bP)XL \) for all \( X \in M_{nm} \), where \( L \in M_n \), \( P \) is an \( n \times n \) permutation matrix, \( P \neq I \), \( a \) and \( b \) are real numbers such that \( ab \leq 0 \), and, if \( n \neq 2 \), \( ab = 0 \). Moreover, if \( n \neq 2 \), then \( aI + bP = cQ \) for some \( c \in \mathbb{R} \) and, hence, \( T(X) = QXK \) for some \( K \in M_n \).

2. Linear preservers of \( \prec_{\ell_\infty} \) on \( \mathbb{R}^n \)

In what follows, \( [T] = [t_{ij}] \) will denote the matrix representation of an operator \( T : \mathbb{R}^n \to \mathbb{R}^n \) with respect to the standard basis \( \{e_1, e_2, \ldots, e_n\} \) of \( \mathbb{R}^n \). Also, \( e = \sum_{j=1}^n e_j \in \mathbb{R}^n \) and

\[
\begin{align*}
a & : = \max\{t_{ij} \mid 1 \leq i, j \leq n\}, \\
b & : = \min\{t_{ij} \mid 1 \leq i, j \leq n\}.
\end{align*}
\]
By Theorem 1.2, the matrix representation of a linear preserver of $\prec_{\ell^s}$ with respect to the standard basis of $\mathbb{R}^2$ is as follows:

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

for some real numbers $a, b$ satisfying $ab \leq 0$.

All linear operators $T: \mathbb{R} \rightarrow \mathbb{R}$ are preservers of $\prec_{\ell^s}$ (for all $x \in \mathbb{R}$ and for all $r \in [0, 1]$). Also, $T = 0$ is a linear preserver of $\prec_{\ell^s}$. Hence, throughout the paper, for a linear operator $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ we shall assume that $T \neq 0$ and $n \geq 2$.

$T: M_{nm} \rightarrow M_{nm}$ is a linear preserver of $\prec_{\ell^s}$ if and only if $T$ is a linear preserver of $\prec_{\ell^s}$ for all nonzero real numbers $a$. Hence without loss of generality we shall assume that $a > 0$ and $|b| \leq a$, where $a$ and $b$ are as in (2.1).

Throughout the paper, for a given vector $x \in \mathbb{R}^n$, $\max x$ and $\min x$ denote the maximum and minimum values of components of $x$, respectively. Also, we write $x_M = \max x$ and $x_m = \min x$.

The following important lemmas are easy consequences of the definitions of $\prec_{\ell^s}$ and $\prec_{\ell^s}$.

**Lemma 2.1.** Let $x, y \in \mathbb{R}^n$. If $x \prec_{\ell^s} y$ then the following assertions are true.

(a) $x_i \in \text{Conv}(\{y_1, \ldots, y_n\} \cup \{0\})$, for all $i$ ($1 \leq i \leq n$).

(b) If $y_m \geq 0$, then $x_m \geq 0$.

(c) If $y_M \leq 0$, then $x_M \leq 0$.

(d) If $y_m \leq 0$ and $y_M \geq 0$, then $y_m \leq x_m \leq x_M \leq y_M$.

**Lemma 2.2.** Let $x, y$ be nonzero vectors in $\mathbb{R}^n$. If $x \sim_{\ell^s} y$, then exactly one of the following occurs:

(a) $x, y$ are entrywise nonnegative and $x_M = y_M$.

(b) $x, y$ are entrywise nonpositive and $x_m = y_m$.

(c) $x_m = y_m \leq 0$ and $x_M = y_M \geq 0$.

Furthermore, if $x, y \in \mathbb{R}^n$ and at least one of the conditions (a), (b) and (c) holds, then $x \sim_{\ell^s} y$.

Theorem 2.3 presents some necessary conditions for a nonzero operator $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $n \geq 2$, to be a linear preserver of $\sim_{\ell^s}$.

**Theorem 2.3.** Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a nonzero linear preserver of $\sim_{\ell^s}$, and assume that $n \geq 2$, and $a$ and $b$ are as in (2.1). Then the following assertions are true:

(a) For each $j \in \{1, 2, \ldots, n\}$, $\max T(e_j) = a$. In particular, every column of $[T]$ contains at least one entry equal to $a$.

(b) $\max T(e) = a$; moreover, if a row of $[T]$ contains an entry equal to $a$, then all other nonnegative entries of that row are zero.

(c) $b = 0$. 
Proof. (a). Without loss of generality, we can assume that \( t_{11} = a \) and \( a > 0 \). \( t_{11} = a \) implies that \( \max T(e_1) = a \). Let \( j \in \{1, 2, \ldots, n\} \) be fixed. Since \( e_j \sim_{\ell_s} e_1 \) and \( T \) preserves \( \sim_{\ell_s} \), hence \( T(e_j) \sim_{\ell_s} T(e_1) \). By Lemma 2.2, \( \max T(e_j) = \max T(e_1) = a \). Since \( j \in \{1, 2, \ldots, n\} \) is arbitrary, \( \max T(e_j) = a \), for all \( j (1 \leq j \leq n) \), therefore, every column of \( [T] \) has at least one entry equal to \( a \).

(b). By Lemma 2.2, \( \Sigma_{j \in J} e_j \sim_{\ell_s} e_1 \), for all \( J \subseteq \{1, \ldots, n\} \) and hence \( \Sigma_{j \in J} T(e_j) \sim_{\ell_s} T(e_1) \). Lemma 2.2 implies that \( \max \Sigma_{j \in J} T(e_j) = a \), for all \( J \subseteq \{1, 2, \ldots, n\} \). Therefore, for all \( J \subseteq \{1, \ldots, n\} \), \( \max \Sigma_{j \in J} t_{ij} = a \) where the maximum is taken over \( i (1 \leq i \leq n) \). Thus, if a row of \([T]\) contains an entry equal to \( a \), then all nonnegative entries of that row are zero. In particular, \( \max T(e) = a \).

(c). From (a), it follows that every column of \([T]\) has at least one entry equal to \( a \). Also, (b) implies that every row of \([T]\) has at most one entry equal to \( a \). Since \([T]\) is \( n \times n \), every row of \([T]\) has exactly one entry equal to \( a \). Hence by (b), all other nonnegative entries of rows of \([T]\) must be zero. Therefore \( b \leq 0 \). If \( b < 0 \), without loss of generality, we may write \( t_{11} = b \). So, \( \max T(e_1) = a > 0 \) and \( \min T(e_1) = b < 0 \). Let \( k \in \{1, \ldots, n\} \) be fixed, since \( e_1 \sim_{\ell_s} e_k \) and \( T \) preserves \( \sim_{\ell_s} \), then \( T(e_1) \sim_{\ell_s} T(e_k) \). Hence by Lemma 2.2, \( \max T(e_k) = \max T(e_1) = a \) and \( \min T(e_k) = \min T(e_1) = b \). Since \( k \) is arbitrary, each column of \([T]\) has at least one entry equal to \( b \). Let \( J \subseteq \{1, \ldots, n\} \).

Since \( \Sigma_{j \in J} e_j \sim_{\ell_s} e_1 \), \( \Sigma_{j \in J} T(e_j) \sim_{\ell_s} T(e_1) \), by Lemma 2.2, \( \min \Sigma_{j \in J} T(e_j) = b \), for all \( J \subseteq \{1, \ldots, n\} \). Thus, if a row of \([T]\) has one entry equal to \( b \), then all its other nonpositive entries of it must be zero. Thus, at most one entry of each row of \([T]\) equals to \( b \). Since \([T]\) is \( n \times n \), each row of \([T]\) has one entry equal to \( b \) and other nonpositive entries are zero. But one entry of each row of \([T]\) is equal to \( a \), which is a contradiction, hence \( b = 0 \).

Theorem 2.4. If \( T \) is such that \( T(x) = aPx \), for all \( x \in \mathbb{R}^n \), for a real number \( a \) and a permutation matrix \( P \in \mathcal{P}(n) \), the operator \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \), \( n \geq 2 \) is a linear preserver of \( \prec_{\ell_s} \).

Proof. Let \( x \in \mathbb{R}^n \) and \( R \) be a row substochastic matrix in \( M_n \). Since \( PR = R'P \) for some row substochastic matrix \( R' \), \( T(Rx) = aPRx = aRx = R'(T(x)) \). Therefore, \( T \) is a linear preserver of \( \prec_{\ell_s} \).

The following theorem follows from Theorem 2.2 and Theorem 2.4.

Theorem 2.5. Let \( n \geq 2 \) and \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a linear operator. Then the following assertions are equivalent:

(a) \( T \) preserves \( \prec_{\ell_s} \),

(b) \( T \) preserves \( \sim_{\ell_s} \),

(c) \( T(x) = aPx \), for all \( x \in \mathbb{R}^n \) and \( a \in \mathbb{R} \).

Theorem 1.1 and Theorem 2.2 imply the following corollary.
Let is not true for every. The linear operator whose matrix representation is

$$[T] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

is a linear preserver of $\prec$ but not a linear preserver of $\prec_{\ell_2}$.

### 3. Linear Preservers of $\prec_{\ell_2}$ on $M_{nm}$

For each $i$ ($1 \leq i \leq m$), define the linear operators $E_i: \mathbb{R}^n \to M_{nm}$ by $E_i(x) = x e_i^T$ for all $x \in \mathbb{R}^n$ and $E_i: M_{nm} \to \mathbb{R}^n$ by $E_i(X) = X e_i$ for all $X \in M_{nm}$, where $\{e_1, \ldots, e_m\}$ denotes the standard basis for $\mathbb{R}^m$.

**Lemma 3.1.** Let $T: M_{nm} \to M_{nm}$ be a linear preserver of $\prec_{\ell_2}$. Then the linear operators $T_{ij} = E_j \circ T \circ E_i$ preserve $\prec_{\ell_2}$ for all $i, j = 1, 2, \ldots, m$.

**Proof.** Let $x \in \mathbb{R}^n$ and $R$ be a row substochastic matrix in $M_n$. $Rx \prec_{\ell_2} x$ implies that $E_i(Rx) \prec_{\ell_2} E_i(x)$. Since $T$ is a linear preserver of $\prec_{\ell_2}$, for every $i$ ($1 \leq i \leq m$), $T(E_i(Rx)) \prec_{\ell_2} T(E_i(x))$. Therefore $E_j(T(E_i(Rx))) \prec_{\ell_2} E_j(T(E_i(x)))$, for all $i, j = 1, 2, \ldots, m$. 

**Theorem 3.2.** Let $T: M_{nm} \to M_{nm}$ be a linear operator. If $T$ preserves $\sim_{\ell_2}$, then $T(X) = PAX$, for all $X \in M_{nm}$, for some $A \in M_n$ and some $n \times n$ permutation matrix $P$.

**Proof.** For each $X = [x_1, x_2, \ldots, x_m] \in M_{nm}$, it is easily seen that

$$T(X) = T([x_1, x_2, \ldots, x_m]) = [\Sigma_{i=1}^m T_{i1}(x_i), \ldots, \Sigma_{i=1}^m T_{im}(x_i)].$$

It follows from Lemma 3.1 that every $T_{ij}$ is a linear preserver of $\sim_{\ell_2}$. Hence, by Theorem 2.4, $T_{ij}(x) = a_{ij} P_{ij} x$ for some permutation matrices $P_{ij}$ and some real numbers $a_{ij}$, where $i, j = 1, 2, \ldots, m$. Since $T \neq 0$, $a_{ij} \neq 0$, for some $i, j$ ($1 \leq i, j \leq m$). Without loss of generality, let $i = j = 1$ and $P = P_{11}$.

We claim that $P_{ij} = P$, for all $i, j = 1, 2, \ldots, m$. Let $r, s \in \{1, \ldots, m\}$, $\alpha, \beta$ be scalars and $(X)$, denote the $i$th column of the matrix $X \in M_{nm}$. Fix $k \in \{1, \ldots, n\}$ and define $X, Y \in M_{nm}$ by $(X)_r = \alpha e, (Y)_r = \alpha e_k, (X)_s = \beta e, (Y)_s = \beta e_k$ and $(X)_i = (Y)_i = 0$, if $i \neq r, i \neq s$. $X \sim_{\ell_2} Y$ implies that $T(X) \sim_{\ell_2} T(Y)$, and hence,

$$[(T(X))_r, (T(X))_s] \sim_{\ell_2} [(T(Y))_r, (T(Y))_s].$$

Therefore,

$$[\alpha a_{rr} e + \beta a_{rs} e, \alpha a_{rr} e + \beta a_{rs} e] \sim_{\ell_2} [\alpha a_{rr} P_{rs} e_k + \beta a_{rs} P_{rs} e_k, \alpha a_{rr} P_{rs} e_k + \beta a_{rs} P_{rs} e_k].$$
If \( a_{rr}a_{rs} \neq 0 \), we prove that \( P_{rr} = P_{rs} \). Let \( \alpha = 1 \) and \( \beta = 0 \). We have \( e = RP_{rr}e_k = RP_{rs}e_k \), for some row substochastic matrix \( R \). Since \( R \) has at most one column equal to \( e \) and \( k \) is arbitrary, \( P_{rr} = P_{rs} \).

Now, suppose \( a_{rr}a_{sr} \neq 0 \). We prove that \( P_{rr} = P_{sr} \). Let \( \alpha, \beta \) be such that \((\alpha a_{rr})(\beta a_{sr}) > 0\). We know that
\[
\alpha a_{rr}e + \beta a_{sr}e \sim_{ts} \alpha a_{rr}P_{rr}e_k + \beta a_{sr}P_{sr}e_k
\]
If \( P_{rr} \neq P_{sr} \), then \( \alpha a_{rr} + \beta a_{sr} \in \text{Conv}(\{\alpha a_{rr}, \beta a_{sr}\} \cup \{0\}) \), which is a contradiction. Therefore, \( P_{rr} = P_{sr} \).

Now suppose that \( a_{rr}a_{ss} \neq 0 \), but \( a_{rs} = a_{sr} = 0 \). Thus,
\[
[\alpha a_{rr}e, \beta a_{ss}e] \sim_{ts} [\alpha a_{rr}P_{rr}e_k, \beta a_{ss}P_{ss}e_k].
\]
Let \( \alpha = \beta = 1 \). Then \( e = RP_{rr}e_k = RP_{ss}e_k \). Since \( k \) is arbitrary and \( R \) has at most one column equal to \( e \), we get \( P_{rr} = P_{ss} \).

We conclude that \( P_{ij} = P \) for all \( i, j \in \{1, \ldots, m\} \). Therefore,
\[
T(X) = \begin{bmatrix} \sum_{i=1}^{m}a_{i1}X_1, \ldots, \sum_{i=1}^{m}a_{im}X_m \end{bmatrix} = P[\sum_{i=1}^{m}a_{i1}X_1, \ldots, \sum_{i=1}^{m}a_{im}X_m] = PXA,
\]
where \( A = [a_{ij}] \).

\[ \Box \]

**Theorem 3.3.** Let \( T : M_{nm} \to M_{nm} \) be a linear operator. Then the following assertions are equivalent:

(a) \( T \) preserves \( \prec_{ts} \),
(b) \( T \) preserves \( \sim_{ts} \),
(c) \( T(X) = PXA \), for all \( X \in M_{nm} \), some \( A \in M_m \), and some \( n \times n \) permutation matrix \( P \).

**Proof.** By Theorem 3.2, it is sufficient to prove that (c) implies (a). Let \( T(X) = PXA \) and \( R \) be a row substochastic matrix. Since \( PR = PR'P \) for some row substochastic matrix \( R' \), \( T(RX) = PRXA = R'PX = R'(T(X)) \). Hence \( T(RX) \prec_{ts} T(X) \).

\[ \Box \]

**Corollary 3.4.** A linear operator \( T : M_{nm} \to M_{nm} \) strongly preserves the majorization relation \( \prec_{ts} \) if and only if there exists \( P \in P(n) \) and an invertible matrix \( L \) in \( M_m \) such that \( T(X) = PXL \) for all \( X \in M_{nm} \).

**Proof.** By Theorem 3.2, there exists \( P \in P(n), L \in M_m \), and a nonzero real number \( a \) such that \( T(X) = aPXL \) for all \( X \in M_{nm} \). Choose \( X \in M_{nm} \) such that \( XL = 0 \). Thus, \( T(X) = aPXL = 0 \prec_{ts} 0 = T(0) \) and therefore, \( X \prec_{ts} 0 \). Hence, \( X = 0 \) which implies that \( L \) is invertible. Replacing \( L \) by \( a^{-1}L \) yields \( T(X) = PXL \) for all \( X \in M_{nm} \), for some \( P \in P(n) \) and an invertible matrix \( L \in M_m \).
Let $T(X) \prec_{\ell s} T(Y)$ for $X, Y \in M_{nm}$. Then $PXL = RPYL$ for some row substochastic matrix $R$. Since $L$ is invertible $PX = RPY$, then $X = RY$ and hence $X \prec_{\ell s} Y$. □

References


(Fatemeh Khalooei) Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran.

E-mail address: f_khalooei@uk.ac.ir