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ADDITIVE MAPS ON C*-ALGEBRAS COMMUTING WITH $|.|^k$ ON NORMAL ELEMENTS

Y. GUAN, C. WANG AND J. HOU*

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. Assume that $\mathcal{A}$ is of real rank zero and unital with unit I and $k > 0$ is a real number. It is shown that if $\Phi : \mathcal{A} \to \mathcal{B}$ is an additive map preserving $|\cdot|^k$ for all normal elements; that is, $\Phi(|A|^k) = |\Phi(A)|^k$ for all normal elements $A \in \mathcal{A}$, $\Phi(I)$ is a projection, and there exists a positive number c such that $\Phi(iI)\Phi(iI)^* \leq c\Phi(I)\Phi(I)^*$, then Φ is the sum of a linear Jordan *-homomorphism and a conjugate-linear Jordan *-homomorphism. If, moreover, the map Φ commutes with $|.|^k$ on $\mathcal{A}$, then Φ is the sum of a linear *-homomorphism and a conjugate-linear *-homomorphism. In the case when $k \neq 1$, the assumption $\Phi(I)$ being a projection can be deleted.
Keywords: C*-algebras, additive maps, Jordan homomorphism, *-homomorphism.
MSC(2010): Primary 47B49; Secondary: 46L05, 47L30.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{A}'$ be algebras over a field $\mathbb{F}$. Recall that a map $\Phi : \mathcal{A} \to \mathcal{A}'$ is called multiplicative if $\Phi(AB) = \Phi(A)\Phi(B)$ for all $A, B \in \mathcal{A}$; it is called a Jordan multiplicative map if $\Phi(AB + BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)$ for all $A, B \in \mathcal{A}$; the mapping Φ is called a (Jordan) homomorphism if it is additive and (Jordan) multiplicative; moreover, if the algebras involved are *-algebras and Φ is *-preserving, then Φ is called a (Jordan) *-homomorphism.

In the present paper we always assume that $\mathcal{A}, \mathcal{B}$ are C*-algebras [9, 10]. Thus a linear (i.e., $\mathbb{C}$-linear) Jordan *-homomorphism is exactly a C*-homomorphism. For a real number $k > 0$, a map $\Phi : \mathcal{A} \to \mathcal{B}$ is said to be k'th-power absolute value preserving if

$$\Phi(|A|^k) = |\Phi(A)|^k \tag{1.1}$$

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for all $A \in \mathcal{A}$, where $|A| = (A^*A)^{\frac{1}{2}}$; that is, Φ commutes with $|\cdot|^k$ on $\mathcal{A}$. Note that we do not assume here that k is a positive integer.

Let H and K be complex Hilbert spaces, and let $\mathcal{B}(H)$ and $\mathcal{B}(K)$ denote the algebras of all bounded linear operators on H and K, respectively. Assume that $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is an additive map with range containing all finite rank operators and $k > 0$ is a natural number. It was shown by Molnár in [12] that, if Φ satisfies (1.1), then there exists a positive real number c and an additive *-automorphism Ψ such that $\Phi = c\Psi$; moreover, if $k > 1$, then $\Phi = \Psi$. This result was generalized to the maps from a von Neumann algebra into $\mathcal{B}(H)$ by Bai and Hou in [2]. It was shown by Radjabalipour, Seddighi and Taghavi in [16] that if an additive map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ satisfies $\Phi(|A|) = |\Phi(A)|$ for all A in $\mathcal{B}(H)$, and if $\overline{\Phi(iI)K} \subset \phi(I)K$ and $\Phi(I)$ is a projection, then Φ is the sum of two *-homomorphisms, one of which is linear and the other is conjugate-linear. For the case that $\mathcal{A}$ is a von Neumann algebra, Radjabalipour proved in [17] that if $\varphi : \mathcal{A} \to \mathcal{B}(K)$ satisfies $|\varphi(A)| = \varphi(|A|)$ $(A \in \mathcal{A})$ and if $\mathcal{A}$ contains no nonzero abelian central projection, then there exists a space decomposition $K = K_0 \oplus K_+ \oplus K_-$, *-homomorphism $\varphi_\pm : \mathcal{A} \to \mathcal{B}(K_\pm)$ with φ_+ linear and φ_- conjugate-linear, and positive operator $C_\pm \in \mathcal{B}(K_\pm)$ such that $\varphi(A) = C_+\varphi_+(A) \oplus C_-\varphi_-(A)$ for all $A \in \mathcal{A}$. Recently, Taghavi [19] considered further the question for additive maps between C*-algebras. Let $\mathcal{A}$ be a unital C*-algebra and $\mathcal{B}$ be a C*-algebras of real rank zero. It is shown in [19] that if $\Phi : \mathcal{A} \to \mathcal{B}$ is an additive surjective map satisfying $\Phi(|A|) = |\Phi(A)|$ for every $A \in \mathcal{A}$ and $\Phi(I)$ is a projection, then Φ is a linear or conjugate-linear *-homomorphism.

The aim of this paper is to continue the studying of characterizing additive maps between C*-algebras satisfying (1.1). We generalize some known results mentioned above by two aspects. Firstly, for given $k > 0$ instead of positive integer, we consider the map Φ satisfying $\Phi(|A|^k) = |\Phi(A)|^k$, where Φ is an additive map between C*-algebras; secondly, we assume that $\Phi(|A|^k) = |\Phi(A)|^k$ is satisfied only for normal elements A.

Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. Denote by $\mathcal{A}_s$ and $\mathcal{A}_N$ respectively the set of all self-adjoint elements in $\mathcal{A}$ and the set of all normal elements in $\mathcal{A}$, that is, for $A \in \mathcal{A}$, $A \in \mathcal{A}_s \Leftrightarrow A = A^*$ and $A \in \mathcal{A}_N \Leftrightarrow A^*A = AA^*$. Assume that $\mathcal{A}$ is of real rank zero and unital with unit I, $k > 0$ is a real number. We show that if an additive map $\Phi : \mathcal{A} \to \mathcal{B}$ with $\Phi(I)$ a projection satisfies $\Phi(|A|^k) = |\Phi(A)|^k$ for all A in $\mathcal{A}_N$, and there exists a positive real number $c > 0$ such that $\Phi(iI)\Phi(iI)^* \leq c\Phi(I)\Phi(I)^*$, then Φ is the sum of a linear Jordan *-homomorphism and a conjugate-linear Jordan *-homomorphism; if $\Phi(|A|^k) = |\Phi(A)|^k$ for all $A \in \mathcal{A}$, then Φ is the sum of a linear *-homomorphism and a conjugate-linear *-homomorphism. Moreover, if $k \neq 1$, the assumption that $\Phi(I)$ is a projection can be omitted.

2. **Main results and corollaries**

The following is our main result which states that, under some soft assumptions, every additive map from a C*-algebra of real rank zero into a C*-algebra which preserves the k'th powers of the absolute values of normal operators is the sum of a linear Jordan *-homomorphism and a conjugate-linear Jordan *-homomorphism.

For a unital C*-algebra $\mathcal{A}$, recall that the real rank of $\mathcal{A}$ is the smallest integer, $RR(\mathcal{A})$, such that for each n-tuple $(x_1, \ldots, x_n)$ of self-adjoint elements in $\mathcal{A}$, with $n \leq RR(\mathcal{A}) + 1$, and every $\varepsilon > 0$, there is an n-tuple $(y_1, \ldots, y_n)$ in $\mathcal{A}_s$ such that $\sum {y_k}^2$ is invertible and $\| \sum (x_k - y_k)^2 \| < \varepsilon$. By definition $RR(\mathcal{A}) = 0$ if and only if every self-adjoint element in $\mathcal{A}$ can be approximated by an invertible self-adjoint element. It is well known that, if $\mathcal{A}$ is of real rank zero, then the set of all elements that can be written as real linear combination of mutually orthogonal projections is dense in $\mathcal{A}_s$ (Ref. [3]).

Theorem 2.1. *Let $\mathcal{A}$, $\mathcal{B}$ be C*-algebras. Denote $\mathcal{A}_N = \{A \in \mathcal{A} : AA^* = A^*A\}$. Assume that $\mathcal{A}$ is of real rank zero and unital with unit I, $k > 0$ is a positive number. If $\Phi : \mathcal{A} \to \mathcal{B}$ is an additive map satisfying*

(i) $\Phi(|A|^k) = |\Phi(A)|^k$ for all $A \in \mathcal{A}_N$;

(ii) $\Phi(I)$ is a projection;

(iii) there exists a positive real number $c > 0$ such that $\Phi(iI)\Phi(iI)^* \leq c\Phi(I)\Phi(I)^*$,

*then there exist projections $Q_1, Q_2 \in \mathcal{B}$ with $Q_1Q_2 = 0$, a linear Jordan *-homomorphism $\Phi_1 : \mathcal{A} \to Q_1\mathcal{B}Q_1$ and a conjugate-linear Jordan *-homomorphism $\Phi_2 : \mathcal{A} \to Q_2\mathcal{B}Q_2$ such that*

$$\Phi(A) = \Phi_1(A) + \Phi_2(A)$$

for all $A \in \mathcal{A}$. Moreover, in the case when $k \neq 1$, the assumption (ii) can be deleted.

Proof. We prove the theorem by firstly considering the case of $k = 1$ and secondly considering the case of $k \neq 1$.

Part I. The case of $k = 1$.

The proof is divided into several steps.

Step 1. Φ sends self-adjoint (resp, positive) elements of $\mathcal{A}$ into self-adjoint (resp, positive) elements of $\mathcal{B}$. Moreover, $\Phi|_{\mathcal{A}_s}$ is a continuous $\mathbb{R}$-linear map.

The idea is similar to that in [12, Proof of Theorem 1]. Indeed, if A is a positive element, then $\Phi(A) = \Phi(|A|) = |\Phi(A)|$ is also a positive operator. So Φ maps positive elements of $\mathcal{A}$ into positive elements of $\mathcal{B}$. Since Φ is additive and every self-adjoint element in a C*-algebra is the difference of two orthogonal positive ones, Φ preserves self-adjointness. It follows that Φ is order-preserving. Let $A \geq 0$ and $\lambda \in \mathbb{R}$ be fixed for a moment and consider arbitrary rational numbers r, s with $r < \lambda < s$. Since Φ is additive, it is $\mathbb{Q}$-linear. Consequently, we have $r\Phi(A) = \Phi(rA) \leq \Phi(\lambda A) \leq \Phi(sA) = s\Phi(A)$.

This gives that $\Phi(\lambda A) = \lambda\Phi(A)$. So Φ is a real linear map from $\mathcal{A}_s$ into $\mathcal{B}_s$. Let us turn to the continuity of $\Phi|_{\mathcal{A}_s}$. For any $A \in \mathcal{A}_s$, by the inequality $|A| \leq \||A|\|I = \|A\|I$, we get $|\Phi(A)| = \Phi(|A|) \leq \|A\|\Phi(I)$. Since the norm of a positive operator is equal to its numerical radius, we arrive at $\|\Phi(A)\| = \||\Phi(A)|\| \leq \|A\|\|\Phi(I)\|$, which yields the continuity of Φ when restricted to the real linear Banach space $\mathcal{A}_s$. Recall that, for any $T \in \mathcal{B}$, the numerical range of T is the set $W(T) = \{f(T) : f \in \mathcal{S}(\mathcal{B})\}$ and the numerical radius is $w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$, where $\mathcal{S}(\mathcal{B})$ is the set of all states of $\mathcal{B}$.

Step 2. For any pair of commuting self-adjoint operators $A, B \in \mathcal{A}_s$, we have

(1°) $\Phi(iB)^*\Phi(A) = -\Phi(A)\Phi(iB)$ and

(2°) if $AB = 0$, then $\Phi(A)\Phi(B) = 0$

The proof is similar to the Step 2 of the proof in [17], we omit it here.

Step 3. Φ preserves projections and orthogonality. Moreover,

$$\Phi(P) = \Phi(P)\Phi(I) = \Phi(I)\Phi(P)$$

holds for all projections $P \in \mathcal{A}$.

For every projection P, as Φ is order-preserving, we have $\Phi(P) \leq \Phi(I)$. It follows from the assumption (ii) that $\Phi(I)$ is a projection and thus $\Phi(I)\Phi(P) = \Phi(P)$. Now, by the assertion (2°) of Step 2, we can conclude that Φ sends projections to projections and preserves orthogonality of projections since

$$\Phi(P)^2 = \Phi(P)\Phi(I) = \Phi(I)\Phi(P) = \Phi(P).$$

Step 4. The restriction of Φ to $\mathcal{A}_s$ is a real linear Jordan homomorphism.

Note that $\mathcal{A}, \mathcal{B}$ are C*-algebras, one only needs to check that $\Phi(A^2) = \Phi(A)^2$ holds for all $A \in \mathcal{A}_s$. Instead of using the spectral theorem in [16], here we use the assumption that the C*-algebra $\mathcal{A}$ is of real rank zero. Suppose $A = \sum_{i=1}^{n} \lambda_i P_i$, where $\lambda_i \in \mathbb{R}$, $\{P_i\}$ are mutually orthogonal projections. Then $\Phi(A) = \sum_{i=1}^{n} \lambda_i \Phi(P_i)$, and $\Phi(A^2) = \sum_{i=1}^{n} \lambda_i^2 \Phi(P_i)$ as $A^2 = \sum_{i=1}^{n} \lambda_i^2 P_i$. Since $\{\Phi(P_i)\}$ are also mutually orthogonal projections, we have

$$\Phi(A)^2 = (\sum_{i=1}^{n} \lambda_i \Phi(P_i))^2 = \sum_{i=1}^{n} \lambda_i^2 \Phi(P_i).$$

So $\Phi(A^2) = \Phi(A)^2$. Now, as $\mathcal{A}$ is of real rank zero, every element in $\mathcal{A}_s$ can be approximated by elements of the form $\sum_{i=1}^{n} \lambda_i P_i$ as above. Then, by the continuity of Φ, we arrive at $\Phi(A^2) = \Phi(A)^2$ for any $A \in \mathcal{A}_s$. Thus the restriction of Φ to $\mathcal{A}_s$ is a Jordan homomorphism.

Step 5. For every $A \in \mathcal{A}$, $\Phi(A) = \Phi(A)\Phi(I) = \Phi(I)\Phi(A)$.

We first prove that, for every self-adjoint $S \in \mathcal{A}_s$,

$$\Phi(S) = \Phi(S)\Phi(I) = \Phi(I)\Phi(S). \tag{2.1}$$

As $\mathcal{A}$ is a C*-algebra of real rank zero and $\Phi|_{\mathcal{A}_s}$ is continuous and real linear, Eq. (2.1) follows from Step 3.

Now let us turn to show that

$$\Phi(iT) = \Phi(iT)\Phi(I) = \Phi(I)\Phi(iT) \tag{2.2}$$

holds for every self-adjoint operator T.

By the representation theorem of C*-algebras, we may assume that $\mathcal{A} \subset \mathcal{B}(H)$ and $\mathcal{B} \subset \mathcal{B}(K)$ for some complex Hilbert spaces H and K. Since, for any projection $P \in \mathcal{A}$,

$$\Phi(P) = \Phi(P)^2 = |\Phi(iP)|^2 = \Phi(iP)^*\Phi(iP),$$

we see that $|\Phi(iP)| = \Phi(P)$ and, by [7, Theorem 1], we have

$$\mathrm{ran}\Phi(iP)^* = \mathrm{ran}\Phi(P) \subseteq \mathrm{ran}\Phi(I).$$

So $\Phi(I)\Phi(iP)^* = \Phi(iP)^*$ as $\Phi(I)$ is a projection, which implies that

$$\Phi(iP)\Phi(I) = \Phi(iP). \tag{2.3}$$

Similar to the argument in [16, Proof of Theorem 1], we let $\Phi(iI) = V\Phi(I)$ be the polar decomposition of $\Phi(iI)$, where $V \in \mathcal{B}(K)$ is a partial isometry with the initial space $\mathrm{ran}\Phi(I)$ and the final space $\overline{\mathrm{ran}\Phi(iI)}$. For any projection P in $\mathcal{A}$, write $\Phi(iP) = V_P\Phi(P)$ and $\Phi(i(I-P)) = V_{I-P}\Phi(I-P)$ the polar decompositions of $\Phi(iP)$ and $\Phi(i(I-P))$, respectively. Note that, as $\mathcal{B}$ is a C*-algebra, V, V_p, V_{I-P} may not be in $\mathcal{B}$. In fact, one of the task below is to show that $V \in \mathcal{B}$. Clearly, we have

$$V_P\Phi(P) + V_{I-P}\Phi(I-P) = \Phi(iI) = V\Phi(P) + V\Phi(I-P).$$

Multiplying $\Phi(P)$ from the right of the above equations and applying the assertion of Step 1, one gets $V_P\Phi(P) = V\Phi(P)$. So,

$$\Phi(iP) = V\Phi(P)$$

for every projection P. Again, since $\mathcal{A}$ is of real rank zero, we also get

$$\Phi(iS) = V\Phi(S)$$

for every $A \in \mathcal{A}_s$. Now, by the hypotheses (ii) and (iii), there exists a positive real number c such that

$$\Phi(iI)\Phi(iI)^* \leq c\Phi(I)\Phi(I)^* = c\Phi(I)^2 = c\Phi(I).$$

It follows from [7] that $\mathrm{ran}\Phi(iI) \subseteq \mathrm{ran}\Phi(I)$, and consequently, $\Phi(I)\Phi(iI) = \Phi(iI)$. Thus we get

$$\Phi(iI) = V\Phi(I) = V = \Phi(I)V \tag{2.4}$$

and

$$\Phi(iP) = V\Phi(P) = \Phi(iI)\Phi(P) \quad \text{for every projection} \quad P \in \mathcal{A}.$$

Hence $V \in \mathcal{B}$ and

$$\Phi(I)\Phi(iP) = \Phi(I)\Phi(iI)\Phi(P) = \Phi(iI)\Phi(P) = \Phi(iP),$$

which, together with (2.3), shows that

$$\Phi(iP)\Phi(I) = \Phi(iP) = \Phi(I)\Phi(iP) \tag{2.5}$$

holds for every projection $P \in \mathcal{A}$.

Now using the assumption $\mathcal{A}$ is of real rank zero, we conclude from (2.5) that

$$\Phi(iT)\Phi(I) = \Phi(iT) = \Phi(I)\Phi(iT) \tag{2.6}$$

holds for every self-adjoint element T in $\mathcal{A}$.

Finally, for any $A \in \mathcal{A}$, we can write A in the form $A = S + iT$, where S and T are self-adjoint operators. Thus, by Eqs.(2.1) and (2.6), it is easily checked that

$$\Phi(A)\Phi(I) = \Phi(A) = \Phi(I)\Phi(A)$$

holds for all $A \in \mathcal{A}$, that is, the assertion of Step 5 is true.

Step 6. Let V be the partial isometry as in Step 5. Then, for every $A \in \mathcal{A}$, we have $\Phi(iA) = V\Phi(A)$ and $\Phi(A)V = V\Phi(A)$.

It is clear that $\text{ran}V^* = \text{ran}\Phi(I)$ because the initial space of V is $\text{ran}\Phi(I)$.

On the other hand, substitute $A = B = I$ in (1°) of Step 2, one gets $\Phi(I)V\Phi(I) = -\Phi(I)V^*\Phi(I)$. Hence we must have

$$V = -V^* \tag{2.7}$$

and consequently,

$$\overline{\text{ran}\Phi(iI)} = \text{ran}V^* = \text{ran}V = \text{ran}\Phi(I).$$

It follows from (2.7) that

$$-V^2 = V^*V = VV^* = \Phi(I). \tag{2.8}$$

For every $A \in \mathcal{A}$, write $A = S + iT$, where $S, T \in \mathcal{A}_s$. By Step 5 and (2.8), one gets

$$\Phi(iA) = \Phi(iS - T) = V\Phi(S) - \Phi(T)$$

and

$$\begin{aligned} V\Phi(A) = &\quad V(\Phi(S) + \Phi(iT)) = V\Phi(S) + V^2\Phi(T) \\ = &\quad V\Phi(S) - \Phi(I)\Phi(T) = V\Phi(S) - \Phi(T). \end{aligned}$$

Therefore,

$$\Phi(iA) = V\Phi(A) \quad \text{for every } A \in \mathcal{A}. \tag{2.9}$$

By substituting $B = I$ in (1°) of Step 2 and applying (2.9) we obtain

$$\Phi(A)V\Phi(I) = -\Phi(I)V^*\Phi(A), \quad \forall A \in \mathcal{A}_s.$$

Thus, by (2.7), we have $\Phi(A)V\Phi(I) = \Phi(I)V\Phi(A)$, which, together with (2.9), implies that $\Phi(A)V = V\Phi(A)$ and $\Phi(iA)V = V\Phi(iA)$ holds for all $A \in \mathcal{A}_s$ as $\Phi(I)V = V\Phi(I) = V$. Now it is obvious that $\Phi(A)V = V\Phi(A)$ holds for every $A \in \mathcal{A}$, completing the proof of Step 6.

Step 7. Φ is a real linear Jordan *-homomorphism from $\mathcal{A}$ into $\mathcal{B}$.

By Step 4, Φ is a real linear Jordan homomorphism from $\mathcal{A}_s$ into $\mathcal{B}_s$.

For any $\lambda \in \mathbb{R}$ and any $A \in \mathcal{A}$, write $A = S + iT$, where $S, T \in \mathcal{A}_s$. We have

$$\begin{aligned}\Phi(\lambda A) = & \quad \Phi(\lambda S) + \Phi(i(\lambda T)) = \lambda\Phi(S) + V\Phi(\lambda T) \\ = & \quad \lambda(\Phi(S) + V\Phi(T)) = \lambda\Phi(A).\end{aligned}$$

So, Φ is real linear on $\mathcal{A}$.

For any $B \in \mathcal{A}_s$, iB is a skew self-adjoint element in $\mathcal{A}$, that is $(iB)^* = -iB$. By Step 5 and (1°) of Step 2 with $A = I$, one gets $\Phi(iB)^* = \Phi(iB)^*\Phi(I) = -\Phi(I)\Phi(iB) = -\Phi(iB) = \Phi((iB)^*)$. Then it follows from the additivity of Φ that

$$\Phi(A^*) = \Phi(A)^* \quad \text{for all } A \in \mathcal{A},$$

that is, Φ is *-preserving.

Furthermore, for any self-adjoint operator $T \in \mathcal{A}$, by Steps 5-6 and (2.8), we have

$$\Phi((iT)^2) = -\Phi(T^2) = -\Phi(T)^2 = (V\Phi(T))^2 = \Phi(iT)^2.$$

Now, since

$$\Phi(A^2) = \Phi(S^2 + iST + iTS + (iT)^2) = \Phi(S^2) + V\Phi(ST + TS) + \Phi((iT)^2)$$

and

$$\begin{aligned}\Phi(A)^2 = & \quad \Phi(S)^2 + \Phi(S)\Phi(iT) + \Phi(iT)\Phi(S) + \Phi(iT)^2 \\ = & \quad \Phi(S)^2 + \Phi(S)V\Phi(T) + V\Phi(T)\Phi(S) + \Phi(iT)^2 \\ = & \quad \Phi(S)^2 + V\Phi(S)\Phi(T) + V\Phi(T)\Phi(S) + \Phi(iT)^2,\end{aligned}$$

one sees that

$$\Phi(A^2) = \Phi(A)^2$$

holds for all $A \in \mathcal{A}$, that is, Φ is a real linear Jordan *-homomorphism.

Step 8. There exist two orthogonal projections $Q_1, Q_2 \in \mathcal{B}$ and a linear Jordan *-homomorphism $\Phi_1 : \mathcal{A} \to Q_1\mathcal{B}Q_1$, a conjugate-linear Jordan *-homomorphism $\Phi_2 : \mathcal{A} \to Q_2\mathcal{B}Q_2$ such that $\Phi(A) = \Phi_1(A) + \Phi_2(A)$ for all A in $\mathcal{A}$.

Let $\mathcal{B}_1 = \Phi(I)\mathcal{B}\Phi(I)$. Then, $I_1 = \Phi(I)$ is the unit of $\mathcal{B}_1$ and, by Step 5, it is easily seen that $\Phi(\mathcal{A}) \subseteq \mathcal{B}_1$. So, we can regard Φ as a unital real linear Jordan *-homomorphism from $\mathcal{A}$ into $\mathcal{B}_1$.

Since $\Phi(iI)$ is skew self-adjoint, $V = \Phi(iI) = iW$ for some self-adjoint element $W \in \mathcal{B}_1$. Furthermore, $-W^2 = \Phi(iI)^2 = -\Phi(I) = -I_1$ gives $W^2 = I_1$. Then the spectrum of W as an element in $\mathcal{B}_1$, $\sigma^{\mathcal{B}_1}(W) \subseteq \{-1, 1\}$, which implies that there exists a projection Q in $\mathcal{B}_1$ so that $W = 2Q - I_1$ and $\Phi(iI) = i(2Q - I_1)$.

It is clear by Step 6 that $iW\Phi(A) = \Phi(A)iW$ for all $A \in \mathcal{A}$. So we have $W\Phi(A) = \Phi(A)W$ for all A, which entails that $Q\Phi(A) = \Phi(A)Q$ for each $A \in \mathcal{A}$. Then

$$\Phi(A) = Q\Phi(A) + (I_1 - Q)\Phi(A)$$

holds for all A. Now let $Q_1 = Q$, $Q_2 = I_1 - Q$ and define Φ_1, Φ_2 by $\Phi_1(A) = Q_1\Phi(A)$, $\Phi_2(A) = Q_2\Phi(A)$. Then, $Q_1Q_2 = 0$, $\Phi_i : \mathcal{A} \to Q_i\mathcal{B}_1Q_i$ is real linear Jordan *-homomorphism, $i = 1, 2$. In addition, as $\Phi(iI) = Q_1\Phi(iI)Q_1 + Q_2\Phi(iI)Q_2$, we see that $\Phi_1(iI) = iQ_1$ and $\Phi_2(iI) = -iQ_2$. Hence Φ_1 is linear and Φ_2 is conjugate-linear. This completes the proof of the theorem for the case when $k = 1$.

Part II. The case when $k \neq 1$.

For any positive element $A \in \mathcal{A}$, there is a unique positive element $B \in \mathcal{A}$ such that $A = B^k$. Thus $\Phi(A) = \Phi(B^k) = \Phi(|B|^k) = |\Phi(B)|^k \geq 0$. So $\Phi(A)$ is positive. Moreover, for any normal element A, we have $|\Phi(A)|^k = \Phi(|A|^k) = \Phi(|A|)^k$. As the positive k'th root of a positive element is unique, we see that $|\Phi(A)| = \Phi(|A|)$. Thus, Φ satisfies the hypothesis (i).

Next we show that $\Phi(I)$ is automatically a projection and hence the assumption (ii) is superfluous.

If $P \in \mathcal{A}$ is a projection, then $\Phi(P)^k = |\Phi(P)|^k = \Phi(|P|^k) = \Phi(P^k) = \Phi(P)$. Let $\Phi(P) = \int_0^{\|\Phi(P)\|} t dE_t$ be the spectral resolution of $\Phi(P)$ regarded as a positive operator in $\mathcal{B}(K)$. Thus we have $\int_0^{\|\Phi(P)\|} t^k dE_t = \Phi(P)^k = \Phi(P) = \int_0^{\|\Phi(P)\|} t dE_t$, which implies that $t^k = t$ almost everywhere. As $k \neq 1$, the support of the spectral measure is a subset of $\{0, 1\}$. Then it follows that $t^2 = t$ a.e. and hence $\Phi(P)^2 = \Phi(P)$. Thus Φ is projection preserving. Particularly, $\Phi(I)^2 = \Phi(I)$. So, Φ meets all the assumptions of Part I, and then, has the form stated in the theorem, completing the proof. □

If we strengthen the assumption on Φ by letting $\Phi(|A|^k) = |\Phi(A)|^k$ holds for all $A \in \mathcal{A}$, then Φ is the sum of a linear *-homomorphism and a conjugate-linear *-homomorphism, as stated in the following result.

Theorem 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. Assume that $\mathcal{A}$ is of real rank zero and unital with unit I, k is a positive number. Then $\Phi : \mathcal{A} \to \mathcal{B}$ is an additive map satisfying that*

(i) $\Phi(|A|^k) = |\Phi(A)|^k$ for all $A \in \mathcal{A}$;

(ii) $\Phi(I)$ is a projection;

(iii) there exists a positive real number $c > 0$ such that $\Phi(iI)\Phi(iI)^* \leq c\Phi(I)\Phi(I)^*$,

*if and only if there exist projections $Q_1, Q_2 \in \mathcal{B}$ with $Q_1Q_2 = 0$, a linear *-homomorphism $\Phi_1 : \mathcal{A} \to Q_1\mathcal{B}Q_1$, a conjugate-linear *-homomorphism $\Phi_2 : \mathcal{A} \to Q_2\mathcal{B}Q_2$ such that $\Phi(A) = \Phi_1(A) + \Phi_2(A)$ for all $A \in \mathcal{A}$. Moreover, in the case when $k \neq 1$, the assumption (ii) can be omitted.*

Proof. The "if" part is obvious. In fact, if Φ has the form $\Phi(A) = \Phi_1(A) + \Phi_2(A)$ as stated in Theorem 2.2, where Φ_1 is a linear *-homomorphism and Φ_2 is a conjugate-linear *-homomorphism with $\Phi_1(A)\Phi_2(A) = \Phi_2(A)\Phi_1(A) = 0$ for every A, then it is clear that $\Phi(I)$ is a projection and

$$\begin{aligned}\Phi(iI)\Phi(iI)^* = &\ \Phi_1(iI)\Phi_1(iI)^* + \Phi_2(iI)\Phi_2(iI)^* \\ = &\ i\Phi_1(I)(i\Phi_1(I))^* - i\Phi_2(I)(-i\Phi_2(I))^* = \Phi(I)\Phi(I)^*,\end{aligned}$$

that is, Φ satisfies the conditions (ii) and (iii).

Obviously, $\Phi(S^k) = \Phi(S)^k$ for every positive element $S \geq 0$. So, for any element $A \in \mathcal{A}$, we have

$$|\Phi(A)|^k = (\Phi(A)^*\Phi(A))^{\frac{k}{2}} = \Phi(A^*A)^{\frac{k}{2}} = \Phi(|A|^k)$$

for every $A \in \mathcal{A}$. So (i) also holds.

Next we check the "only if" part. By Theorem 2.1, there exist projections $Q_1, Q_2 \in \mathcal{B}$ with $Q_1Q_2 = 0$, a linear Jordan *-homomorphism $\Phi_1 : \mathcal{A} \to Q_1\mathcal{B}Q_1$, a conjugate-linear Jordan *-homomorphism $\Phi_2 : \mathcal{A} \to Q_2\mathcal{B}Q_2$ such that $\Phi(A) = \Phi_1(A) + \Phi_2(A)$ for all $A \in \mathcal{A}$. We have to show that $\Phi(AB) = \Phi(A)\Phi(B)$ holds for any $A, B \in \mathcal{A}$.

Similar to the argument in the Part II of the proof of Theorem 2.1, it is easily checked that

$$\Phi(|A|) = |\Phi(A)| \quad \text{holds for all } A \in \mathcal{A} \tag{2.10}$$

Since $\Phi(S)^2 = \Phi(S^2)$ for all self-adjoint elements $S \in \mathcal{A}_s$, by (2.10), we have

$$\Phi(A^*A) = \Phi(|A|^2) = \Phi(|A|)^2 = |\Phi(A)|^2 = \Phi(A)^*\Phi(A) \tag{2.11}$$

holds for any $A \in \mathcal{A}$. Replacing A in (2.11) once by $A + B$ and once by $A + iB$ shows that

$$\Phi(A^*B) = \Phi(A)^*\Phi(B)$$

for all $A, B \in \mathcal{A}$, which entails that $\Phi(AB) = \Phi(A)\Phi(B)$ for all $A, B \in \mathcal{A}$. Hence, Φ is a *-homomorphism and has the form stated in Theorem 2.2. $\square$

Remark 2.3. The assumption (iii) in Theorem 2.1 and Theorem 2.2 can not be omitted. For example, let $\mathcal{A}$ be a unital C*-algebra of real rank zero and consider the map $\Phi : \mathcal{A} \to \mathcal{A} \otimes M_2$ defined by $\Phi(A) = \begin{pmatrix} 0 & \mathrm{Im}A \\ 0 & \mathrm{Re}A \end{pmatrix}$. It is clear that Φ is additive and $\Phi(I)$ is a projection. Obviously, Φ breaks the condition (iii). For any $A \in \mathcal{A}_N$, write $A = S + iT$, where $S = \mathrm{Re}A = \frac{A+A^*}{2}$ and $T = \mathrm{Im}A = \frac{A-A^*}{2i}$. As A is normal, we have $ST = TS$ and $|A| = \sqrt{S^2 + T^2}$. Then it is easily checked that $|\Phi(A)| = \Phi(|A|)$. Thus Φ satisfies the conditions (i) of Theorem 2.1. But Φ is not of the form stated in Theorem 2.1. For an example for Theorem 2.2, let $\mathcal{A}$ be a commutative algebra.

However, if the map is "surjective" in some sense, the assumption (iii) is superfluous.

Corollary 2.4. *Let $\mathcal{A}$ be a unital C*-algebra of real rank zero and $\mathcal{B}$ be a C*-algebra. Assume that k is a positive number and $\Phi : \mathcal{A} \to \mathcal{B}$ is an additive map with $\mathcal{B}_N \subseteq \Phi(\mathcal{A}_N)$. If Φ satisfies that $\Phi(|A|^k) = |\Phi(A)|^k$ for all $A \in \mathcal{A}_N$ and if $\Phi(I)$ is a projection, then $\Phi(I) = I$, there exists a central projection $Q \in \mathcal{B}$, a linear Jordan *-homomorphism $\Phi_1 : \mathcal{A} \to Q\mathcal{B}$ and a conjugate-linear Jordan *-homomorphism $\Phi_2 : \mathcal{A} \to (I - Q)\mathcal{B}$ such that $\Phi(A) = \Phi_1(A) + \Phi_2(A)$ for every $A \in \mathcal{A}$. Particularly, if $\mathcal{B}$ is a factor C*-algebra, then Φ is either a linear or a conjugate-linear Jordan *-homomorphism. Moreover, in the case when $k \neq 1$, the assumption $\Phi(I)$ being a projection can be omitted.*

Proof. Note that, $\Phi(|A|) = |\Phi(A)|$ for all normal elements A and the assertions of Step 1-Step 4 in the proof of Theorem 2.1 are still true for Φ here. For any positive element $B \in \mathcal{B}$, as $\mathcal{B}_N \subseteq \Phi(\mathcal{A}_N)$, there is some $A \in \mathcal{A}_N$ such that $B = \Phi(A)$. Thus $B = |B| = |\Phi(A)| = \Phi(|A|)$, which means every positive element is a Φ-image of a positive element, and consequently, $\mathcal{B}_s \subseteq \Phi(\mathcal{A}_s)$. Since $\Phi(I)\Phi(A) = \Phi(A)\Phi(I) = \Phi(A)$ holds for every $A \in \mathcal{A}_s$, one has that $\Phi(I)B = B\Phi(I) = B$ for every $B \in \mathcal{B}_s$. Hence $\mathcal{B}$ must be unital and $\Phi(I) = I$ is its unit. Thus Φ meets all conditions (i)-(iii) and has the form of Theorem 2.1 with $Q_1 + Q_2 = \Phi(I) = I$. It is clear that both Q_1 and Q_2 commute with all elements of $\mathcal{B}$. So, Q_1, Q_2 are central projections of $\mathcal{B}$ and $\mathcal{B} = Q_1\mathcal{B}Q_1 + Q_2\mathcal{B}Q_2$. Thus, if $\mathcal{B}$ is a factor, then either $Q_1 = I, Q_2 = 0$, in this case Φ is a linear Jordan *-homomorphism; or $Q_1 = 0, Q_2 = I$, in this case Φ is a conjugate-linear Jordan *-homomorphism. □

The following corollary is immediate by Theorem 2.2 and the proof of Corollary 2.4.

Corollary 2.5. *Let $\mathcal{A}$ be a unital C*-algebra of real rank zero and $\mathcal{B}$ be a C*-algebra. Assume that k is a positive number and $\Phi : \mathcal{A} \to \mathcal{B}$ is an additive surjective map with $\Phi(I)$ a projection. Then Φ satisfies that $\Phi(|A|^k) = |\Phi(A)|^k$ for all $A \in \mathcal{A}$ if and only if there exist a central projection $Q \in \mathcal{B}$, a linear *-homomorphism $\Phi_1 : \mathcal{A} \to Q\mathcal{B}$ and a conjugate-linear *-homomorphism $\Phi_2 : \mathcal{A} \to (I - Q)\mathcal{B}$ such that $\Phi(A) = \Phi_1(A) + \Phi_2(A)$ for every $A \in \mathcal{A}$. Particularly, if $\mathcal{B}$ is a factor, then Φ is either a linear or conjugate linear *-homomorphism. Moreover, in the case when $k \neq 1$, the assumption $\Phi(I)$ being a projection can be omitted.*

If the map is bijective, one can say more.

Corollary 2.6. *Let $\mathcal{A}$ be a unital C*-algebra of real rank zero and $\mathcal{B}$ be a C*-algebra. Assume that k is a positive number and $\Phi : \mathcal{A} \to \mathcal{B}$ is an additive bijective map with $\Phi(I)$ a projection. Then Φ satisfies that $\Phi(|A|^k) = |\Phi(A)|^k$ for all $A \in \mathcal{A}$ if and only if there exist a central projection $P \in \mathcal{A}$, with $Q = \Phi(P)$ central projection in $\mathcal{B}$, a linear *-homomorphism $\Phi_1 : P\mathcal{A}P \to Q\mathcal{B}Q$, a conjugate-linear *-homomorphism $\Phi_2 : (I - P)\mathcal{A}(I - P) \to (I - Q)\mathcal{B}(I - Q)$,* *such that $\Phi = \Phi_1 \oplus \Phi_2$. Particularly, if $\mathcal{B}$ is a factor, then Φ is either a linear or conjugate linear *-homomorphism. Moreover, in the case when $k \neq 1$, the assumption $\Phi(I)$ being a projection can be omitted.*

Proof. If $\Phi(|A|^k) = |\Phi(A)|^k$ for all $A \in \mathcal{A}$, then it has the form stated in Corollary 2.5. Since Φ is injective, we have $\ker\Phi_1 \cap \ker\Phi_2 = \{0\}$. Let $\mathcal{A}_1 = \Phi^{-1}(Q\mathcal{B})$, $\mathcal{A}_2 = \Phi^{-1}((I-Q)\mathcal{B})$. Since $Q\mathcal{B}+(I-Q)\mathcal{B} = \mathcal{B}$, we have $\mathcal{A} = \mathcal{A}_1+\mathcal{A}_2$. Also, $Q_1\mathcal{B}\cap Q_2\mathcal{B} = \{0\}$ entails that $\mathcal{A}_1 \cap \mathcal{A}_2 = \{0\}$. For any $A_i \in \mathcal{A}_i$, $i \in \{1, 2\}$, one has $\Phi(A_1A_2) = \Phi(A_1)\Phi(A_2) = Q\Phi_1(A_1)(I - Q)\Phi_2(A_2) = 0$, which implies that $A_1A_2 = A_2A_1 = 0$ by the injectivity of Φ. Let $P = \Phi^{-1}(Q) = {\Phi_1}^{-1}(Q)$. Clearly P is a projection. Write $P_1 = P$ and $P_2 = I - P$; also $Q_1 = Q$ and $Q_2 = I - Q$. We claim that $\mathcal{A}_i = P_i\mathcal{A} = \mathcal{A}P_i$, $i = 1, 2$. As $\Phi(I) = I$, we have $P_2 = \Phi^{-1}(Q_2)$. Clearly, $P_i \in \mathcal{A}_i$, and hence $P_i\mathcal{A}P_i \subseteq \mathcal{A}_i$. Since $\mathcal{A}_2 \subseteq \ker\Phi_1$, $\mathcal{A}_1 \subseteq \ker\Phi_2$, for any $A \in \mathcal{A}$, we have $\Phi(P_1AP_1) = \Phi_1(P_1A_1P_1) = \Phi_1(P_1)\Phi_1(A)\Phi_1(P_1) = Q_1\Phi_1(A) = \Phi_1(P_1A) = \Phi_1(A)Q_1 = \Phi_1(AP_1)$, which implies that $P_1AP_1 = P_1A = AP_1$ holds for all $A \in \mathcal{A}$. Thus $P_1 \in \mathcal{Z}(\mathcal{A})$, the center of $\mathcal{A}$. So $\mathcal{A}_1 = \Phi^{-1}(Q\mathcal{B}) = P_1\mathcal{A} = \mathcal{A}P_1$, and P_1 is the unit of $\mathcal{A}_1$. Similarly, $\mathcal{A}_2 = P_2\mathcal{A} = \mathcal{A}P_2$. Thus we may regard Φ_i as a bijective map from $\mathcal{A}_i$ onto $\mathcal{B}_i = Q_i\mathcal{B}$ and $\Phi = \Phi_1 \oplus \Phi_2$, where Φ_1 is a linear *-homomorphism and Φ_2 is a conjugate-linear *-homomorphism. □

As an application of Theorems 2.1 and 2.2, in the following we characterize additive maps preserving absolute values of skew products.

Corollary 2.7. *Let $\mathcal{A}$ be a unital C^*-algebra of real rank zero and $\mathcal{B}$ be a C^*-algebra. Assume that $\Phi : \mathcal{A} \to \mathcal{B}$ is an additive map. If Φ satisfies that $\Phi(|A^*B|) = |\Phi(A)^*\Phi(B)|$ for any $A, B \in \mathcal{A}$ with $A^*B = BA^*$, then there exist projections $Q_1, Q_2 \in \mathcal{B}$ with $Q_1Q_2 = 0$, a linear Jordan *-homomorphism $\Phi_1 : \mathcal{A} \to Q_1\mathcal{B}Q_1$, a conjugate-linear Jordan *-homomorphism $\Phi_2 : \mathcal{A} \to Q_2\mathcal{B}Q_2$ such that $\Phi(A) = \Phi_1(A) + \Phi_2(A)$ for all $A \in \mathcal{A}$.*

Proof. For any $A \in \mathcal{A}_N$ we have $\Phi(|A|^2) = \Phi(|A^*A|) = |\Phi(A)^*\Phi(A)| = |\Phi(A)|^2$. Thus Φ meets the condition (i) of Theorem 2.1 with $k = 2$. It follows that Φ is a $\mathbb{R}$-linear Jordan homomorphism from $\mathcal{A}_s$ into $\mathcal{B}_s$ and $\Phi(I)$ is a projection as proved in the Part II of the proof of Theorem 2.1. Moreover, $\Phi(|A|) = |\Phi(A)|$ for all $A \in \mathcal{A}_N$. By an argument similar to Step 2-Step 5 of Theorem 2.1, one obtains that

$$\Phi(iS)^*\Phi(A) = -\Phi(A)\Phi(iS) \tag{2.12}$$

and

$$\Phi(iS)^*\Phi(iS) = |\Phi(iS)|^2 = \Phi(|S|)^2 = |\Phi(S)|^2 = \Phi(S)^2 \tag{2.13}$$

hold for all commuting self-adjoint operator $A, S \in \mathcal{A}_s$, and

$$\Phi(A) = \Phi(I)\Phi(A) = \Phi(A)\Phi(I) \tag{2.14}$$

holds for all $A \in \mathcal{A}_s$.

Without loss of generality, we may assume that $\mathcal{B} \subseteq \mathcal{B}(K)$ for some Hilbert space K. Thus, with respect to the space decomposition $K = K_0 \oplus {K_0}^{\perp}$ with $K_0 = \ker\Phi(\mathrm{I})$, we have $\Phi(I) = \begin{pmatrix} 0 & 0 \\ 0 & I_1 \end{pmatrix}$ and $\Phi(S)$ has the form $\Phi(S) = \begin{pmatrix} 0 & 0 \\ 0 & \xi(S) \end{pmatrix}$ if S is self-adjoint. Write $\Phi(iS) = \begin{pmatrix} C & \eta(iS) \\ D & \xi(iS) \end{pmatrix}$, where $S \in \mathcal{A}_s$. It follows from (2.12) by letting $A = I$ that $D = 0$. By (2.13) and (2.14), one gets $C = 0$. So there exist $\mathbb{R}$-linear continuous maps ξ and η from $\mathcal{A}$ into respectively $B({K_0}^{\perp})$ and $B({K_0}^{\perp}, K_0)$ such that

$$\Phi(A) = \begin{pmatrix} 0 & \eta(A) \\ 0 & \xi(A) \end{pmatrix}.$$

Moreover $\eta(S) = 0$ for any $S \in \mathcal{A}_s$. Particularly, $\Phi(iI) = \begin{pmatrix} 0 & \eta(iI) \\ 0 & \xi(iI) \end{pmatrix}$. Let $A = iI$, $B = iI$; by the assumption of Φ, one gets $|\Phi(iI)^*\Phi(iI)| = \Phi(I)$. That is to say $\eta(iI)^*\eta(iI) + \xi(iI)^*\xi(iI) = I_1$. Similarly, let $A = I$ and $B = iI$, we have $\xi(iI)^*\xi(iI) = I_1$. Then, it follows that $\eta(iI) = 0$. So we have $\Phi(iI) = \begin{pmatrix} 0 & 0 \\ 0 & \xi(iI) \end{pmatrix}$, which entails that there exists a positive real number $c > 0$ such that $\Phi(iI)\Phi(iI)^* \leq c\Phi(I)\Phi(I)^*$. Thus Φ meets all the assumptions of Theorem 2.1 and has the desired form. □

Corollary 2.8. *Let $\mathcal{A}$ be a unital C*-algebra of real rank zero and $\mathcal{B}$ be a C*-algebra. Assume that $\Phi : \mathcal{A} \to \mathcal{B}$ is an additive map. Then Φ satisfies that $\Phi(|A^*B|) = |\Phi(A)^*\Phi(B)|$ for all $A, B \in \mathcal{A}$ if and only if there exist projections $Q_1, Q_2 \in \mathcal{B}$ with $Q_1Q_2 = 0$, a linear *-homomorphism $\Phi_1 : \mathcal{A} \to Q_1\mathcal{B}Q_1$, a conjugate-linear *-homomorphism $\Phi_2 : \mathcal{A} \to Q_2\mathcal{B}Q_2$ such that $\Phi(A) = \Phi_1(A) + \Phi_2(A)$ for all $A \in \mathcal{A}$.*

Proof. The "if" part is clear. In fact, by Theorem 2.2, Φ is a ring *-homomorphism and $|\Phi(A)| = \Phi(|A|)$ for all $A \in \mathcal{A}$. Thus, $|\Phi(A)^*\Phi(B)| = |\Phi(A^*)\Phi(B)| = |\Phi(A^*B)| = \Phi(|A^*B|)$.

Next we check the "only if" part. By Corollary 2.7, Φ is the sum of a linear Jordan *-homomorphism and a conjugate-linear Jordan *-homomorphism. So, we only need to show that Φ is multiplicative, that is, $\Phi(AB) = \Phi(A)\Phi(B)$ holds for any $A, B \in \mathcal{A}$.

By the proof of Corollary 2.6, one has

$$\Phi(|A|) = |\Phi(A)| \tag{2.15}$$

$$\text{for all } A \in \mathcal{A} \text{ and } \Phi(S)^2 = \Phi(S^2) \tag{2.16}$$

for all self-adjoint elements $S \in \mathcal{A}_s$. Hence, by (2.15) (2.16), for any $A \in \mathcal{A}$,

$$\Phi(A^*A) = \Phi(|A|^2) = \Phi(|A|)^2 = |\Phi(A)|^2 = \Phi(A)^*\Phi(A). \tag{2.17}$$

Replacing A in (2.17) once by $A+B$ and once by $A+iB$ derives

$$\Phi(A^*B) = \Phi(A)^*\Phi(B)$$

for all $A, B \in \mathcal{A}$, which entails that $\Phi(AB) = \Phi(A)\Phi(B)$ for all $A, B \in \mathcal{A}$. $\square$

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