ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Special Issue of the

Bulletin of the
Iranian Mathematical Society

in Honor of Professor Heydar Radjavi's 80th Birthday

Vol. 41 (2015), No. 7, pp. 99–106

Title:

A Haar wavelets approach to Stirling's formula

Author(s):

M. Ahmadinia and H. Naderi Yeganeh

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 7, pp. 99–106
Online ISSN: 1735-8515

A HAAR WAVELETS APPROACH TO STIRLING'S FORMULA

M. AHMADINIA* AND H. NADERI YEGANEH

(Communicated by Peter Rosenthal)

Dedicated to Professor Heydar Radjavi on his 80th birthday

Abstract. This paper presents a proof of Stirling's formula using Haar wavelets and some properties of Hilbert space, such as Parseval's identity. The present paper shows a connection between Haar wavelets and certain sequences.
Keywords: Haar wavelets, Parseval's identity, Stirling's formula.
MSC(2010): Primary: 15-xx; Secondary: 40-xx.

1. Introduction

Stirling's formula ($\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1$) plays an important role in statistics and probability; its main use is estimating the value of $n!$. The first proofs of Stirling's formula were presented by de Moivre and Stirling (see [4,8]). Estimates of $n!$ have been obtained via upper and lower bounds of $n!$ by various authors (for example see [3], [5] and [6]). Some papers prove Stirling's formula by different techniques as well (for example see [2]). The present paper introduces a new method of proving Stirling's formula by using Haar wavelets. Although there are some proofs of Stirling's formula that are simpler than the present one, we feel that it is of interest to know that this formula can be established using Haar wavelets.

The outline of the paper is as follows: In Section 2, we introduce some notation and present two lemmas related to Parseval's identity and the Haar wavelet basis. In Section 3, we present an application of the two lemmas, and in the last section we give a proof of Stirling's formula and of a lower and upper bound for $n!$.

Article electronically published on December 31, 2015.
Received: 22 November 2014, Accepted: 18 April 2015.
*Corresponding author.

2. **Preliminaries and Haar Wavelets**

This section introduces Haar wavelets and some related notation. The following notation is fixed throughout the paper:

$$Z^* := N \cup \{0\},$$

(N is the set of natural numbers). The bijective function P and the surjective function Q are defined as follows:

$$P : S \to N, \quad P(n,k) := 2^n + k,$$

and

$$Q : W \to Z^*, \quad Q(r,n) := n,$$

where

$$S := \{(n,k) \in Z^* \times Z^* | 0 \le k \le 2^n - 1\}, \;\&\; W := \{(r,n) \in Z^* \times Z^* | r \le n\}.$$

Consider the following functions

$$\phi(x) := \begin{cases} 1, & 0 \le x < 1, \\ 0, & otherwise, \end{cases}$$

and

$$\psi(x) := \begin{cases} 1, & 0 \le x < \frac{1}{2}, \\ -1, & \frac{1}{2} \le x < 1, \\ 0, & otherwise, \end{cases}$$

and also

$$\psi_{n,k}(x) := 2^{\frac{n}{2}} \psi(2^n x - k), \qquad (n,k) \in S.$$

The real Hilbert space $L^2([0,1])$ with inner product $\langle f, g\rangle := \int_0^1 f(x)g(x)dx$, has the following orthonormal basis, which is called "the Haar wavelets basis":

$$\{\phi(x)\} \cup \{\psi_{n,k}(x) \mid (n,k) \in S\}\,. \tag{2.1}$$

Parseval's identity implies that

$$\langle f, g\rangle = \langle f, \phi\rangle\langle g, \phi\rangle + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} \langle f, \psi_{n,k}\rangle\langle g, \psi_{n,k}\rangle, \tag{2.2}$$

(for details see [1] and [7]). For convenience, we employ the following notation:

$$\int_{n,k} f := \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} f(x)dx.$$

Lemma 2.1. *If f and g are in $L^2([0,1])$, then*

$$\langle f, g\rangle = 2^r \sum_{k=0}^{2^r-1} \int_{r,k} f \int_{r,k} g + \sum_{n=r}^{\infty} \sum_{k=0}^{2^n-1} \langle f, \psi_{n,k}\rangle\langle g, \psi_{n,k}\rangle, \tag{2.3}$$

for all $r \in Z^*$.

Proof. This lemma is proved by induction. If $r = 0$, then (2.3) and (2.2) are the same. If the equation (2.3) holds for r, then

$$\langle f, g\rangle = 2^r \sum_{k=0}^{2^r-1} \int_{r,k} f \int_{r,k} g + \sum_{k=0}^{2^r-1} \langle f, \psi_{r,k}\rangle\langle g, \psi_{r,k}\rangle + \sum_{n=r+1}^{\infty} \sum_{k=0}^{2^n-1} \langle f, \psi_{n,k}\rangle\langle g, \psi_{n,k}\rangle. \tag{2.4}$$

Let h be an arbitrary function in $L^2([0,1])$. Then

$$\int_{r,k} h = \int_{r+1,2k} h + \int_{r+1,2k+1} h, \tag{2.5}$$

and

$$\langle h, \psi_{r,k}\rangle = 2^{\frac{r}{2}} \left(\int_{r+1,2k} h - \int_{r+1,2k+1} h \right). \tag{2.6}$$

We can derive the following equation by using (2.5) and (2.6) in the left side of the equation:

$$2^r \sum_{k=0}^{2^r-1} \int_{r,k} f \int_{r,k} g + \sum_{k=0}^{2^r-1} \langle f, \psi_{r,k}\rangle\langle g, \psi_{r,k}\rangle = 2^{r+1} \sum_{k=0}^{2^{r+1}-1} \int_{r+1,k} f \int_{r+1,k} g. \tag{2.7}$$

Substitution of (2.7) into (2.4) yields (2.3) for $r + 1$. $\square$

Lemma 2.2. *If f and g are continuous functions on $[0,1]$ that are differentiable on $(0,1)$, and if also $|f'(x)| < M_1$ and $|g'(x)| < M_2$ for all $x \in (0,1)$, then the following assertions are true:*

(i) $|\langle f, \psi_{n,k}\rangle| < \frac{M_1}{2^{\frac{3n}{2}+1}}$ *and* $|\langle g, \psi_{n,k}\rangle| < \frac{M_2}{2^{\frac{3n}{2}+1}}$, $(n,k) \in S$,

(ii) $|\int_{r,k} fg - 2^r \int_{r,k} f \int_{r,k} g| < \frac{M_1 M_2}{3 \cdot 2^{3r}}$, $(r,k) \in S$.

Proof. The mean value theorem for integrals and (2.6) imply

$$\langle f, \psi_{n,k}\rangle = 2^{\frac{n}{2}} \left(\int_{n+1,2k} f - \int_{n+1,2k+1} f \right) = \frac{f(c_1)}{2^{\frac{n}{2}+1}} - \frac{f(c_2)}{2^{\frac{n}{2}+1}},$$

for some $c_1 \in (\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}})$ and $c_2 \in (\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}})$. The mean value theorem for derivatives yields

$$\langle f, \psi_{n,k}\rangle = \frac{f'(c)(c_1 - c_2)}{2^{\frac{n}{2}+1}},$$

for some $c \in (c_1, c_2)$. Note that $|c_1 - c_2| < \frac{1}{2^n}$. Then

$$|\langle f, \psi_{n,k}\rangle| \leq \frac{|f'(c)|}{2^{\frac{3n}{2}+1}} < \frac{M_1}{2^{\frac{3n}{2}+1}}.$$

The proof of statement (i) is completed by applying a similar proof for g. To prove statement (ii), consider the following functions on $[0,1]$ with $(r,k) \in S$ arbitrary and fixed:

$$f_{_1}(x) := f(\frac{x+k}{2^r}), \quad \& \quad g_{_1}(x) := g(\frac{x+k}{2^r}).$$

Then

$$\int_{r,k} fg - 2^r \int_{r,k} f \int_{r,k} g = \frac{1}{2^r}\langle f_{_1}, g_{_1}\rangle - \frac{1}{2^r}\langle f_{_1}, \phi\rangle\langle g_{_1}, \phi\rangle$$

$$= \frac{1}{2^r}\sum_{n=0}^{\infty}\sum_{l=0}^{2^n-1}\langle f_{_1}, \psi_{n,l}\rangle\langle g_{_1}, \psi_{n,l}\rangle. \tag{2.8}$$

(The first equation is clear and the second equation is obtained from (2.2).) Since $\frac{M_1}{2^r}$ and $\frac{M_2}{2^r}$ are the bounds for $f_{_1}'$ and $g_{_1}'$ respectively, statement (i) implies

$$|\langle f_{_1}, \psi_{n,k}\rangle| < \frac{M_1}{2^{\frac{3n}{2}+1+r}} \quad \& \quad |\langle g_{_1}, \psi_{n,k}\rangle| < \frac{M_2}{2^{\frac{3n}{2}+1+r}}. \tag{2.9}$$

Using (2.8) and (2.9) yields

$$\left|\int_{r,k} fg - 2^r \int_{r,k} f \int_{r,k} g\right| < \sum_{n=0}^{\infty}\frac{1}{2^{3n+2}}\sum_{l=0}^{2^n-1}\frac{M_1M_2}{2^{3r}} = \frac{M_1M_2}{3\cdot 2^{3r}}.$$

□

3. **An Application: An Infinite Product**

An infinite product can be obtained by applying the above two lemmas, in the special case where $f = \frac{1}{1+x}$ and $g = 1+x$.

Theorem 3.1. *The following equation holds:*

$$\frac{e}{\sqrt{2}}\prod_{m=1}^{\infty} e(\frac{m}{m+1})^{\frac{2m+1}{2}} = \sqrt{\pi}. \tag{3.1}$$

Proof. Consider the following functions on $[0,1]$:

$$f(x) := \frac{1}{1+x}, \quad g(x) := 1+x.$$

The inner product $\langle f, g\rangle$ can be written as follows, for all $r \in Z^*$:

$$\langle f, g\rangle = \sum_{k=0}^{2^r-1}\int_{r,k} fg, \tag{3.2}$$

Lemma 2.1 and (3.2) imply that

$$\sum_{k=0}^{2^r-1}\left(\int_{r,k} fg - 2^r\int_{r,k} f\int_{r,k} g\right) = \sum_{n=r}^{\infty}\sum_{k=0}^{2^n-1}\langle f,\psi_{n,k}\rangle\langle g,\psi_{n,k}\rangle, \tag{3.3}$$

The following equation is clear, by (3.3),

$$\sum_{r=0}^{\infty}2^r\sum_{k=0}^{2^r-1}\left(\int_{r,k} fg - 2^r\int_{r,k} f\int_{r,k} g\right) = \sum_{r=0}^{\infty}2^r\sum_{n=r}^{\infty}\sum_{k=0}^{2^n-1}\langle f,\psi_{n,k}\rangle\langle g,\psi_{n,k}\rangle, \tag{3.4}$$

Note that the left hand side of (3.4) is convergent by part (ii) of Lemma 2.2. Also, part (i) of Lemma 2.2 implies that the right hand side of (3.4) is absolutely convergent. Thus this series can be rearranged using the surjective function Q as follows:

$$\begin{aligned}
\sum_{r=0}^{\infty}2^r\sum_{n=r}^{\infty}\sum_{k=0}^{2^n-1}\langle f,\psi_{n,k}\rangle\langle g,\psi_{n,k}\rangle &= \sum_{r=0}^{\infty}2^r\sum_{n=r}^{\infty}\sum_{k=0}^{2^{Q(r,n)}-1}\langle f,\psi_{Q(r,n),k}\rangle\langle g,\psi_{Q(r,n),k}\rangle\\
&= \sum_{i=0}^{\infty}\sum_{Q(t,u)=i}2^t\sum_{k=0}^{2^i-1}\langle f,\psi_{i,k}\rangle\langle g,\psi_{i,k}\rangle\\
&= \sum_{i=0}^{\infty}\left(\sum_{t=0}^{i}2^t\right)\sum_{k=0}^{2^i-1}\langle f,\psi_{i,k}\rangle\langle g,\psi_{i,k}\rangle\\
&= \sum_{i=0}^{\infty}(2^{i+1}-1)\sum_{k=0}^{2^i-1}\langle f,\psi_{i,k}\rangle\langle g,\psi_{i,k}\rangle.
\end{aligned}$$

Hence, (3.4) can be written as

(3.5)

$$\sum_{r=0}^{\infty}2^r\sum_{k=0}^{2^r-1}\left(\int_{r,k} fg - 2^r\int_{r,k} f\int_{r,k} g\right) = \sum_{r=0}^{\infty}(2^{r+1}-1)\sum_{k=0}^{2^r-1}\langle f,\psi_{r,k}\rangle\langle g,\psi_{r,k}\rangle.$$

Together, (2.2) and (3.5) yield:

$$\langle f,g\rangle - \langle f,\phi\rangle\langle g,\phi\rangle + A = B, \tag{3.6}$$

where

$$A = \sum_{r=0}^{\infty}2^r\sum_{k=0}^{2^r-1}\left(\int_{r,k} fg - 2^r\int_{r,k} f\int_{r,k} g\right),$$

and

$$B = \sum_{r=0}^{\infty}2^{r+1}\sum_{k=0}^{2^r-1}\langle f,\psi_{r,k}\rangle\langle g,\psi_{r,k}\rangle.$$

Parts (ii) and (i) of Lemma 2.2 show that the series A and B are absolutely convergent. Therefore, we can rearrange these series using the bijective function P to get:

$$
\begin{aligned}
A &= \sum_{r=0}^{\infty}\sum_{k=0}^{2^r-1}\left(1-\frac{2(2^r+k)+1}{2}\ln\left(\frac{2^r+k+1}{2^r+k}\right)\right)\\
&= \sum_{r=0}^{\infty}\sum_{k=0}^{2^r-1}\left(1-\frac{2P(r,k)+1}{2}\ln\left(\frac{P(r,k)+1}{P(r,k)}\right)\right)\\
&= \sum_{m=1}^{\infty}\left(1-\frac{2m+1}{2}\ln\left(\frac{m+1}{m}\right)\right), \qquad (3.7)
\end{aligned}
$$

and

$$
\begin{aligned}
B &= \frac{1}{2}\sum_{r=0}^{\infty}\sum_{k=0}^{2^r-1}\ln\left(\frac{(2^{r+1}+2k)(2^{r+1}+2k+2)}{(2^{r+1}+2k+1)^2}\right)\\
&= \frac{1}{2}\sum_{r=0}^{\infty}\sum_{k=0}^{2^r-1}\ln\left(\frac{2P(r,k)(2P(r,k)+2)}{(2P(r,k)+1)^2}\right)\\
&= \frac{1}{2}\sum_{m=1}^{\infty}\ln\left(\frac{2m(2m+2)}{(2m+1)^2}\right). \qquad (3.8)
\end{aligned}
$$

Substitution of (3.7) and (3.8) into (3.6) implies that

$$
1-\frac{3}{2}\ln 2+\sum_{m=1}^{\infty}\left(1-\frac{2m+1}{2}\ln\left(\frac{m+1}{m}\right)\right)=\frac{1}{2}\sum_{m=1}^{\infty}\ln\left(\frac{2m(2m+2)}{(2m+1)^2}\right). \qquad (3.9)
$$

Using the function $\exp(x)$ in (3.9) yields

$$
\frac{e}{2\sqrt{2}}\prod_{m=1}^{\infty}e\left(\frac{m}{m+1}\right)^{\frac{2m+1}{2}}=\left(\prod_{m=1}^{\infty}\frac{2m(2m+2)}{(2m+1)^2}\right)^{\frac{1}{2}}. \qquad (3.10)
$$

The square of the right hand side of (3.10) is $\frac{\pi}{4}$ (which is the Wallis' Product). This completes the proof. $\square$

4. Stirling's Formula

Now, we can prove Stirling's formula. Equation (3.1) implies

$$
\lim_{M\to\infty}\frac{e^{M+1}}{\sqrt{2\pi}}\prod_{m=1}^{M}\left(\frac{m}{m+1}\right)^{\frac{2m+1}{2}}=1. \qquad (4.1)
$$

The following equation is clear:

$$\prod_{m=1}^{M}(\frac{m}{m+1})^{2m+1}=\frac{(M!)^2}{(M+1)^{2M+1}}. \tag{4.2}$$

Substitution of (4.2) into (4.1) yields

$$\lim_{M\to\infty}\frac{(M+1)!e^{M+1}}{(M+1)^{M+1}\sqrt{2\pi(M+1)}}=1,$$

which is Stirling's formula. Finally, we obtain a lower and upper bound of $\frac{n!}{n^n e^{-n}\sqrt{2\pi n}}$. Equations (3.1) and (4.2) yield

$$\prod_{m=n}^{\infty}e^{-1}(\frac{m+1}{m})^{\frac{2m+1}{2}}=\Theta(n), \quad n=1,2,3,\ldots$$

where

$$\Theta(n):=\frac{n!}{n^n e^{-n}\sqrt{2\pi n}}.$$

The following inequalities follow from elementary calculus:

$$\int_n^{\infty}-1+\frac{2x+1}{2}\ln(\frac{x+1}{x})dx\le\sum_{m=n}^{\infty}\left(-1+\frac{2m+1}{2}\ln(\frac{m+1}{m})\right),$$

and

$$\sum_{m=n}^{\infty}\left(-1+\frac{2m+1}{2}\ln(\frac{m+1}{m})\right)\le\int_{n-1}^{\infty}-1+\frac{2x+1}{2}\ln(\frac{x+1}{x})dx.$$

Since $\ln(\Theta(n))=\sum_{m=n}^{\infty}\left(-1+\frac{2m+1}{2}\ln(\frac{m+1}{m})\right)$, the last two inequalities imply that

$$\frac{-n-n^2}{2}\ln(\frac{n+1}{n})+\frac{2n+1}{4}\le\ln(\Theta(n))\le\frac{n-n^2}{2}\ln(\frac{n}{n-1})+\frac{2n-1}{4}.$$

Hence

$$e^{\frac{2n+1}{4}}\left(\frac{n+1}{n}\right)^{\frac{-n-n^2}{2}}\le\Theta(n)\le e^{\frac{2n-1}{4}}\left(\frac{n}{n-1}\right)^{\frac{n-n^2}{2}},$$

for $n=2,3,\ldots$.

References

[1] I. Daubechies, Ten lectures on wavelets, Regional Conference Series in Applied Mathematics, 61, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1992.

[2] P. Diaconis and D. Freedman, An elementary proof of Stirling's formula, *Amer. Math. Monthly* **93** (1986), no. 2. 123–125.

[3] A. J. Maria, A remark on Stirling's formula, *Amer. Math. Monthly* **72** (1965) 1096–1098.

[4] A. de Moivre, Miscellanea Analytica de Seriebus et Quadraturis, London, 1730.

[5] T. S. Nanjundiah, Note on Stirling's formula, *Amer. Math. Monthly* **66** (1965) 701–703.

[6] H. Robbins, A remark on Stirling's formula, *Amer. Math. Monthly* **62** (1955) 26–29.

[7] J. Stoer and R. Bulirsch, Introduction to numerical analysis, Translated from the German by R. Bartels, W. Gautschi and C. Witzgall, Third Edition, Texts in Applied Mathematics, 12, Springer-Verlag, New York, 2002.

[8] I. Tweddle, James Stirling's Methodus differentialis. An annotated translation of Stirling's text. Sources and Studies in the History of Mathematics and Physical Sciences. Springer-Verlag London, Ltd., London, 2003.

(Mahdi Ahmadinia) Department of Mathematics, University of Qom, P.O. Box 37185-3766, Qom, Iran.

E-mail address: mahdiahmadinia72@gmail.com & m-ahmadinia@qom.ac.ir

(Hamid Naderi Yeganeh) Department of Mathematics, University of Qom, P.O. Box 37185-3766, Qom, Iran.

E-mail address: hamid@hamidnaderiyeganeh.id.ir