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# ADDITIVITY OF MAPS PRESERVING JORDAN $\eta_*$-PRODUCTS ON $C^*$-ALGEBRAS

A. TAGHAVI*, H. ROHI AND V. DARVISH

(Communicated by Peter Rosenthal)

*Dedicated to Professor Heydar Radjavi on his 80th birthday*

Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be two $C^*$-algebras such that $\mathcal{B}$ is prime. In this paper, we investigate the additivity of maps $\Phi$ from $\mathcal{A}$ onto $\mathcal{B}$ that are bijective, unital and that satisfy $\Phi(AP + \eta PA^*) = \Phi(A)\Phi(P) + \eta\Phi(P)\Phi(A)^*$, for all $A \in \mathcal{A}$ and $P \in \{P_1, I_{\mathcal{A}} - P_1\}$ where $P_1$ is a nontrivial projection in $\mathcal{A}$. If $\eta$ is a non-zero complex number such that $|\eta| \neq 1$, then $\Phi$ is additive. Moreover, if $\eta$ is rational, then $\Phi$ is $*$-additive.
**Keywords:** Maps preserving Jordan $\eta_*$-product, Additive, Prime $C^*$-algebras.
**MSC(2010):** Primary: 47B48; Secondary: 46L10.

## 1. Introduction

Let $\mathcal{R}$ and $\mathcal{R}'$ be rings. We say that the map $\Phi : \mathcal{R} \to \mathcal{R}'$ "preserves products" or is "multiplicative" if $\Phi(AB) = \Phi(A)\Phi(B)$ for all $A, B \in \mathcal{R}$. The question of when a multiplicative map is additive was discussed by several authors: see [20] and references therein. Motivated by this, many authors paid more attention to maps on rings (and algebras) that preserve the Lie product $[A, B] = AB - BA$ or the Jordan product $A \circ B = AB + BA$. (See, for example, [1, 2, 6, 10, 13, 16–18, 24, 25].) These results show that, in some sense, the Jordan product or Lie product structure is enough to determine the ring or algebraic structure. Historically, many mathematicians devoted themselves to the study of additive or linear Jordan or Lie product preservers between rings or operator algebras. Such maps are called Jordan homomorphisms or Lie homomorphisms respectively. See, for example, [3–5, 11, 12, 14, 20–22].

Let $\mathcal{R}$ be a $*$-ring and $\eta$ be a non-zero complex scalar. For $A, B \in \mathcal{R}$, denote the Jordan $\eta_*$-product of $A$ and $B$ by $A \bullet_\eta B = AB + \eta BA^*$ and the

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Lie $\eta_*$-product by $[A,B]_*^\eta = AB - \eta BA^*$. In particular, $A \bullet_1 B = AB + BA^*$ and $[A,B]_*^1 = AB - BA^*$ are the Jordan $1_*$-product and the Lie $1_*$-product, respectively. These products are playing an increasingly important role, and have been studied by many authors (for example, see [7,23,26]). A fundamental problem is to determine the conditions under which maps preserving Jordan $\eta_*$-products or Lie $\eta_*$-products on rings or algebras are isomorphisms. In [8], J. Cui and C. K. Li proved that a bijective map on a factor von Neumann algebra which preserves the Lie $1_*$-product ($[A,B]_*^1$) must be a $*$-isomorphism. Moreover, in [15] C. Li et al, discussed nonlinear bijective maps preserving Jordan $1_*$-product ($A \bullet_1 B$). They proved that such a mapping on factor von Neumann algebras is also a $*$-ring isomorphism. These two articles also discussed new products for arbitrary operators on factor von Neumann algebras. In addition, A. Taghavi et al [27], proved that a bijective unital map (not necessarily linear) on a prime $C^*$-algebra which preserves both the Lie $1_*$-product and the Jordan $1_*$-product in the case where one of the operators is a projection must be $*$-additive (i.e., additive and star-preserving). In a recent paper [9], L. Dai and F. Lu proved that a bijective map on a von Neumann algebra which preserves the Jordan $\eta_*$-product is a linear $*$-isomorphism if $\eta$ is not real, it is and is a sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism if $\eta$ is real.

In this paper, we will consider when a bijective unital map on a prime $C^*$-algebra which preserves the Jordan $\eta_*$-product $\Phi(A \bullet_\eta P) = \Phi(A) \bullet_\eta \Phi(P)$ is additive. We denote the real part of an operator $T$ by $\Re(T)$; i.e., $\Re(T) = \frac{T+T^*}{2}$. A $C^*$-algebra $\mathcal{A}$ is prime if $A\mathcal{A}B = 0$ for $A, B \in \mathcal{A}$ implies that either $A = 0$ or $B = 0$.

## 2. Main results

We need the following lemmas.

**Lemma 2.1.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two $C^*$-algebras and $\Phi : \mathcal{A} \to \mathcal{B}$ be a map which satisfies $\Phi(A \bullet_\eta P) = \Phi(A) \bullet_\eta \Phi(P)$ for all $A \in \mathcal{A}$ and some $P \in \mathcal{A}$ where $\eta$ is a non-zero complex number such that $|\eta| \neq 1$. Let $A, B$ and $T$ be in $\mathcal{A}$ such that $\Phi(T) = \Phi(A) + \Phi(B)$. Then we have*

$$\Phi(T \bullet_\eta P) = \Phi(A \bullet_\eta P) + \Phi(B \bullet_\eta P). \tag{2.1}$$

*Proof.* Multiply the equations $\Phi(T) = \Phi(A) + \Phi(B)$ and $\Phi(T)^* = \Phi(A)^* + \Phi(B)^*$ by $\Phi(P)$ from the right and $\eta\Phi(P)$ from the left. We get

$$\Phi(T)\Phi(P) = \Phi(A)\Phi(P) + \Phi(B)\Phi(P),$$

and

$$\eta\Phi(P)\Phi(T)^* = \eta\Phi(P)\Phi(A)^* + \eta\Phi(P)\Phi(B)^*.$$

By adding the two equations, we get

$$\Phi(T)\Phi(P)+\eta\Phi(P)\Phi(T)^* = \Phi(A)\Phi(P)+\eta\Phi(P)\Phi(A)^*+\Phi(B)\Phi(P)+\eta\Phi(P)\Phi(B)^*.$$

□

**Lemma 2.2.** *Let $\mathcal{A}$ be a $C^*$-algebra. Suppose $T \in \mathcal{A}$ and $\eta$ is a non-zero complex number such that $|\eta| \neq 1$. If $T + \eta T^* = 0,?$ then $T = 0$.*

*Proof.* Let $T + \eta T^* = 0$. Then $T^* + \bar{\eta} T = 0$. By an easy computation we have $(1 - \bar{\eta}\eta)T = 0$. Thus $T = 0$. □

**Lemma 2.3.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two $C^*$-algebras with identities and $\Phi : \mathcal{A} \to \mathcal{B}$ be a bijective map which satisfies $\Phi(A \bullet_\eta P) = \Phi(A) \bullet_\eta \Phi(P)$ for all $A \in \mathcal{A}$ and $P \in \{P_1, I - P_1\}$, where $P_1$ is a nontrivial projection in $\mathcal{A}$ and $\eta$ is a non-zero complex number such that $|\eta| \neq 1$. Then $\Phi(0) = 0$.*

*Proof.* Let $\Phi(T) = 0$. We prove that $T = 0$. To show this, apply Lemma 2.1 to $\Phi(T) = 0$ for $P_1$ and $P_2 = I - P_1$. This gives

$$\Phi(T \bullet_\eta P_1) = 0,$$

and

$$\Phi(T \bullet_\eta P_2) = 0.$$

So, by injectivity of $\Phi$, we obtain $T \bullet_\eta P_1 = T \bullet_\eta P_2$. By the definition of Jordan $\eta_*$-product, we have $TP_1 + \eta P_1 T^* = TP_2 + \eta P_2 T^*$; i.e., $(TP_1 - TP_2) + \eta(TP_1 - TP_2)^* = 0$. Now, applying Lemma 2.2, we obtain $TP_1 = TP_2$. We multiply the latter equation by $P_2$ on the right; it follows that $TP_2 = 0$. Thus $TP_1 = 0$ also. Now we put $I - P_2$ instead of $P_1$ in $TP_1 = 0$, obtaining $T(I - P_2) = 0$. Therefore $T = 0$ since $TP_2 = 0$. □

Our main theorem is as follows:

**Main Theorem.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two $C^*$-algebras such that $\mathcal{B}$ is prime, with $I_\mathcal{A}$ and $I_\mathcal{B}$ their respective identities. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a unital bijective map which satisfies $\Phi(A \bullet_\eta P) = \Phi(A) \bullet_\eta \Phi(P)$ for all $A \in \mathcal{A}$ and $P \in \{P_1, I_\mathcal{A} - P_1\}$, where $P_1$ is a nontrivial projection in $\mathcal{A}$ and $\eta$ is a non-zero complex number such that $|\eta| \neq 1$. Then $\Phi$ is additive. Moreover, if $\eta$ is rational, then $\Phi$ is $*$-additive.*

*Proof of Main Theorem.* Let $P_2 = I_\mathcal{A} - P_1$ and let $\mathcal{A}_{ij} = P_i \mathcal{A} P_j,\ i, j = 1, 2$. Then $\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}$. For every $A \in \mathcal{A}$ we may write $A = A_{11} + A_{12} + A_{21} + A_{22}$. In all that follows, when we write $A_{ij}$, it indicates that $A_{ij} \in \mathcal{A}_{ij}$. To show additivity of $\Phi$ on $\mathcal{A}$ we will use the above partition of $\mathcal{A}$ and establish that $\Phi$ is additive on each $\mathcal{A}_{ij},\ i, j = 1, 2$. □

**Claim 2.4.** *For every $A_{11} \in \mathcal{A}_{11}$ and $D_{22} \in \mathcal{A}_{22}$, we have*

$$\Phi(A_{11} + D_{22}) = \Phi(A_{11}) + \Phi(D_{22}).$$

Since $\Phi$ is surjective, we can find an element $T = T_{11} + T_{12} + T_{21} + T_{22} \in \mathcal{A}$ such that

$$\Phi(T) = \Phi(A_{11}) + \Phi(D_{22}). \tag{2.2}$$

We now show $T = A_{11} + D_{22}$. We apply Lemma 2.1 to (2.2) for $P_1$. We write

$$\Phi(T \bullet_\eta P_1) = \Phi(A_{11} \bullet_\eta P_1) + \Phi(D_{22} \bullet_\eta P_1),$$

so

$$\Phi(T_{11} + T_{21} + \eta T_{11}^* + \eta T_{21}^*) = \Phi(A_{11} + \eta A_{11}^*).$$

By the injectivity of $\Phi$, we get $T_{11}+T_{21}+\eta T_{11}^*+\eta T_{21}^* = A_{11}+\eta A_{11}^*$. Multiply the latter equation by $P_2$ on the right and left side, respectively, to obtain $T_{21} = T_{21}^* = 0$. Thus we have $T_{11}+\eta T_{11}^* = A_{11}+\eta A_{11}^*$, or $(T_{11}-A_{11})+\eta(T_{11}-A_{11})^* = 0$. By Lemma 2.2, we get $T_{11} = A_{11}$. Similarly, applying Lemma 2.1 to (2.2) for $P_2$ gives

$$\Phi(T \bullet_\eta P_2) = \Phi(A_{11} \bullet_\eta P_2) + \Phi(D_{22} \bullet_\eta P_2),$$

and hence

$$\Phi(T_{12} + T_{22} + \eta T_{12}^* + \eta T_{22}^*) = \Phi(D_{22} + \eta D_{22}^*).$$

By the injectivity of $\Phi$, we get $T_{12} + T_{22} + \eta T_{12}^* + \eta T_{22}^* = D_{22} + \eta D_{22}^*$. Multiply the latter equation by $P_1$ on the right and left side respectively, to obtain $T_{12} = T_{12}^* = 0$. Thus we have $T_{22} + \eta T_{22}^* = D_{22} + \eta D_{22}^*$, or $(T_{22} - D_{22}) + \eta(T_{22} - D_{22})^* = 0$. By Lemma 2.2, we get $T_{22} = D_{22}$.

**Claim 2.5.** *For every $B_{12} \in \mathcal{A}_{12}$, $C_{21} \in \mathcal{A}_{21}$, we have*

$$\Phi(B_{12} + C_{21}) = \Phi(B_{12}) + \Phi(C_{21}).$$

Let $T = T_{11} + T_{12} + T_{21} + T_{22} \in \mathcal{A}$ be such that

$$\Phi(T) = \Phi(B_{12}) + \Phi(C_{21}). \tag{2.3}$$

Applying Lemma 2.1 to (2.3) for $P_1$, we have obtain

$$\begin{aligned}\Phi(T \bullet_\eta P_1) &= \Phi(B_{12} \bullet_\eta P_1) + \Phi(C_{21} \bullet_\eta P_1)\\ &= \Phi(C_{21} \bullet_\eta P_1).\end{aligned}$$

Thus, by the injectivity of $\Phi$, we have $T \bullet_\eta P_1 = C_{21} \bullet_\eta P_1$. It follows that

$$T_{11} + T_{21} + \eta T_{11}^* + \eta T_{21}^* = C_{21} + \eta C_{21}^*.$$

Multiplying the above equation by $P_2$ on the left, gives $T_{21} = C_{21}$ and $T_{11} = 0$. Similarly, we can obtain $T_{12} = B_{12}$ and $T_{22} = 0$ by applying Lemma 2.1 to (2.3) for $P_2$.

**Claim 2.6.** *For every $A_{11} \in \mathcal{A}_{11}$ and $B_{12} \in \mathcal{A}_{12}$, we have*

$$\Phi(A_{11} + B_{12}) = \Phi(A_{11}) + \Phi(B_{12}).$$

Since $\Phi$ is surjective, we can find an element $T = T_{11} + T_{12} + T_{21} + T_{22} \in \mathcal{A}$ such that

$$\Phi(T) = \Phi(A_{11}) + \Phi(B_{12}). \tag{2.4}$$

We show that $T = A_{11} + B_{12}$. We apply Lemma 2.1 to (2.4) for $P_1$, we can write $T_{11} = A_{11}$ and $T_{21} = 0$.

Similarly, we apply Lemma 2.1 to (2.4) for $P_2$, we will have $T_{12} = B_{12}$ and $T_{22} = 0$. So, $T = A_{11} + B_{12}$.

Note that the equation $\Phi(C_{21} + D_{22}) = \Phi(C_{21}) + \Phi(D_{22})$, where $C_{21} \in \mathcal{A}_{21}$ and $D_{22} \in \mathcal{A}_{22}$, can be obtained as above.

**Claim 2.7.** *For every $A_{11} \in \mathcal{A}_{11}$ and $C_{21} \in \mathcal{A}_{21}$, we have*

$$\Phi(A_{11} + C_{21}) = \Phi(A_{11}) + \Phi(C_{21}).$$

Since $\Phi$ is surjective, we can find an element $T = T_{11} + T_{12} + T_{21} + T_{22} \in \mathcal{A}$ such that

$$\Phi(T) = \Phi(A_{11}) + \Phi(C_{21}). \tag{2.5}$$

We show that $T = A_{11} + C_{12}$. We apply Lemma 2.1 to (2.5) for $P_2$; then we have $\Phi(T \bullet_\eta P_2) = 0$. This means that $\Phi(T_{12} + T_{22} + \eta T_{12}^* + \eta T_{22}^*) = 0$. By Lemma 2.3 and Lemma 2.2, we obtain $T_{12} = T_{22} = 0$. On the other hand, we apply Lemma 2.1 to (2.5) for $P_1$. Then, by Claim 2.6, we have

$$\begin{aligned}
\Phi(T \bullet_\eta P_1) &= \Phi(A_{11} \bullet_\eta P_1) + \Phi(C_{21} \bullet_\eta P_1) \\
&= \Phi(A_{11} + \eta A_{11}^*) + \Phi(C_{21} + \eta C_{21}^*) \\
&= \Phi(A_{11} + \eta A_{11}^*) + \Phi(2\Re(C_{21}^*) + (\eta - 1)C_{21}^*) \\
&= \Phi(A_{11} + \eta A_{11}^* + 2\Re(C_{21}^*) + (\eta - 1)C_{21}^*)
\end{aligned}$$

We get the following:

$$\Phi(T_{11} + T_{21} + \eta T_{11}^* + T_{21}^*) = \Phi(A_{11} + \eta A_{11}^* + C_{21} + \eta C_{21}^*).$$

Now we use Lemma 2.2 to obtain $T_{11} = A_{11}$ and $T_{21} = C_{21}$.

Note that the fact that $\Phi(B_{12} + D_{22}) = \Phi(B_{12}) + \Phi(D_{22})$, where $B_{12} \in \mathcal{A}_{12}$ and $D_{22} \in \mathcal{A}_{22}$, can be obtained as above.

**Claim 2.8.** *For every $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ such that $1 \leq i \neq j \leq 2$, we have*

$$\Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij}).$$

Let $T = T_{11} + T_{12} + T_{21} + T_{22} \in \mathcal{A}$ be such that

$$\Phi(T) = \Phi(A_{ij}) + \Phi(B_{ij}). \tag{2.6}$$

Applying Lemma 2.1 to (2.6) for $P_i$, we get

$$\Phi(TP_i + \eta P_i T^*) = \Phi(A_{ij}P_i + \eta P_i A_{ij}^*) + \Phi(B_{ij}P_i + \eta P_i B_{ij}^*) = \Phi(0) = 0.$$

Therefore, $\Phi(T_{ii} + T_{ji} + \eta T_{ii}^* + \eta T_{ji}^*) = 0$. So, by Lemma 2.2, we have $T_{ii} = T_{ji} = 0$. On the other hand, we apply Lemma 2.1 to (2.6) for $P_j$ again. By

Claim 2.5, it we see that

$$\begin{aligned}\Phi(TP_j + \eta P_j T^*) &= \Phi(A_{ij}P_j + \eta P_j A_{ij}^*) + \Phi(B_{ij}P_j + \eta P_j B_{ij}^*)\\ &= \Phi(A_{ij} + \eta A_{ij}^*) + \Phi(B_{ij} + \eta B_{ij}^*)\\ &= \Phi(2\Re(A_{ij}^*) + (\eta-1)A_{ij}^*) + \Phi(2\eta\Re(B_{ij}) + (1-\eta)B_{ij})\\ &= \Phi(2\Re(A_{ij}^*) + (\eta-1)A_{ij}^* + 2\eta\Re(B_{ij}) + (1-\eta)B_{ij})\\ &= \Phi(A_{ij} + \eta A_{ij}^* + B_{ij} + \eta B_{ij}^*).\end{aligned}$$

So,

$$T_{ij} + T_{jj} + \eta T_{ij}^* + \eta T_{jj}^* = A_{ij} + \eta A_{ij}^* + B_{ij} + \eta B_{ij}^*.$$

By Lemma 2.2, we have $T_{jj} = 0$ and $T_{ij} = A_{ij} + B_{ij}$.

**Claim 2.9.** *For every $A_{11} \in \mathcal{A}_{11}$, $B_{12} \in \mathcal{A}_{12}$, $C_{21} \in \mathcal{A}_{21}$, we have*

$$\Phi(A_{11} + B_{12} + C_{21}) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}).$$

Let $T = T_{11} + T_{12} + T_{21} + T_{22} \in \mathcal{A}$ be such that

$$\Phi(T) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}). \tag{2.7}$$

Applying Lemma 2.1 to (2.7) for $P_2$, we have

$$\begin{aligned}\Phi(T \bullet_\eta P_2) &= \Phi(A_{11} \bullet_\eta P_2) + \Phi(B_{12} \bullet_\eta P_2) + \Phi(C_{21} \bullet_\eta P_2)\\ &= \Phi(B_{12} \bullet_\eta P_2).\end{aligned}$$

Thus, by injectivity of $\Phi$, we have $T \bullet_\eta P_2 = B_{12} \bullet_\eta P_2$. It follows that

$$T_{22} + T_{12} + \eta T_{22}^* + \eta T_{12}^* = B_{12} + \eta B_{12}^*.$$

Multiply the above equation by $P_1$ on the left to get $T_{12} = B_{12}$ and $T_{22} = 0$. Similarly, we apply Lemma 2.1 to (2.7) for $P_1$. By Claim 2.6, we have the following

$$\begin{aligned}\Phi(T \bullet_\eta P_1) &= \Phi(A_{11} \bullet_\eta P_1) + \Phi(B_{12} \bullet_\eta P_1) + \Phi(C_{21} \bullet_\eta P_1)\\ &= \Phi(A_{11} \bullet_\eta P_1) + \Phi(C_{21} \bullet_\eta P_1)\\ &= \Phi(A_{11} + \eta A_{11}^*) + \Phi(C_{21} + \eta C_{21}^*)\\ &= \Phi(A_{11} + \eta A_{11}^*) + \Phi(2\Re(C_{21}^*) + (\eta-1)C_{21}^*)\\ &= \Phi(A_{11} + \eta A_{11}^* + 2\Re(C_{21}^*) + (\eta-1)C_{21}^*)\\ &= \Phi(A_{11} + \eta A_{11}^* + C_{21} + \eta C_{21}^*)\end{aligned}$$

Injectivity of $\Phi$ implies $T_{11} + T_{21} + \eta T_{11}^* + \eta T_{21}^* = A_{11} + C_{21} + \eta A_{11}^* + \eta C_{21}^*$. We have $T_{11} = A_{11}$ and $T_{21} = C_{21}$.

**Claim 2.10.** *For every $A_{11} \in \mathcal{A}_{11}$, $B_{12} \in \mathcal{A}_{12}$, $C_{21} \in \mathcal{A}_{21}$ and $D_{22} \in \mathcal{A}_{22}$, we have*

$$\Phi(A_{11} + B_{12} + C_{21} + D_{22}) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22}).$$

Assume $T = T_{11} + T_{12} + T_{21} + T_{22}$. Then

$$\Phi(T) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22}). \tag{2.8}$$

Applying Lemma 2.1 to (2.8) for $P_1$, Claim 2.5 and Claim 2.9, we obtain

$$\begin{aligned}
\Phi(T \bullet_\eta P_1) &= \Phi(A_{11} \bullet_\eta P_1) + \Phi(B_{12} \bullet_\eta P_1) + \Phi(C_{21} \bullet_\eta P_1) + \Phi(D_{22} \bullet_\eta P_1) \\
&= \Phi(A_{11} \bullet_\eta P_1) + \Phi(C_{21} \bullet_\eta P_1) \\
&= \Phi(A_{11} + \eta A_{11}^*) + \Phi(C_{21} + \eta C_{21}^*) \\
&= \Phi(A_{11} + \eta A_{11}^*) + \Phi(C_{21}) + \Phi(\eta C_{21}^*) \\
&= \Phi(A_{11} + \eta A_{11}^* + C_{21} + \eta C_{21}^*)
\end{aligned}$$

Since $\Phi$ is injective we have $T_{11} + T_{21} + \eta T_{11}^* + \eta T_{21}^* = A_{11} + C_{21} + \eta A_{11}^* + \eta C_{21}^*$. We obtain $T_{11} = A_{11}$ and $T_{21} = C_{21}$. Similarly, apply Lemma 2.1 to (2.8) for $P_2$ and the same computation as above to easily obtain $T_{12} = B_{12}$ and $T_{22} = D_{22}$. So, $\Phi(A_{11} + B_{12} + C_{21} + D_{22}) = \Phi(A_{11}) + \Phi(B_{12}) + \Phi(C_{21}) + \Phi(D_{22})$.

**Lemma 2.11.** *Let $\Phi$ satisfy the assumptions of the Main Theorem. Then, for every $A \in \mathcal{A}$ we have the following*

$$\Phi(A \bullet_\eta I) = \Phi(A) \bullet_\eta \Phi(I).$$

*Proof.* The definition of the Jordan $\eta_*$-product implies that

$$\Phi(A \bullet_\eta P_1) = \Phi(A) \bullet_\eta \Phi(P_1), \tag{2.9}$$

and

$$\Phi(A \bullet_\eta P_2) = \Phi(A) \bullet_\eta \Phi(P_2). \tag{2.10}$$

Add Equations (2.9) and (2.10) together, to get

$$\Phi(A \bullet_\eta P_1) + \Phi(A \bullet_\eta P_2) = \Phi(A) \bullet_\eta (\Phi(P_1) + \Phi(P_2)).$$

By Claims 2.4, 2.8 and 2.10, we obtain

$$\Phi(A \bullet_\eta P_1 + A \bullet_\eta P_2) = \Phi(A) \bullet_\eta \Phi(P_1 + P_2),$$

Equivalently,

$$\Phi(A \bullet_\eta (P_1 + P_2)) = \Phi(A) \bullet_\eta \Phi(P_1 + P_2),$$

so, $\Phi(A \bullet_\eta I) = \Phi(A) \bullet_\eta \Phi(I)$. □

**Lemma 2.12.** *Let $\Phi$ satisfy the assumptions of the Main Theorem. Then $\Phi(P_1)$ and $\Phi(P_2)$ are nontrivial orthogonal projections in $\mathcal{B}$.*

*Proof.* Let $P \in \{P_1, P_2\}$, where $P_i$ for $1 \le i \le 2$ are nontrivial projections in $\mathcal{A}$. By Lemma 2.11 and the definition of the Jordan $\eta_*$-product we have

$$\begin{aligned}
\Phi(P \bullet_\eta I) &= \Phi(P) \bullet_\eta \Phi(I) \\
\Phi(I \bullet_\eta P) &= \Phi(I) \bullet_\eta \Phi(P).
\end{aligned}$$

Since $\Phi$ is unital, the above equations give us

$$\begin{aligned}\Phi(P+\eta P) &= \Phi(P)+\eta\Phi(P)^* \\ \Phi(P+\eta P) &= \Phi(P)+\eta\Phi(P),\end{aligned} \tag{2.11}$$

so we obtain $\Phi(P)=\Phi(P)^*$. On the other hand, $\Phi(P\bullet_\eta P)=\Phi(P)\bullet_\eta\Phi(P)$. Then $\Phi(P+\eta P)=\Phi(P)^2+\eta\Phi(P)^2$. It follows from (2.11) that $\Phi(P)+\eta\Phi(P)=\Phi(P)^2+\eta\Phi(P)^2$. Now, Lemma 2.2 implies $\Phi(P)^2=\Phi(P)$. We show that $\Phi(P_1)$ and $\Phi(P_2)$ are orthogonal. Let $\Phi(P_1)=Q_1$ and $\Phi(P_2)=Q_2$. Then, by Claim 2.4, $Q_1+Q_2=I$. Also, $Q_1.Q_2=\Phi(P_1).\Phi(P_2)=\Phi(P_1).(\Phi(I)-\Phi(P_1))=0$. □

**Lemma 2.13.** *Let $\Phi$ satisfy the assumptions of the Main Theorem. Then*

$$\Phi(AP_i)=\Phi(A)\Phi(P_i), \tag{2.12}$$

*for $1\le i\le 2$.*

*Proof.* It is easy to check that $\Phi(AP_i\bullet_\eta I)=\Phi(A\bullet_\eta P_i)$. The above equation can be written $\Phi(AP_i)+\eta\Phi(AP_i)^*=\Phi(A)\Phi(P_i)+\eta\Phi(P_i)\Phi(A)^*$. Equivalently,

$$(\Phi(AP_i)-\Phi(A)\Phi(P_i))+\eta(\Phi(AP_i)-\Phi(A)\Phi(P_i))^*=0,$$

Applying Lemma 2.2, we get $\Phi(AP_i)=\Phi(A)\Phi(P_i)$. □

Now, lemma 2.12 ensures that there exist nontrivial projections $Q_i$ $(i=1,2)$ such that $\Phi(P_i)=Q_i$ and $Q_1+Q_2=I$. We can write $\mathcal{B}=\sum_{i,j=1}^2\mathcal{B}_{ij}$ where $\mathcal{B}_{ij}=Q_i\mathcal{B}Q_j,\ i,j=1,2$.

The primeness property of $\mathcal{B}$ is only used in the following claim.

**Claim 2.14.** *For every $A_{ii},B_{ii}\in\mathcal{A}_{ii}$, $1\le i\le 2$, we have*

$$\Phi(A_{ii}+B_{ii})=\Phi(A_{ii})+\Phi(B_{ii}).$$

First, we will prove that $\Phi(P_iA+P_iB)=\Phi(P_iA)+\Phi(P_iB)$ for every $A,B\in\mathcal{A}$.
By Lemma 2.13, Claim 2.8 and for every $T_{ji}\in\mathcal{B}_{ji}$ such that $i\neq j$ we obtain

$$\begin{aligned}(\Phi(P_iA+P_iB)-\Phi(P_iA)-\Phi(P_iB))T_{ji} &= (\Phi(P_iA+P_iB)-\Phi(P_iA)\\ &\quad -\Phi(P_iB))Q_jTQ_i\\ &= (\Phi(P_iA+P_iB)Q_j-\Phi(P_iA)Q_j\\ &\quad -\Phi(P_iB)Q_j)TQ_i\\ &= (\Phi(P_iAP_j+P_iBP_j)-\Phi(P_iAP_j)\\ &\quad -\Phi(P_iBP_j))TQ_i\\ &= (\Phi(A_{ij}+B_{ij})-\Phi(A_{ij})\\ &\quad -\Phi(B_{ij}))TQ_i\\ &= (\Phi(A_{ij})+\Phi(B_{ij})-\Phi(A_{ij})\\ &\quad -\Phi(B_{ij}))TQ_i\\ &= 0.\end{aligned}$$

By the primeness of $\mathcal{B}$, we have

$$\Phi(P_iA + P_iB) = \Phi(P_iA) + \Phi(P_iA). \tag{2.13}$$

Now, multiply the right side of equation (2.13) by $\Phi(P_i) = Q_i$ and use Lemma 2.13 to obtain

$$\Phi(P_iAP_i + P_iBP_i) = \Phi(P_iAP_i) + \Phi(P_iBP_i).$$

So, additivity of $\Phi$ follows from Claims 2.8, 2.10 and 2.14. It remains to prove that $\Phi$ is $*$-preserving for non-zero rational numbers $\eta$ such that $|\eta| \neq 1$. Since $\Phi$ is Jordan $\eta_*$-product preserving, we have

$$\Phi(A \bullet_\eta I) = \Phi(A) \bullet_\eta \Phi(I).$$

Since $\Phi$ is unital,

$$\Phi(A + \eta A^*) = \Phi(A) + \eta\Phi(A)^*.$$

Additivity of $\Phi$ and the above equation imply

$$\Phi(\eta A^*) = \eta\Phi(A)^*. \tag{2.14}$$

Let $\eta = \frac{a}{b}$, where $a, b$ are integers. It is easy to see that

$$\Phi(\frac{1}{b}A^*) = \frac{1}{b}\Phi(A^*). \tag{2.15}$$

Now, by additivity of $\Phi$ and (2.15), we have $\Phi(\frac{a}{b}A^*) = \frac{a}{b}\Phi(A^*)$. It follows that $\Phi(\eta A^*) = \eta\Phi(A^*)$. Hence, by the latter equation and equation (2.14), we get $\Phi(A^*) = \Phi(A)^*$.

$\square$

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