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# A note on lifting projections

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# A NOTE ON LIFTING PROJECTIONS

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(Communicated by Peter Rosenthal)

*Dedicated to Professor Heydar Radjavi on his 80th birthday, the most lovable mathematician and humor critic I know.*

Abstract. Suppose $\pi : \mathcal{A} \to \mathcal{B}$ is a surjective unital $*$-homomorphism between C*-algebras $\mathcal{A}$ and $\mathcal{B}$, and $0 \leq a \leq 1$ with $a \in \mathcal{A}$. We give a sufficient condition that ensures there is a proection $p \in \mathcal{A}$ such that $\pi(p) = \pi(a)$. An easy consequence is a result of [L. G. Brown and G. k. Pedersen, C*-algebras of real rank zero, *J. Funct. Anal.* 99 (1991) 131–149] that such a $p$ exists when $\mathcal{A}$ has real rank zero.
**Keywords:** C*-algebra, projection.
**MSC(2010):** Primary: 05C38, 15A15; Secondary: 05A15, 15A18.

## 1. Introduction

In this paper we address the problem of lifting projections. One statement of the problem is if $\mathcal{J}$ is a closed ideal in a unital C*algebra and $P$ is a projection in $\mathcal{A}/\mathcal{J}$, when does there exist a projection $p \in \mathcal{A}$ such that $p + \mathcal{J} = P$? An equivalent formulation is if $\mathcal{A}$ and $\mathcal{B}$ are unital C*-algebras and $\pi : \mathcal{A} \to \mathcal{B}$ is a surjective unital $*$-homomorphism and $a \in \mathcal{A}$ and $\pi(a)$ is a projection, when is there a projection $p$ in $\mathcal{A}$ such that $\pi(p) = \pi(a)$? This problem was solved affirmatively by C. Olsen [4] when $\mathcal{A} = B(\ell^2)$ and $\mathcal{J}$ is the ideal of compact operators on $\ell^2$.

One of the best results [1], due to L. G. Brown and G. Pedersen, is that the answer is affirmative when $\mathcal{A}$ has *real rank zero*, i.e., every selfadjoint element is a norm limit of invertible selfadjoint elements. It is this result that interests us the most. We have obtained a sufficient condition for lifting projections that easily implies the Brown-Pedersen result. On the surface, the condition seems much weaker than being real rank zero, in that it doesn't require arbitrary close approximations. However, we have not yet found an example of a C*-algebra satisfying our condition that is not real rank zero.

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G. Pedersen and C. Olsen [5] proved that nilpotents always lift, i.e., if $B \in \mathcal{B}$ and $B^n = 0$, then there is a $b \in \mathcal{A}$ such that $b^n = 0$ and $\pi(b) = B$. T. Shulman [6] showed that $b$ can be chosen so that $\|b\| = \|B\|$. Later the author [2] showed that whenever projections always lift, then all algebraic elements lift, i.e., if $D \in \mathcal{B}$ and $f(x)$ is a nonzero polynomial such that $f(D) = 0$, then there is a $d \in \mathcal{A}$ such that $\pi(d) = D$. Also T. Shulman [7] and T. Loring and T. Shulman [3] proved that $d$ can be chosen so that $\|d\| = \|D\|$.

It is easy to show that if $\pi(x)$ is a projection, there is an $a \in \mathcal{A}$ such that $0 \le a \le 1$ such that $\pi(a) = \pi(x)$. Hence we only need to consider the case when $0 \le a \le 1$ and $\pi(a) = p$ is a projection.

Before we get to our main result, suppose $\pi : \mathcal{A} \to \mathcal{B}$ is a surjective unital $*$-homomorphism and $p \in \mathcal{B}$ is a projection. Suppose $a \in \mathcal{A}$, $0 \le a \le 1$ and $\pi(a) = p$. Suppose $0 < \lambda < 1$ and the spectral projection $e = \chi_{[\lambda,\infty)}(a) \in \mathcal{A}$. This will happen when $\mathcal{A}$ is a von Neumann algebra or when $\lambda \notin \sigma(a)$. Then $\pi(e)$ and $\pi(a)$ are commuting projections, so $\pi\left(e^{\perp}\right)\pi(a)$ is a projection whose norm is at most $\left\|e^{\perp}a\right\| \le \lambda < 1$. Hence $\pi(e)^{\perp}\pi(a) = 0$, so $\pi(a) = \pi(e)\pi(a)$. However,

$$\lambda e \le ea \le e$$

implies

$$\lambda\pi(e) \le \pi(ea) = \pi(a) \le \pi(e).$$

Since $\pi(a)$ and $\pi(e)$ are projections, we see that $\pi(a) = \pi(e)$, and we have lifted $\pi(a) = p$ to the projection $e$. This means that if a lifting is impossible, then $\sigma(a) = [0,1]$. If $\mathcal{A}$ has real rank 0, then we can approximate $a$ with selfadjoint elements $a_n$ whose spectrum does not contain $\lambda$, so the projections $\chi_{[\lambda,\infty)}(a_n) = e_n \in \mathcal{A}$ and we can modify the argument. Our result below does not require arbitrarily close approximation.

We can assume that $\mathcal{A} \subset B(H)$ for some Hilbert space $H$. If $e$ is a projection, relative to the decomposition $H = e(H) \oplus e^{\perp}(H)$ we can write

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } a = \begin{pmatrix} a_e & b \\ b^* & a_{e^{\perp}} \end{pmatrix}.$$

Note that condition (1) in the next theorem says $\|b\| = \|ea - ae\| < 1/25$ and that $2\|ea - ae\|^{\frac{1}{2}} < 2/5$. Conditions (2) and (3) merely say $\sigma(a_e) \subset (\sqrt{2}\|ea - ae\|^{\frac{1}{2}}, 1]$ and $\sigma(a_{e^{\perp}}) \subset [0, 1 - \sqrt{2}\|ea - ae\|^{\frac{1}{2}})$. It follows that the intersection of these intervals contains an open interval containing $\left[\sqrt{2}/5, 1-\sqrt{2}/5\right]$ and the smaller $\|ea - ae\|$ is, the bigger this intersection gets. So there is nothing here forcing $\sigma(a_e)$ and $\sigma(a_{e^{\perp}})$ to be disjoint. In fact if $b = 0$, conditions (2) and (3) say $\sigma(a_e) \subset (0,1]$ and $\sigma(a_{e^{\perp}}) \subset [0,1)$.

**Theorem 1.1.** *Suppose $\mathcal{A}$ and $\mathcal{B}$ are unital $C^*$-algebras and $\pi : \mathcal{A} \to \mathcal{B}$ is a surjective unital $*$-homomorphism and suppose $a \in \mathcal{A}$, $0 \le a \le 1$ and $\pi(a) = P$ is a projection. Suppose there is a projection $e \in \mathcal{A}$ such that*

*(1)* $\|ea - ae\| < \frac{1}{25}$,

*(2)* $eae - (2\|ea - ae\|)^{\frac{1}{2}} e \geq \delta e$ *for some* $\delta > 0$.

*(3)* $\|e^{\perp} a e^{\perp}\| < 1 - (2\|ea - ae\|)^{\frac{1}{2}}$.

*Then there is a projection* $p \in \mathcal{A}$ *such that* $\pi(p) = \pi(a) = P$.

*Proof.* Let $\varepsilon = \|ea - ae\| < 1/25$. Let $Q = \pi(e)$, and let $A = QPQ + Q^{\perp}PQ^{\perp}$. Then $P$ and $Q$ are projections and

$$\varepsilon \geq \|PQ - QP\| = \left\|P - \left[QPQ + Q^{\perp}PQ^{\perp}\right]\right\| = \|P - A\|.$$

Thus

$$\left\|P - A^2\right\| = \left\|P^2 - A^2\right\| \leq \|P(P - A)\| + \|(P - A)A\| \leq 2\varepsilon.$$

Since $A^2 = (QPQ)^2 + \left(Q^{\perp}PQ^{\perp}\right)^2$, we obtain

$$\left\|P - \left[(QPQ)^2 + \left(Q^{\perp}PQ^{\perp}\right)^2\right]\right\| \leq 2\varepsilon.$$

Multiplying the last expression inside the norm on both sides by $Q$ and by $Q^{\perp}$, we obtain

$$\left\|QPQ - (QPQ)^2\right\| \leq 2\varepsilon, \text{ and}$$

$$\left\|Q^{\perp}PQ^{\perp} - \left(Q^{\perp}PQ^{\perp}\right)^2\right\| \leq 2\varepsilon.$$

If $\lambda \in \sigma(QPQ)$ or $\lambda \in \sigma\left(Q^{\perp}PQ^{\perp}\right)$ we see that

$$\left|\lambda - \lambda^2\right| = |\lambda|\,|1 - \lambda| \leq 2\varepsilon,$$

which implies $|\lambda| \leq \sqrt{2\varepsilon}$ or $|1 - \lambda| \leq \sqrt{2\varepsilon}$. Hence $\sigma(QPQ) \cup \sigma\left(Q^{\perp}PQ^{\perp}\right) \subset \left[-\sqrt{2\varepsilon}, \sqrt{2\varepsilon}\right] \cup \left[1 - \sqrt{2\varepsilon}, 1 + \sqrt{2\varepsilon}\right]$. Choose $g : \mathbb{R} \to [0, 1]$ to be continuous so that $g = 0$ on $(-\infty, \sqrt{2\varepsilon}]$ and $g = 1$ on $[1 - \sqrt{2\varepsilon}, \infty)$. Since$\sqrt{2\varepsilon} < \sqrt{2\varepsilon} + \delta$, we can also choose $g$ so that $g = 1$ on $[\delta + \sqrt{2\varepsilon}, \infty)$.

Since, on $\sigma(QPQ) \cup \sigma\left(Q^{\perp}PQ^{\perp}\right)$ we have $g = \bar{g} = g^2$ and $|g(t) - t| \leq \sqrt{2\varepsilon}$, we see that $g(QPQ)$ and $g\left(Q^{\perp}PQ^{\perp}\right)$ are projections and

$$\|QPQ - g(QPQ)\| \leq \sqrt{2\varepsilon}, \text{ and}$$

$$\left\|Q^{\perp}PQ^{\perp} - g\left(Q^{\perp}PQ^{\perp}\right)\right\| \leq \sqrt{2\varepsilon}.$$

It follows from (3) that

$$\left\|Q^{\perp}PQ^{\perp}\right\| = \left\|\pi\left(e^{\perp}ae^{\perp}\right)\right\| < 1 - \sqrt{2\varepsilon}.$$

Hence,

$$\left\|g\left(Q^{\perp}PQ^{\perp}\right)\right\| \leq \left\|Q^{\perp}PQ^{\perp}\right\| + \left\|Q^{\perp}PQ^{\perp} - g\left(Q^{\perp}PQ^{\perp}\right)\right\| < \left(1 - \sqrt{2\varepsilon}\right) + \sqrt{2\varepsilon} = 1.$$

But $g\left(Q^{\perp}PQ^{\perp}\right)$ is a projection, so we conclude that $g\left(Q^{\perp}PQ^{\perp}\right) = 0$. Thus

$$\left\|Q^{\perp}PQ^{\perp}\right\| = \left\|Q^{\perp}PQ^{\perp} - g\left(Q^{\perp}PQ^{\perp}\right)\right\| \leq \sqrt{2\varepsilon}.$$

It follows that

$$\|P - g(QPQ)\| \leq$$
$$\left\|P - \left[QPQ + Q^{\perp}PQ^{\perp}\right]\right\| + \|QPQ - g(QPQ)\| + \left\|Q^{\perp}PQ^{\perp}\right\| \leq$$
$$\varepsilon + \sqrt{2\varepsilon} + \sqrt{2\varepsilon} < 1/25 + 2\sqrt{2/25} < 1.$$

Since $P$ and $g(QPQ)$ are projections, if we define $S = Pg(QPQ) + P^{\perp}g(QPQ)^{\perp}$, we get $1 - S = (2P - 1)(P - g(QPQ))$ and, since $2P - 1$ is unitary, we have

$$\|1 - S\| < 1, \text{ and}$$
$$PS = Sg(QPQ).$$

Since $\|1 - S\| < 1$, $\sigma(S) \subset D(1,1)$ on which $\log z$ is analytic, we can write $C = \log(S) \in C^*(P, g(QPQ)) \subseteq \mathcal{B}$ and $S = e^C$. If we let $U = S(SS^*)^{-\frac{1}{2}}$, then $U$ is unitary. Also

$$[PS]S^* = [Sg(QPQ)]S^* = S[Sg(QPQ)]^* = SS^*P.$$

Hence $P(SS^*)^{1/2} = (SS^*)^{1/2}P$, which implies $PU = Ug(QPQ)$. If we choose $c \in \mathcal{A}$ such that $\pi(c) = C$ and define $s = e^c \in \mathcal{A}$ and $u = s(ss^*)^{-\frac{1}{2}}$, then $u \in \mathcal{A}$ is unitary and

$$\pi(ug(eae)u^*) = Ug(QPQ)U^* = P.$$

However, condition (2) implies

$$eae \geq \left(\delta + \sqrt{2\varepsilon}\right)e$$

Since $g = 1$ on $[\delta + \sqrt{2\varepsilon}, \infty)$, we see that $g(eae) = e$. Thus $p = ueu^*$ is a projection and $\pi(p) = P = \pi(a)$. □

**Remark 1.2.** Note that, in Theorem 1.1, if $h : [0,1] \to [0,1]$ is continuous, $h(0) = 0$ and $h(1) = 1$, then $\pi(h(a)) = h(\pi(a)) = \pi(a)$. Thus if there is a projection $e \in \mathcal{A}$ such that conditions (1)-(3) hold with $a$ replaced with $h(a)$, then the conclusion of Theorem 1.1 still holds.

Note that in the following theorem that if $\varepsilon$ gets close to 0, then $\lambda$ could be very close to 0 or 1.

**Theorem 1.3.** *Suppose $\mathcal{A}$ is a unital C*-algebra such that, for every $a \in \mathcal{A}$ with $0 \leq a \leq 1$ there is a continuous function $h : [0,1] \to [0,1]$ fixing 0 and 1, and a selfadjoint element $b \in \mathcal{A}$ such that if $\varepsilon = \|b - h(a)\|$ and $\alpha = \varepsilon + \sqrt{2\varepsilon}$, then*

*(1) $\varepsilon < 1/25$,*
*(2) The interval $(\alpha, 1 - \alpha)$ is not contained in $\sigma(b)$.*

*If $\mathcal{B}$ is any unital C*-algebra and if $\pi : \mathcal{A} \to \mathcal{B}$ is a surjective unital *-homomorphism, and $P$ is a projection in $\mathcal{B}$, then there is a projection $p \in \mathcal{A}$ such that $\pi(p) = P$.*

*Proof.* First choose $a \in \mathcal{A}$ such that $\pi(a) = P$ and $0 \le a \le 1$. Now choose $h$ and $b$ as in the hypothesis. Choose $\lambda \in \left(\varepsilon + \sqrt{2\varepsilon}, 1 - \varepsilon - \sqrt{2\varepsilon}\right) \backslash \sigma(b)$. Thus the spectral projection $e$ of $b$ for the set $[\lambda, \infty)$ is in $C^*(b) \subset \mathcal{A}$. Then

$$\|eh(a) - h(a)e\| = \|e(h(a) - b) - (h(a) - b)e\| \le \|h(a) - b\| < 1/25.$$

Also

$$\left\|e^\perp h(a) e^\perp\right\| \le \left\|e^\perp b e^\perp\right\| + \left\|e^\perp (h(a) - b) e^\perp\right\| < \lambda + \varepsilon < \\ 1 - \varepsilon - \sqrt{2\varepsilon} + \varepsilon = 1 - \sqrt{2\varepsilon}.$$

We also have

$$eh(a)e - \sqrt{2}\,\|eh(a) - h(a)e\|^{1/2} e \ge ebe + e(h(a) - b)e - \sqrt{2\varepsilon}e > \\ \left(\lambda - \|h(a) - b\| - \sqrt{2\varepsilon}\right) e.$$

However, $\lambda - \|h(a) - b\| - \sqrt{2\varepsilon} > \varepsilon + \sqrt{2\varepsilon} - \varepsilon - \sqrt{2\varepsilon} = 0$. Since $\pi(h(a)) = h(\pi(a)) = h(P) = P$, Theorem 1.3 applies to get a projection $p \in \mathcal{A}$ such that $\pi(p) = P$. $\square$

A C*-algebra has real rank zero if every selfadjoint element is a limit of invertible selfadjoints. It follows that if $0 \le a \le 1$, and $\{b_n\}$ is a sequence of invertible selfadjoint elements converging to $a - 1/2$, then $\{1/2 + b_n\}$ is a sequence of selfadjoints with spectrum not containing $1/2$ that is converging to $a$. Once we have $\|b_n + 1/2 - a\| < 1/50$, we see that Theorem 1.3 easily applies with $h(t) = t$.

**Corollary 1.4** (Brown-Pedersen [1]). *If $\mathcal{A}$ has real rank 0 and $\pi : \mathcal{A} \to \mathcal{B}$ is a surjective unital $*$-homomorphism, and $P$ is a projection in $\mathcal{B}$, then there is a projection $p \in \mathcal{A}$ such that $\pi(p) = P$.*

Here is the question that most concerns us here.

**Question.** Is there a unital C*-algebra satisfying the hypothesis of Theorem 1.3 that does not have real rank 0?.

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