ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Special Issue of the

Bulletin of the Iranian Mathematical Society

in Honor of Professor Heydar Radjavi's 80th Birthday

Vol. 41 (2015), No. 7, pp. 123–132

Title:

Toeplitz transforms of Fibonacci sequences

Author(s):

L. Connell, M. Levine, B. Mathes and J. Sukiennik

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 41 (2015), No. 7, pp. 123–132
Online ISSN: 1735-8515

TOEPLITZ TRANSFORMS OF FIBONACCI SEQUENCES

L. CONNELL, M. LEVINE, B. MATHES* AND J. SUKIENNIK

(Communicated by Bamdad Yahaghi)

Dedicated to Professor Heydar Radjavi on his 80th birthday

Abstract. We introduce a matricial Toeplitz transform and prove that the Toeplitz transform of a second order recurrence sequence is another second order recurrence sequence. We investigate the injectivity of this transform and show how this distinguishes the Fibonacci sequence among other recurrence sequences. We then obtain new Fibonacci identities as an application of our transform.
Keywords: Hankel transform, Fibonacci numbers, Fibonacci identities.
MSC(2010): Primary: 15B36; Secondary: 65F35.

1. **Introduction**

Many authors have investigated determinants of Hankel and Toeplitz matrices ([2,5,8–10,15,19,20]). More recently, matrices whose entries are Fibonacci numbers have been popular ([1,3,6,12,16–18]). There has also been a recent trend to consider the sequence of Hankel determinants as a sequential transform ([4,5,7,14]). The contents of this note continues to build on these developments, with an investigation of the transforms of Fibonacci sequences obtained from the sequence of determinants of corresponding symmetric Toeplitz matrices.

Let a unilateral complex sequence $s = (s_0, s_1, \ldots)$ be given, and define, for each natural number n, a corresponding symmetric Toeplitz matrix

$$T_n(s) = \begin{pmatrix} s_0 & s_1 & \ldots & s_{n-2} & s_{n-1} \\ s_1 & s_0 & s_1 & \ldots & s_{n-2} \\ \vdots & \ddots & \ddots & & \vdots \\ s_{n-2} & \ldots & & \ldots & s_1 \\ s_{n-1} & s_{n-2} & \ldots & s_1 & s_0 \end{pmatrix}.$$

Article electronically published on December 31, 2015.
Received: 28 April 2015, Accepted: 28 June 2015.
*Corresponding author.

We define the *Toeplitz transform* of s to be the sequence

$$\tau(s)_n = \det T_n(s) \qquad (n = 1, 2, \ldots).$$

Our definition of the Toeplitz transform is akin to the sequential Hankel transform defined in [14]. What we found intriguing is the relationship between this Hankel transform and the Catalan sequence (see OEIS A000108): the Catalan sequence is completely determined as the unique sequence whose Hankel transform gives constant ones, and whose shifted transform also gives constant ones [7]. That there is a unique sequence with this property is trivial. With $|A|$ denoting the determinant of A, one can recursively solve the equations

$$\left| \, c_0 \, \right| = 1, \left| \, c_1 \, \right| = 1, \begin{vmatrix} c_0 & c_1 \\ c_1 & c_2 \end{vmatrix} = 1, \begin{vmatrix} c_1 & c_2 \\ c_2 & c_3 \end{vmatrix} = 1, \begin{vmatrix} c_0 & c_1 & c_2 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{vmatrix} = 1, \ldots$$

to obtain $\{c_0, c_1, \ldots\}$. The surprise here is that the solution is the Catalan sequence. As we will see, our Toeplitz transform appears particularly well suited for recurrence sequences, with the surprise that the celebrated Fibonacci sequence is distinguished by this transform, much as the Catalan sequence is by the Hankel transform.

We refer the reader to [13] for a delightful exposition of second order recurrence sequences. For complex numbers a and b we let $\mathfrak{R}(a, b)$ denote the set of bilateral complex sequences (s_i) with

$$bs_{i-2} + as_{i-1} = s_i.$$

When the polynomial $x^2 - ax - b$ has two distinct roots, we will denote them with ϕ and ψ. The geometric sequences (ϕ^i) and (ψ^i) then give a basis of $\mathfrak{R}(a, b)$. We refer to these as the *geometric elements* of $\mathfrak{R}(a, b)$. This basis allows one to write any element of $\mathfrak{R}(a, b)$ in a closed form. We refer to the element $s \in \mathfrak{R}(a, b)$ with $s_0 = 0$ and $s_1 = 1$ as the Fibonacci element of $\mathfrak{R}(a, b)$, and the element with $s_0 = 2$ and $s_1 = a$ is called the Lucas element.

2. The transform of recurrence sequences

Theorem 2.1. *Assume that $s \in \mathfrak{R}(a, b)$. Then $\tau(s)$ is a second order recurrence sequence that satisfies*

$$(bs_1 + as_0 - s_1)b(s_{-1} - s_1)\tau(s)_{i-2} + (s_0 - bs_2 - 2as_1 + a(bs_1 + as_0))\tau(s)_{i-1} = \tau(s)_i$$

for all $i \geq 3$.

Proof. Use the recurrence relation, and the multi-linearity of the determinant, on the last column of $T_n(s)$ to write

$$\tau(s)_n = |T_n(s)| = \begin{vmatrix} s_0 & s_1 & \dots & s_{n-2} & 0 \\ s_1 & s_0 & s_1 & \dots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ s_{n-3} & & & & 0 \\ s_{n-2} & \dots & & s_0 & s_1 - bs_1 - as_0 \\ s_{n-1} & s_{n-2} & \dots & s_1 & s_0 - bs_2 - as_1 \end{vmatrix}.$$

It follows that

$$|T_n(s)| = (s_0 - bs_2 - as_1)|T_{n-1}(s)| - (s_1 - bs_1 - as_0)|W|,$$

with W given by

$$W = \begin{pmatrix} s_0 & s_1 & \dots & s_{n-2} \\ s_1 & & \dots & s_{n-3} \\ \vdots & \ddots & \ddots & \vdots \\ s_{n-3} & s_{n-4} & \dots & s_1 \\ s_{n-1} & s_{n-2} & \dots & s_1 \end{pmatrix}.$$

Now use the multi-linearity again, this time on the last row, to write

$$\begin{aligned} |W| &= b\begin{vmatrix} s_0 & s_1 & \dots & s_{n-2} \\ s_1 & & \dots & s_{n-3} \\ \vdots & \ddots & \ddots & \vdots \\ s_{n-3} & s_{n-4} & \dots & s_1 \\ s_{n-3} & s_{n-4} & \dots & s_{-1} \end{vmatrix} + a\begin{vmatrix} s_0 & s_1 & \dots & s_{n-2} \\ s_1 & & \dots & s_{n-3} \\ \vdots & \ddots & \ddots & \vdots \\ s_{n-3} & s_{n-4} & \dots & s_1 \\ s_{n-2} & s_{n-3} & \dots & s_0 \end{vmatrix} \\ &= b\begin{vmatrix} s_0 & s_1 & \dots & s_{n-2} \\ s_1 & & \dots & s_{n-3} \\ \vdots & \ddots & \ddots & \vdots \\ s_{n-3} & s_{n-4} & \dots & s_1 \\ 0 & 0 & \dots & s_{-1} - s_1 \end{vmatrix} + a\begin{vmatrix} s_0 & s_1 & \dots & s_{n-2} \\ s_1 & & \dots & s_{n-3} \\ \vdots & \ddots & \ddots & \vdots \\ s_{n-3} & s_{n-4} & \dots & s_1 \\ s_{n-2} & s_{n-3} & \dots & s_0 \end{vmatrix} \\ &= b(s_{-1} - s_1)\tau(s)_{n-2} + a\tau(s)_{n-1}. \end{aligned}$$

Substituting this expression into the most recent equation for $|T_n(s)|$ above completes the proof. □

It is worthwhile to isolate two special cases of our theorem, that reduce the recurrence formula to a more manageable expression. In the case of $\mathfrak{R}(k,1)$, the elements have frequently been referred to as *k-Fibonacci sequences* ([16], [17]), and our theorem takes the following form.

Corollary 2.2. *If* $s \in \mathfrak{R}(k,1)$*, then* $\tau(s) \in \mathfrak{R}(-2ks_1 + k^2 s_0, -k^2 s_0^2)$.

In the space $\mathfrak{R}(1,1)$, that has long been referred to as the space of *generalized Fibonacci sequences* [11], our theorem is the following.

Corollary 2.3. *If* $s \in \mathfrak{R}(1,1)$*, then* $\tau(s) \in \mathfrak{R}(-2s_1 + s_0, -s_0^2)$.

Example 2.4. Let (f_i) denote the classical Fibonacci sequence, i.e. $(f_0, f_1, f_2, \ldots)$ is the sequence

$$0, 1, 1, 2, 3, \ldots,$$

an element of $\mathfrak{R}(1,1)$. It follows that $\tau(f)$ is the element of $\mathfrak{R}(-2,0)$ whose first two terms are $\tau(f)_1 = 0$ and $\tau(f)_2 = -1$. Thus we have

$$\tau(f)_n = (-1)^{n-1} 2^{n-2}$$

for $n \geq 2$. This particular transform leads to a Fibonacci identity for the difference between the n^{th} Fibonacci number and 2^{n-1}, which we give later in this note.

Example 2.5. Let (f_i) denote the Fibonacci element of $\mathfrak{R}(2,1)$, the Pell sequence

$$0, 1, 2, 5, 12, 29, \ldots.$$

It follows that $\tau(f)$ is the element of $\mathfrak{R}(-4,0)$ given by

$$\tau(f)_n = (-1)^{n-1} 4^{n-2}$$

for $n \geq 2$.

Example 2.6. Let (s_i) denote the shifted classical Fibonacci sequence, i.e. $(s_0, s_1, s_2, \ldots)$ is the sequence

$$1, 1, 2, 3, \ldots.$$

This time $\tau(s) \in \mathfrak{R}(-1,-1)$ and begins with $\tau(s)_1 = 1$ and $\tau(s)_2 = 0$, so $\tau(s)$ is

$$1, 0, -1, 1, 0, -1, \ldots.$$

This gives inspiration for a nice Putnam problem: form the symmetric Toeplitz matrix using the Fibonacci numbers, and prove that the resulting $n \times n$ matrix is singular if and only if $n = 2 \pmod 3$.

Example 2.7. Assume (f_i) is the Fibonacci element of $\mathfrak{R}(11,-10)$, the sequence of repunits,

$$0, 1, 11, 111, 1111, \ldots.$$

Then the Toeplitz transform is the sequence $\tau(f)_n = (n-1)(-1)^{n-1} 11^{n-2}$ for $n \geq 2$.

Example 2.8. Let (f_i) be the Fibonacci element of $\mathfrak{R}(2,-1)$, the arithmetic progression

$$0, 1, 2, 3, 4, \ldots.$$

We have $\tau(f)_n = (n-1)(-1)^{n-1}2^{n-2}$ for $n \geq 2$. A little more generally, if (s_i) is the arithmetic progression

$$0, k, 2k, \ldots,$$

then $\tau(s)_n = k^n \tau(f)_n = k^n (n-1)(-1)^{n-1}2^{n-2}$ for $n \geq 2$.

3. **Injectivity of the Toeplitz transform**

Let a complex sequence $s = (s_i)$ be given. We are interested in investigating the extent of injectivity of the transform τ at s. We will say τ is k-injective at s if the inverse image of $\tau(s)$ contains exactly k sequences. When $k = 1$, we will say that τ is injective at s, instead of saying "1-injective", and we will use the term *bi-injective* at s instead of 2-injective at s. If the inverse image of $\tau(s)$ is infinite, we will say that τ is completely non-injective at s.

For each natural number n we define a polynomial $p_n(s)$ as follows: in the matrix $T_n(s)$ replace the s_{n-1} in both the northeast and the southwest corners with a variable x, and call the resulting matrix $T_n^x(s)$, then define

$$p_n(s) = |T_n^x(s)| - |T_n(s)|.$$

In this way we manufacture a polynomial, of degree less than or equal to two, which must have s_{n-1} as a root. We will use this sequence of polynomials to detect injectivity, but first let us dispense with a triviality.

Lemma 3.1. *If α is a non-zero complex number, then τ is k-injective at s if and only if τ is k-injective at αs.*

Proof. This follows from the equality

$$\tau(\alpha s)_n = \alpha^n \tau(s)_n.$$

□

In terms of the polynomials $p_n(s)$, a sufficient condition for τ to be k-injective at s is that $p_n(s)$ be eventually a perfect square, i.e. that eventually, s_{n-1} be a double root of $p_n(s)$. This is exactly what happens when s is either the Fibonacci element of $\mathfrak{R}(a, b)$, or s is a geometric element of $\mathfrak{R}(a, b)$.

Theorem 3.2. *Assume that $s \in \mathfrak{R}(a, b)$ with $b = 1$. Then, for all $n \geq 4$, $p_n(s)$ is a perfect square, if and only if s is a multiple of the Fibonacci element, or s is a multiple of a geometric element.*

Proof. In anticipation of using the determinant's multi-linearity on the first and last columns, let us denote with u and v the first and last columns of $T_n(s)$, with u_x and v_x the first and last columns of $T_n^x(s)$, let l_x be the column vector whose last entry is $x - s_{n-1}$ and zeros elsewhere, and let r_x be the column vector whose first entry is $x - s_{n-1}$ and zeros elsewhere. If we group the unchanging central columns into a matrix M, then the equalities $u_x = u+l_x$ and $v_x = v+r_x$ let us relate the determinants $|T_n(s)|$ and $|T_n^x(s)|$ via the equation

$$\begin{aligned}|T_n^x(s)| &= |((u+l_x), M, (v+r_x))| \\ &= |T_n(s)| + |(l_x, M, v)| + |(u, M, r_x)| + |(l_x, M, r_x)|.\end{aligned}$$

It follows that

$$\begin{aligned}p_n(s) &= |(l_x, M, v)| + |(u, M, r_x)| + |(l_x, M, r_x)| \\ &= 2|(l_x, M, v)| + |(l_x, M, r_x)|.\end{aligned}$$

As $|(l_x, M, r_x)|$ is a constant multiple of $(x - s_{n-1})^2$, we need to see when $|(l_x, M, v)| = 0$. Using the recurrence relation on the last three columns gives

$$|(l_x, M, v)| = \begin{vmatrix} 0 & s_1 & \dots & s_{n-2} & 0 \\ 0 & s_0 & s_1 & \dots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & & & & 0 \\ 0 & \dots & & s_0 & s_1 - bs_1 - as_0 \\ x - s_{n-1} & s_{n-2} & \dots & s_1 & s_0 - bs_2 - as_1 \end{vmatrix} = 0$$

if and only if

$$\begin{vmatrix} s_1 & \dots & s_{n-2} & 0 \\ s_0 & s_1 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ & & & 0 \\ \dots & & s_0 & s_1 - bs_1 - as_0 \end{vmatrix} = 0.$$

Continue using the recurrence relation on the last three columns until you have reduced the matrix to a 2×2, and you see this determinant equal to $(s_1 - bs_1 - as_0)^{n-2}(s_1^2 - s_0 s_2)$, so $p_n(s)$ is a perfect square if and only if

$$(s_1 - bs_1 - as_0)^{n-2}(s_1^2 - s_0 s_2) = 0.$$

Assuming $b = 1$, this happens if and only if $s_0 = 0$ (s is a multiple of the Fibonacci element) or $s_1^2 = s_0 s_2$ (s is a multiple of the geometric element). □

Comment 3.3. In case $b \neq 1$, the preceding proof shows that the distinction enjoyed by the Fibonacci element shifts elsewhere. For example, if $b = 2$ and $a = -1$, the proof shows that the element of $\mathfrak{R}(a, b)$ beginning with $s_0 = 1$ and $s_1 = 1$ has perfect square $p_n(s)$ for $n \geq 4$. In this case (s_n) is the constant 1 sequence, and the inverse image of $\tau(s)$ contains exactly two elements, the additional one being the alternating sequence $1, -1, 1, -1, \dots$.

Corollary 3.4. *If $b = \pm 1$, then τ is bi-injective at the Fibonacci element of $\mathfrak{R}(a, b)$. For any a and b, τ is bi-injective at a non-zero geometric element of $\mathfrak{R}(a, b)$. In the inverse image of $\tau(s)$, the additional element is the alternating sequence $(-1)^i(s_i) \in \mathfrak{R}(a, -b)$.*

Proof. Note that $(-1)^i(s_i)$ has the same transform as (s_i) follows from that fact that the corresponding Toeplitz matrices are similar, via the invertible diagonal operator with alternating 1 and -1 on the diagonal. For the Fibonacci sequence in $\mathfrak{R}(a,b)$, one has $p_3(s) = 2(x-a)$, and higher order $p_n(s)$ all have a single (repeated) root, so there can be no other elements of the inverse image. For the geometric element (ϕ^i), we get $p_3(s) = (\phi^2 - x)^2$, with repeated roots for all higher n. $\square$

Example 3.5. Assume that ϕ a root of the polynomial x^2+x-1, so $s = (\phi^i)$ is a geometric element of $\mathfrak{R}(-1,1)$. It follows that $\tau(s) \in \mathfrak{R}(2\phi+1,-1)$, and

$$(-1)(\phi^{k-2}) + (2\phi+1)(\phi^{k-1}) = \phi^{k-2}(\phi^2 + (\phi^2+\phi-1)) = \phi^k$$

shows that $s = (\phi^i)$ is also a geometric element of $\mathfrak{R}(2\phi+1,-1)$, and in particular, s is a fixed point of τ. Thus there are exactly two fixed points of τ, and τ is bi-injective at each fixed point.

Example 3.6. Let s be the Lucas element of $\mathfrak{R}(1,1)$. It follows that $\tau(s)_1 = 2$, $\tau(s)_2 = 3$, $\tau(s)_3 = -8$, and $\tau(s)_4 = -12$. Seeking other recurrence sequences with the same transform, begin by solving the system of equations

$$\begin{array}{c}(bs_1+as_0-s_1)b(s_{-1}-s_1)\tau(s)_1 + (s_0 - bs_2 - 2as_1 + a(bs_1+as_0))\tau(s)_2 = \tau(s)_3 \\ (bs_1+as_0-s_1)b(s_{-1}-s_1)\tau(s)_2 + (s_0 - bs_2 - 2as_1 + a(bs_1+as_0))\tau(s)_3 = \tau(s)_4\end{array}$$

to find that

$$\begin{array}{c}(bs_1+as_0-s_1)b(s_{-1}-s_1) = -4 \\ s_0 - bs_2 - 2as_1 + a(bs_1+as_0) = 0\end{array}.$$

It must be that $s_0 = 2$ and $s_1 = \pm 1$, so we substitute $s_0 = 2$ and $s_1 = 1$ into the above getting

$$\begin{array}{c}(b+2a-1)^2 = 4 \\ 2 + 2(a^2-b^2) - 2a = 0\end{array}$$

(be aware that $s_{-1} = (1/b)(s_1 - as_0)$ and $s_2 = bs_0 + as_1$). Solving for a and b gives the four solutions $(a,b) = (-\frac{5}{3},\frac{7}{3}), (\frac{8}{3},-\frac{7}{3}), (0,-1)$ and $(1,1)$. We get four more solutions with $s_1 = -1$, which shows that τ is 8-injective at s when restricted to the recurrence sequences with $b \neq 0$. All our evidence suggests that, in general, τ is completely non-injective at s, but we have not been able to prove this.

Comment 3.7. We have had success using the polynomials $p_n(s)$ to obtain an upper bound on the size of an inverse image, but are frustrated when using the polynomials to obtain a lower bound on the cardinality of an inverse image. When s is the Lucas sequence, as in the previous example, we can show that $p_n(s)$ has two distinct roots for every $n \geq 2$, so one might think that recursively selecting a root could lead to new elements of the inverse image. A problem with this strategy becomes apparent when showing that τ is injective at the zero element. In this case, $p_n(s)$ is the constant zero polynomial as soon as $n \geq 3$, so one could presumably choose a non-zero number α at the n^{th} step, seeking a non-zero sequence in the pre-image. After this, it is not until the $(2n)^{th}$ step that our choice of α returns to haunt us, when we try to select an x that makes

$$|T^x_{2n}(s)| = \begin{vmatrix} 0 & A \\ A^* & 0 \end{vmatrix} = 0,$$

with upper triangular A given by

$$A = \begin{pmatrix} \alpha & * & * & \cdots \\ 0 & \alpha & * & \cdots \\ 0 & 0 & \alpha & * \\ \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & \alpha \end{pmatrix}.$$

But clearly, $|T^x_{2n}(s)| = \alpha^2$.

4. Fibonacci identities

The amazing identities that decorate Fibonacci sequences are certainly one of their main attractions. We have two new identities that relate the traditional Fibonacci sequence $(\ldots, f_0, f_1, f_2, \ldots) \in \mathfrak{R}(1,1)$ to the geometric sequence 2^n. Our first one is a trivial restatement of Theorem 2.1 and Example 2.4, but it is still striking and worth isolating. The formula on the left of the equality is just the standard definition of determinant applied to the matrix $T_n(f)$. In particular, the summation is over all permutations σ.

Identity 4.1. For $n \geq 2$,

$$\sum_{\sigma} \prod_{i=0}^{n-1} \mathrm{sgn}_{\sigma} f_{|i-\sigma_i|} = (-1)^{n-1} 2^{n-2}.$$

Our second identity was discovered while in a frenzy of column additions of $T_n(f)$. What resulted from the column additions we describe by writing, for $k \geq 1$,

$$F_k^{(0)} = f_k,\ F_k^{(1)} = \sum_{i=0}^{k-1} f_i,\ F_k^{(2)} = \sum_{i=0}^{k-1} F_i^{(1)},$$

and in general

$$F_k^{(n)} = \sum_{i=0}^{k-1} F_i^{(n-1)}.$$

We define $F_0^{(i)} = 0$ for all non-negative integers i.

Identity 4.2. For $n \geq 3$,

$$2^n - f_{n+1} = \sum_{i=0}^{n-1} F_n^{(i)}.$$

Our original proof, via determinants, is long and arduous, and it includes more than one questionable step. Upon reflection we found a much simpler proof, without the mysterious leaps of faith.

Proof. Let C denote the matrix of the operator that transforms the sequence (f_i) into $(F_i^{(1)})$, i.e.

$$C = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \\ \vdots & & \ddots & \ddots \end{pmatrix}.$$

It follows that

$$\sum_{i=0}^{n-1} F_n^{(i)} = \sum_{i=0}^{n-1} \left(C^i f\right)_n = \left(\sum_{i=0}^{n-1} C^i f\right)_n = \left((1-C^n)(1-C)^{-1} f\right)_n.$$

We need to compute the n^{th} coordinate of $(1-C^n)(1-C)^{-1}f$, so begin by noting that the n^{th} row of $(1-C^n)$ is

$$(-1\ 0\ 0\ \cdots\ 0\ 1\ 0\ 0\ \cdots),$$

so we need to find the first and n^{th} coordinate of $(1-C)^{-1}f$. The matrix of $(1-C)^{-1}$ is lower triangular Toeplitz, with the generating sequence $1, 1, 2, 2^2, 2^3, \ldots$, from which we see the first coordinate of $(1-C)^{-1}f$ is f_0 and the n^{th} coordinate is

$$2^{n-1}f_0 + 2^{n-2}f_1 + \ldots + 2f_{n-2} + f_{n-1} + f_n.$$

That $f_0 = 0$ leaves us with proving

$$\sum_{i=1}^{n-1} 2^{n-1-i} f_i + f_n = 2^n - f_{n+1}$$

for all $n \geq 3$. When $n = 3$ we have that $2f_1 + f_2 + f_3 = 2^3 - f_4$, which is certainly true, so assume the induction hypothesis and compute

$$\begin{aligned} \textstyle\sum_{i=1}^{n} 2^{n-i} f_i + f_{n+1} &= 2(\textstyle\sum_{i=1}^{n-1} 2^{n-i-1} f_i) + f_n + f_{n+1} \\ &= 2(2^n - f_{n+1} - f_n) + f_n + f_{n+1} \\ &= 2^{n+1} - f_{n+2} \end{aligned}.$$

□

References

[1] S. Solak and M. Bahsi, On the spectral norms of Hankel matrices with Fibonacci and Lucas numbers, *Selçuk J. Appl. Math.* **12** (2011), no. 1, 71–76.

[2] G. Baxter and P. Schmidt, Determinants of a certain class of non-hermitian Toeplitz matrices, *Math. Scand.* **9** (1961) 122–28.

[3] D. Bozkurt and M. Akbulak, On the norms of Toeplitz matrices involving Fibonacci and Lucas numbers, *Hacet. J. Math. Stat.* **37** (2008), no. 2, 89–95.

[4] N. T. Cameron and A. C. M. Yip, Hankel determinants of sums of consecutive Motzkin numbers, *Linear Algebra Appl.* **434** (2011), no. 3, 712–722.

[5] A. Cvetkovic, P. Rajkovic and M. Ivkovic, Catalan numbers, the Hankel transform, and Fibonacci numbers, *J. Integer Seq.* **5** (2002), no. 1, 8 pages.

[6] J. Dixon and B. Mathes, and D. Wheeler, An application of matricial Fibonacci identities to the computation of spectral norms, *Rocky Mountain J. Math.* **44** (2014), no. 3, 877–887.

[7] M. Dougherty, C. French, W. Qian and B. Saderholm, Hankel transforms of linear combinations of catalan numbers, *J. Integer Seq.* **14** (2011), no. 5, 20 pages.

[8] B. Germano and P. E. Ricci, Computing Hankel determinants, *Pubbl. Dip. Metod. Model. Mat. Sci. Appl. Univ. Stud. Roma* **22** (1984) 21–26.

[9] U. Greenlander and G. Segö, Toeplitz Forms and their Applications, University of California Press, Berkeley-Los Angeles, 1958.

[10] T. Hoholdt and J. Justesen, Determinants of a class of Toeplitz matrices, *Math. Scand.* **43** (1978), no. 2, 250–258.

[11] A. F. Horadam, A generalized Fibonacci sequence, *Amer. Math. Monthly* **68** (1961) 455–459.

[12] A. Ipek, On the spectral norms of circulant matrices with classical Fibonacci and Lucas numbers entries, *Appl. Math. Comput.* **217** (2011), no. 12, 6011–6012.

[13] D. Kalman and R. Mena, The Fibonacci numbers – exposed, *Math. Mag.* **76**, (2003), no. 3, 182–192.

[14] J. W. Layman, The Hankel transform and some of its properties, *J. Integer Seq.* **4** (2001), no. 1, 11 pages.

[15] C. Radoux, Déterminant de Hankel construit sur les polynômes de Hermite, *Ann. Soc. Sci. Bruxelles Sér. I* **104** (1990), no. 2, 59–61.

[16] S. Shen, On the norms of Toeplitz matrices involving k-Fibonacci and k-Lucas numbers, *Int. J. Contemp. Math. Sci.* **7** (2012), no. 5-8, 363–368.

[17] S. Shen, J. Cen, On the spectral norms of r-circulant matrices with the k-Fibonacci and k-Lucas numbers, *Int. J. Contemp. Math. Sci.* **5** (2010), no. 9-12, 569–578.

[18] S. Solak, On the norms of circulant matrices with the Fibonacci and Lucas numbers, *Appl. Math. Comput.* **160** (2005), no. 2, 125–132.

[19] J. L. Ullman, Hankel determinants whose elements are sections of a Taylor series, *Duke Math. J.* **18** (1951) 751–756.

[20] H. Widom, On the eigenvalues of certain Hermitian Operators, *Trans. Amer. Math. Soc.* **88** (1958) 491–522.

(Liam J. Connell) 111 W. Westminster, Lake Forest, IL 60045.
E-mail address: ljrconnell@gmail.com

(Matt Levine) The Catalyst Lofts, 141 41st Street, Pittsburgh, PA 15201.
E-mail address: mjlevine@colby.edu

(Ben Mathes) 5839 Mayflower Hill, Colby College, Waterville, ME 04901.
E-mail address: dbmathes@colby.edu

(Justin Sukiennik) 5839 Mayflower Hill, Colby College, Waterville, ME 04901.
E-mail address: justin.sukiennik@colby.edu