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A NOTE ON APPROXIMATION CONDITIONS, STANDARD TRIANGULARIZABILITY AND A POWER SET TOPOLOGY

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(Communicated by Bamdad Yahaghi)

Dedicated to Professor Heydar Radjavi on his 80th birthday

Abstract. The main result of this article is that for collections of entry-wise non-negative matrices the property of possessing a standard triangularization is stable under approximation. The methodology introduced to prove this result allows us to offer quick proofs of the corresponding results of [B. R. Yahaghi, Near triangularizability implies triangularizability, *Canad. Math. Bull.* 47 (2004), no. 2, 298–313], and [A. A. Jafarian, H. Radjavi, P. Rosenthal and A. R. Sourour, Simultaneous, triangularizability, near commutativity and Rota's theorem, *Trans. Amer. Math. Soc.* 347 (1995), no. 6, 2191–2199] on the approximations and triangularizability of collections of operators and matrices. In conclusion we introduce and explore a related topology on the power sets of metric spaces.
Keywords: Simultaneous triangularizability, positive matrices, standard invariant subspaces, semigroups of operators.
MSC(2010): Primary: 15A30, 15B48; Secondary: 20M20, 47A15, 47A46, 47A58, 47D03, 47H07.

1. Basic terminology

Throughout the paper we write $B(X)$ for the space all operators on a Banach space $(X, \| \ \|)$. Here "*operator*" means a linear function that is Lipschitz with respect to the norm on X. The corresponding Lipschitz constant defines *the operator norm* of the operator, which is our default norm on $B(X)$. Endowed with the operator norm, $B(X)$ is a Banach algebra.

For an invertible operator $T \in B(X)$, *the similarity induced by T on $B(X)$* is the map that sends an operator A to the operator $T^{-1}AT$.

A collection $\mathcal{C}$ of operators in $B(X)$ is said to be *triangularizable*, if there is a maximal (with respect to inclusion) chain of closed subspaces of X each

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of which is invariant under each of the operators in $\mathcal{C}$. In a finite-dimensional setting this is equivalent to having a basis of X with respect to which the matrices of the operators in $\mathcal{C}$ are all upper triangular.

A (not necessarily unital) *semigroup* of operators in $B(X)$ is a collection that is closed under composition. The smallest such semigroup containing a given non-empty collection is said to be the semigroup generated by the collection. This semigroup consists of all finite products of the elements of the original collection.

2. Introduction

In their paper [1], Jafarian, Radjavi, Rosenthal and Sourour explored the connections between triangularizability of a collection of compact operators on a Hilbert space and the property that finite subsets of the collection are in a certain sense close to finite commuting collections of compact normal operators. Among other things, these authors established a version of Rota's theorem for traingularizable collections of compact operators.

Yahaghi, in his paper [10], demonstrated that collections of compact operators on a Banach space are triangularizable whenever all finite subsets of these collections are close to triangularizable collections. Furthermore Yahaghi extended the results of [1] in a finite-dimensional setting by showing that a collection of matrices is triangularizable if and only if every finite subset of it is in a certain sense close to a triangularizable collection.

The purpose of the first part of this note is to offer a framework that uses the existing theory of the traingularizability of operator semigroups, to illuminate the results of [1] and [10], allowing for shorter proofs.

Thereafter we present some new results about standard triangularizability for collections of non-negative matrices. We show that such a collection has a standard triangularization whenever all of its finite subsets are in a certain sense close to collections with standard triangularizations. In that way, near standard triangularizability implies standard triangularizability.

We conclude the note with some topological considerations in an attempt to cast the concepts of "closeness" used in the previous discussion in a topological light.

3. A view on approximation conditions and triangularizability in $B(X)$.

3.1. The case of a general $B(X)$.

(1) Given a subset $\mathcal{T}$ of $B(X)$, the *ϵ-neighbourhood of $\mathcal{T}$* is the set

$$\mathcal{N}_\epsilon(\mathcal{T}) \stackrel{\text{def}}{=} \{\ A \in B(X) \mid \|A - T\| < \epsilon, \ \text{ for some } T \in \mathcal{T}\ \} = \bigcup_{T \in \mathcal{T}} \mathcal{B}_\epsilon(T).$$

As is common, we write $\mathcal{B}_\epsilon(a)$ instead of $\mathcal{N}_\epsilon(\{a\})$, and $\mathcal{B}_r$ instead of $\mathcal{B}_r(0)$.

(2) For a bounded subset $\mathcal{T}$ of $B(X)$, we write

$$\|\mathcal{T}\| \stackrel{\text{def}}{=} \sup\{\ \|T\| \mid T \in \mathcal{T}\ \}.$$

Terminology 3.1. Suppose that $\mathcal{P}$ is a property pertaining to collections of operators in $B(X)$.

(1) A subset $\mathcal{C}$ of $B(X)$ *is close to having property* $\mathcal{P}$ if for every $\epsilon > 0$ there is a set $\mathcal{G}_\epsilon$ with property $\mathcal{P}$ such that

$$\mathcal{C} \subset \mathcal{N}_\epsilon(\mathcal{G}_\epsilon).$$

(2) A subset $\mathcal{C}$ of $B(X)$ *is close to having property* $\mathcal{P}$ *via similarity* if for every $\epsilon > 0$ there is a set $\mathcal{G}_\epsilon$ with property $\mathcal{P}$ and an invertible T_ϵ such that

$$T_\epsilon^{-1}\mathcal{C}T_\epsilon \subset \mathcal{N}_\epsilon(\mathcal{G}_\epsilon).$$

We abbreviate this by saying that $\mathcal{C}$ *is close to having property* $\mathcal{P}$ *v.s.*

(3) A subset $\mathcal{C}$ of $B(X)$ *is close to having property* $\mathcal{P}$ *in a bounded fashion via similarity* if there exists a ball $\mathcal{B}_m$ such that for every $\epsilon > 0$ there is a set $\mathcal{G}_\epsilon \subset \mathcal{B}_m$ with property $\mathcal{P}$ and an invertible T_ϵ such that

$$T_\epsilon^{-1}\mathcal{C}T_\epsilon \subset \mathcal{N}_\epsilon(\mathcal{G}_\epsilon).$$

We shall abbreviate this by saying that $\mathcal{C}$ *is close to having property* $\mathcal{P}$ *b.f.v.s.*

(4) A subset $\mathcal{C}$ of $B(X)$ *is Tukey-close*[1] *to having property* $\mathcal{P}$ *(v.s., b.f.v.s.)* if every finite subset of $\mathcal{C}$ is close to having property $\mathcal{P}$ (v.s., b.f.v.s.).

To illustrate our approach, we start with the following easy proposition, which will be eventually supplanted by Theorem 3.6.

Proposition 3.2. *If $\mathcal{C} \subset B(X)$ is a collection of compact operators that is Tukey-close v.s. to being $\{0\}$, then $\mathcal{C}$ is triangularizable.*

In other words, if for every finite subcollection $\mathcal{F} \subset \mathcal{C}$ and any $\epsilon > 0$, there is an invertible T such that

$$\|T^{-1}\mathcal{F}T\| \le \epsilon, \tag{3.1}$$

then $\mathcal{C}$ is triangularizable.

Our key to a proof of Proposition 3.2 is a lovely theorem of Turovskii regarding semigroups of compact quasinilpotent operators.[2]

[1] This terminology is inspired by the classical and well-known ideas related to the sets of finite character and Tukey's Lemma. The latter is, of course, an equivalent form of the Axiom of Choice. We shall introduce further related terminology in Section 5.

[2] While it was well-known that an algebra of compact quasinilpotent operators on a Banach space is triangularizable, it was a remarkable achievement of Turovskii to have proved that the same is true for semigroups.

This is going to be the pattern throughout this paper: to prove triangularizability results for general collections of operators we will be "pushing off" some triangularizability results for semigroups of operators, i.e., collections closed under composition.

Theorem 3.3 ([8]). *If a semigroup of operators on a Banach space consists of compact quasinilpotent operators, then so does the algebra generated by this semigroup.*

Proof of Proposition 3.2. Suppose $\mathcal{F}$ satisfies the hypothesis. It is clear from the submultiplicativity of the operator norm that if $\mathcal{F}$ satisfies inequality (3.1) with $\epsilon < 1$, then $\|T^{-1}PT\| \leq \epsilon$ for any finite product P of elements of $\mathcal{C}$. In particular the semigroup $\mathcal{S}$ generated by $\mathcal{F}$ satisfies the same hypothesis as $\mathcal{F}$.

Since similarity on $B(X)$ preserves the spectrum, and the spectral radius is dominated by the operator norm, $A \in B(X)$ must be quasinilpotent if for every $\epsilon > 0$ there is an invertible T such that $\|T^{-1}AT\| \leq \epsilon$.

This shows that $\mathcal{S}$ is a semigroup of compact quasinilpotent operators, and therefore $\mathcal{S}$ is triangularizable by Turovskii's theorem. □

The following lemma is the key to the approach we shall take.

Key Lemma 3.4. *Suppose that a given property $\mathcal{P}$, pertaining to collections of operators in $B(X)$, passes to semigroups (resp. linear spans, algebras) and to non-empty subsets; i.e. if a collection of operators has property $\mathcal{P}$, then so does every non-empty subcollection, as well as the semigroup (resp. subspace, algebra) generated by that collection.*

If a non-empty collection $\mathcal{C}$ in $B(X)$ is Tukey-close to having property $\mathcal{P}$ (v.s., b.f.v.s.), then so is the semigroup (resp. subspace, algebra) generated by $\mathcal{C}$.

Proof. We present the proof in the "b.f.v.s" case only; the proofs in the "plain" and "v.s." settings are similar (and are easier).

(1) If $\mathcal{F}$ is a finite subset of the semigroup $\mathcal{S}_{\mathcal{C}}$ generated by $\mathcal{C}$, then $\mathcal{C}$ has a finite subset $\mathcal{K}$ and some $\ell \in \mathbb{N}$ such that every element of $\mathcal{F}$ is a product of no more than ℓ elements of $\mathcal{K}$. By the hypothesis $\mathcal{K}$ is close to having property $\mathcal{P}$ b.f.v.s. We aim to show that so is $\mathcal{F}$.

There is a ball $\mathcal{B}_m$ such that for each $0 < \lambda < 1$ there a set $\mathcal{G}_\lambda \subset \mathcal{B}_m$ with property $\mathcal{P}$ and an invertible T_λ such that

$$T_\lambda^{-1} \mathcal{K} T_\lambda \subset \mathcal{N}_\lambda (\mathcal{G}_\lambda) \subset \mathcal{B}_{m+1}, \tag{3.2}$$

where the second inclusion is automatic.

For $0 < \epsilon < 1$, let

$$\gamma = \frac{\epsilon}{\ell \, (m+1)^{\ell-1}},$$

and let $\prod_{\leq \ell} \mathcal{G}_\gamma$ be the set of the products of at most ℓ elements of $\mathcal{G}_\gamma$.

Note that $\prod_{\le \ell} \mathcal{G}_{\gamma}$ is a subset of $\mathcal{B}_{m^{\ell}}$, as well as a non-empty subset of the semigroup $\mathcal{S}_{\mathcal{G}_{\gamma}}$ generated by $\mathcal{G}_{\gamma}$, so that in particular it has property $\mathcal{P}$.

It is an elementary fact that for operators $A_1, A_2, A_3, \ldots, A_k$ and $B_1, B_2, B_3, \ldots, B_k$ in $\mathcal{B}_r$:

$$\|\Pi_{i=1}^{k} A_i - \Pi_{i=1}^{k} B_i\| \le k\, r^{k-1} \max_i \|A_i - B_i\|.$$

It follows from this and from the inclusions in (3.2) that every product of at most ℓ elements of $T_{\gamma}^{-1} \mathcal{K} T_{\gamma}$ is in $\mathcal{N}_{\epsilon}\left(\prod_{\le \ell} \mathcal{G}_{\gamma}\right)$. In particular

$$T_{\gamma}^{-1} \mathcal{F} T_{\gamma} \subset \mathcal{N}_{\epsilon}\left(\prod_{\le \ell} \mathcal{G}_{\gamma}\right),$$

and we are done.

(2) If $\mathcal{F}$ is a finite subset of the subspace $\mathcal{L}_{\mathcal{C}}$ generated by $\mathcal{C}$, then $\mathcal{C}$ has a finite subset $\mathcal{K}$ and some $\ell \in \mathbb{N}$ such that every element of $\mathcal{F}$ is a linear combination of no more than ℓ elements of $\mathcal{K}$ with coefficients that do not exceed ℓ in modulus. By the hypothesis $\mathcal{K}$ is close to having property $\mathcal{P}$ b.f.v.s. We aim to show that so is $\mathcal{F}$.

There is a ball $\mathcal{B}_m$ such that for each $0 < \lambda < 1$ there exist a set $\mathcal{G}_{\lambda} \subset \mathcal{B}_m$ with property $\mathcal{P}$, and an invertible T_{λ} such that (3.2) holds.

For $0 < \epsilon < 1$, let

$$\gamma = \frac{\epsilon}{\ell^2},$$

and let $\mathfrak{L}_{\le \ell} \mathcal{G}_{\gamma}$ be the set of the linear combinations of at most ℓ elements of $\mathcal{G}_{\gamma}$ with coefficients that do not exceed ℓ in modulus. It is clear that

$$\mathfrak{L}_{\le \ell} \mathcal{G}_{\gamma} \subset \mathcal{B}_{m\ell^2},$$

and being a non-empty subset of $\mathcal{L}_{G_{\gamma}}$, the set $\mathfrak{L}_{\le \ell} \mathcal{G}_{\gamma}$ has property $\mathcal{P}$.

Since for any α_i of modulus at most r and operators A_i and B_i:

$$\|\sum_{i=1}^{k} \alpha_i A_i - \sum_{i=1}^{k} \alpha_i B_i\| \le rk \max_i \|A_i - B_i\|,$$

every linear combination of at most ℓ elements of $T_{\gamma}^{-1} \mathcal{K} T_{\gamma}$ with coefficients that do not exceed ℓ in modulus, is in $\mathcal{N}_{\epsilon}\left(\mathcal{L}_{G_{\gamma}}\right)$. In particular

$$T_{\gamma}^{-1} \mathcal{F} T_{\gamma} \subset \mathcal{N}_{\epsilon}\left(\mathcal{L}_{G_{\gamma}}\right),$$

and we are done.

(3) Since the algebra generated by $\mathcal{C}$ is the subspace generated by the semigroup generated by $\mathcal{C}$, the claim about algebras follows immediately from the already verified claims about semigroups and linear spans.

□

Observation 3.5.

(1) It is easy to see that if in addition to the hypothesis of Key Lemma 3.4 property $\mathcal{P}$ is maintained by a collection after the identity is adjoint to it, then the result of our Key Lemma still holds if one replaces words "semigroup" and "algebra" in it by "unital semigroup" and "unital algebra".

(2) If in giving the definitions of Terminology 3.1 one insists that the (bounded) similarities come from a particular collection thereof (as opposed to being unrestricted), the validity of Key Lemma 3.4 is not affected.

In [1] Jafarian, Radjavi, Rosenthal and Sourour proved the following strengthening of Proposition 3.2, which is stated here in our terminology.

Theorem 3.6 ([1]). *If a collection $\mathcal{C}$ of compact operators in $B(X)$ is Tukey-close b.f.v.s. to being a commuting collection, then $\mathcal{C}$ is triangularizable.*

The key to our quick proof is the following result of Radjavi, Rosenthal and Shulman.[3]

Theorem 3.7 ([7]). *A semigroup $\mathcal{S}$ of compact operators in $B(X)$ is triangularizable if and only if $[A,B] \overset{def}{=} AB-BA$ is quasinilpotent for every $A,B \in \mathcal{S}$.*

Proof of Theorem 3.6. Let the property $\mathcal{P}$ be that of being a commuting collection. It is obvious that $\mathcal{P}$ passes to semigroups and non-empty subsets. By Key Lemma 3.4, the semigroup $\mathcal{S}_c$ is Tukey-close b.f.v.s. to being a commuting collection.

Suppose that $A,B \in \mathcal{S}_c$ are given. Then there is a ball $\mathcal{B}_m$ such that for every $n \in \mathbb{N}$ there is a commuting pair C_n, D_n of the operators in $\mathcal{B}_m$ and an invertible T_n such that

$$\begin{cases} \|T_n^{-1}AT_n - C_n\| \le \frac{1}{n} \\ \|T_n^{-1}BT_n - D_n\| \le \frac{1}{n} \end{cases}$$

Since sequences (C_n) and (D_n) are bounded, so are sequences $\left(T_n^{-1}AT_n\right)$ and $\left(T_n^{-1}BT_n\right)$, which allows us to conclude that

$$T_n^{-1}[A,B]T_n - [C_n,D_n] \longrightarrow 0;$$

i.e. that

$$T_n^{-1}[A,B]T_n \longrightarrow 0,$$

(since $[C_n,D_n]=0$). Since the spectral radius is dominated by operator norm, $[A,B]$ is quasinilpotent, and so an application of Theorem 3.7 completes the proof. □

[3] A corresponding result for algebras of operators was proved by Katavolos and Radjavi ten years earlier in [3].

Example 3.8. The following example in [1] shows that Theorem 3.6 fails even in finite dimensions if "b.f.v.s." is replaced by just "v.s." in the hypothesis.

Let $A_1 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $A_2 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, and $T_\epsilon = \left(\begin{smallmatrix} \epsilon & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $C_1^\epsilon = \begin{pmatrix} 0 & \frac{1}{\epsilon} \\ 0 & 0 \end{pmatrix}$ and $C_2^\epsilon = 0$. Then we have the following inequality for the operator norm:

$$\|T_\epsilon^{-1} A_i T_\epsilon - C_i^\epsilon\| \leq \epsilon;$$

which shows that $\{A_1, A_2\}$ is Tukey-close v.s. to being a commuting collection, but it is clear that $\{A_1, A_2\}$ is not triangularizable, since A_1 and A_2 share no eigenvectors.

On the other hand there is still a pleasant result of Yahaghi in the plain (i.e. non- "b.f./v.s.") approximation case.

Theorem 3.9 ([10])**.** *If a collection $\mathcal{C}$ of compact operators in $\mathcal{B}(\mathcal{X})$ is Tukey-close to being a triangularizable collection, then $\mathcal{C}$ is triangularizable.*

Proof. Here is a quick proof. Let the property $\mathcal{P}$ be that of being a triangularizable collection. It is obvious that $\mathcal{P}$ passes to semigroups and non-empty subsets. By Key Lemma 3.4, the semigroup $\mathcal{S}_\mathcal{C}$ is Tukey-close to being a triangularizable collection. Suppose that $A, B \in \mathcal{S}_\mathcal{C}$ are given.

Just as in the proof of Theorem 3.6 we conclude that there are sequences (C_n) and (D_n) such that

- C_n and D_n are simultaneously triangularizable for each n;
- $\|A - C_n\| \longrightarrow 0$,
- $\|B - D_n\| \longrightarrow 0$.

Since it is clear that (C_n) and (D_n) must be bounded, it follows that

$$[C_n, D_n] \longrightarrow [A, B],$$

so that $[A, B]$ is a limit of quasinilpotents, and being compact, is itself a quasinilpotent by the continuity of spectral radius on the compact operators. The proof is now complete by Theorem 3.7. □

We have not yet been able to answer the following question, which we pose here for the benefit of the reader.

Question 3.10. Does the conclusion of Yahaghi's theorem 3.9 remain valid if one replaces "Tukey-close" by "Tukey-close b.f.v.s." in the hypothesis?

3.2. **In a finite dimensional setting.** As we had mentioned in the introduction, Yahaghi extended Theorem 3.6 to the following result in a finite dimensional case.

Theorem 3.11 ([10])**.** *If a collection $\mathcal{C} \subset \mathbb{M}_n(\mathcal{C})$ is Tukey-close b.f.v.s to being a triangularizable collection, then $\mathcal{C}$ is triangularizable.*

To give a quick proof of Theorem 3.11 we "push off" the following result of Radjavi and Rosenthal.

Theorem 3.12 ([5]). *A unital algebra $\mathcal{A}$ of operators on a finite-dimensional space (over a field of characteristic other than 2) is triangularizable if and only if*

$$trace(AB - BA)^2 = 0$$

for all $A, B \in \mathcal{A}$.

Proof of Theorem 3.11. Let the property $\mathcal{P}$ be that of being a triangularizable collection. It is obvious that $\mathcal{P}$ passes to algebras and non-empty subsets, and is maintained when the identity matrix is adjoined to a collection. By Observation 3.5, the unital algebra $\mathcal{A}_c$ is Tukey-close b.f.v.s to being triangularizable. Suppose that $A, B \in \mathcal{A}_c$ are given.

We proceed just as in the proof of Theorem 3.6 to conclude that there is a sequence (T_n) of invertibles and bounded sequences (C_n) and (D_n) such that

- C_n and D_n are simultaneously triangularizable for each n;
- $\|T_n^{-1}AT_n - C_n\| \longrightarrow 0$,
- $\|T_n^{-1}BT_n - D_n\| \longrightarrow 0$.

Since sequences (C_n) and (D_n) are bounded, so are sequences $\left(T_n^{-1}AT_n\right)$ and $\left(T_n^{-1}BT_n\right)$, which allows us to conclude that

$$\|T_n^{-1}[A, B]^2 T_n - [C_n, D_n]^2\| \longrightarrow 0,$$

and so by the continuity of trace:

$$|trace(T_n^{-1}[A, B]^2 T_n) - trace([C_n, D_n]^2)| \longrightarrow 0.$$

Since $trace([C_n, D_n]^2) = 0$ and

$$trace(T_n^{-1}[A, B]^2 T_n) = trace([A, B]^2),$$

the desired conclusion follows by Lemma 3.12.

□

4. Approximation Conditions And Standard Triangularizability

The idea behind our Key Lemma 3.4 does not just lead to new ways of proving known results. In this section of the paper we use a similar approach to show that a collection of (*entry-wise*) non-negative matrices has a standard triangularization if and only if it can be approximated reasonably well by collections that have standard triangularizations.

A collection of non-negative matrices is said to have a *standard triangularization*, if after a simultaneous application of a similarity generated by a permutation matrix, all matrices in the collection become upper triangular.

Not all non-negative matrices have a standard triangularization individually. The following result characterizes those that do, and yields the fact that the set of such matrices is closed, which we prove below, for the sake of completeness.

Theorem 4.1 (Theorem 5.1.7 in [6]). *A non-negative matrix has a standard triangularization if and only if it becomes nilpotent upon replacement of its diagonal entries by zeros.*

Corollary 4.2. *The set of non-negative matrices in $\mathbb{M}_n$ that possess a standard triangularization is closed.*

Proof. We offer two proofs: the first one uses our methodology, while the second one does not.

(*First proof.*) Obviously since norm convergence in $\mathbb{M}_n$ entails entry-wise convergence, the set of non-negative matrices is closed. Furthermore, since the spectral radius is continuous on $\mathbb{M}_n$, the set of nilpotent matrices is closed.

For a matrix $A \in \mathbb{M}_n$, let $\mathcal{D}(A)$ stand for the diagonal matrix whose diagonal coincides with that of A. We call $\mathcal{D}(A)$ *the diagonal compression of* A. Note that $\mathcal{D} : \mathbb{M}_n \longrightarrow \mathbb{M}_n$ is a continuous map.

Suppose that (A_m) is a sequence of non-negative $n \times n$ matrices, each of which has a standard triangularization, and that $A_n \longrightarrow B$. Then B is non-negative, and

$$\mathcal{D}(A_n) \longrightarrow \mathcal{D}(B),$$

so that

$$A_n - \mathcal{D}(A_n) \longrightarrow B - \mathcal{D}(B).$$

By Theorem 4.1 each $A_n - \mathcal{D}(A_n)$ is nilpotent, and hence so is $B - \mathcal{D}(B)$. One more application of Theorem 4.1 completes the proof.

(*Second proof.*) There are $n!$ permutation matrices in $\mathbb{M}_n$. Given a convergent sequence (A_m) of non-negative $n \times n$ matrices, each of which has a standard triangularization $T_m = P_m^{-1} A_m P_m$, where P_m is a permutation matrix, we can pass to a subsequence (A_{m_k}) such that (P_{m_k}) is a constant sequence (P). Clearly that (T_{m_k}) is convergent. Since the limit of (T_{m_k}) is a non-negative upper triangular matrix T, the limit of (A_{m_k}), and thus of (A_m), is PTP^{-1}, and so is a non-negative matrix possessing a standard triangularization. □

Our "push-off" result in the present context is the following.

Theorem 4.3 (Lemma 5.1.3 in [6]). *A semigroup of non-negative matrices has a standard triangularization if and only if the diagonal compression $\mathcal{D}$ is multiplicative on the semigroup.*

Observation 4.4. As we have noted in Observation 3.5, our Key Lemma 3.4 will remain valid if we were to require that the similarities involved in the definitions given within Terminology 3.1 be induced by the invertible non-negative matrices T with non-negative inverses. The matrix of each such T

is a product of a permutation matrix and an invertible non-negative diagonal matrix, and such T generate the only similarities that leave the whole cone of the non-negative matrices invariant. *For the rest of this section we shall deal with this scenario exclusively.*

Our most substantial new result is the following.

Theorem 4.5. *If $\mathcal{C} \subset \mathbb{M}_n$ is a collection of non-negative matrices which is Tukey-close b.f.v.s.*[4] *to being a non-negative collection with a standard triangularization, then $\mathcal{C}$ has a standard triangularization.*

Proof. Let the property $\mathcal{P}$ be that of having a standard triangularization. Clearly $\mathcal{P}$ passes to semigroups and non-empty subsets. Hence we can invoke our Key Lemma and assume without loss of generality that $\mathcal{C}$ is a semigroup.

It is also obvious that this property $\mathcal{P}$ is invariant under an application of a similarity generated by a permutation matrix. Since a permutation matrix is an isometry with respect to the operator norm, for any permutation P:

$$P^{-1}D^{-1}\mathcal{C}DP \subset \mathcal{N}_\epsilon(\mathcal{G}) \Longleftrightarrow D^{-1}\mathcal{C}D \subset \mathcal{N}_\epsilon\left(P\mathcal{G}P^{-1}\right).$$

Thus we can assume that $\mathcal{C}$ is Tukey-close b.f.v.s. to being a collection with a standard triangularization, via the similarities induced by the set of the non-negative invertible diagonal matrices.

Suppose that $A, B \in \mathcal{C}$ are given. We proceed just as in the proof of Theorem 3.6 to conclude that there is a sequence (T_n) of non-negative invertible diagonal matrices and bounded sequences (C_n) and (D_n) of non-negative matrices such that

- C_n and D_n have a common standard triangularization for each n;
- $\|T_n^{-1}AT_n - C_n\| \longrightarrow 0$,
- $\|T_n^{-1}BT_n - D_n\| \longrightarrow 0$.

In particular, since

$$\mathcal{D}(T_n^{-1}AT_n) = \mathcal{D}(A),$$

and a similar statement holds for B, we conclude that

$$\mathcal{D}(C_n) \longrightarrow \mathcal{D}(A),$$

and

$$\mathcal{D}(D_n) \longrightarrow \mathcal{D}(B),$$

as the linear map $\mathcal{D}$ is contractive with respect to the operator norm.

Since sequences (C_n) and (D_n) are bounded, so are sequences $\left(T_n^{-1}AT_n\right)$ and $\left(T_n^{-1}BT_n\right)$. Thus

$$\|T_n^{-1}ABT_n - C_nD_n\| \longrightarrow 0,$$

from where

$$\mathcal{D}(C_nD_n) \longrightarrow \mathcal{D}(AB).$$

[4] In the sense described in Observation 4.4.

Yet

$$\mathcal{D}(C_n)\mathcal{D}(D_n) = \mathcal{D}(C_nD_n)$$

by Theorem 4.3, since we have assumed that C_n and D_n have a common standard triangularization for each n, and in particular so does the semigroup they generate.

Since sequences $(\mathcal{D}(C_n))$ and $(\mathcal{D}(D_n))$ are convergent,

$$\mathcal{D}(AB) \longleftarrow \mathcal{D}(C_nD_n) = \mathcal{D}(C_n)\mathcal{D}(D_n) \longrightarrow \mathcal{D}(A)\mathcal{D}(B),$$

so that

$$\mathcal{D}(AB) = \mathcal{D}(A)\mathcal{D}(B),$$

and the proof is complete by Theorem 4.3.

□

Note that Example 3.8 shows that we cannot drop "b.f." from the hypothesis of Theorem 4.5. On the other hand, we can drop "b.f.v.s." and arrive at a valid theorem (an analogue of Theorem 3.9).

Theorem 4.6. *If $\mathcal{C} \subset \mathbb{M}_n$ is a collection of non-negative matrices which is Tukey-close to being a non-negative collection with a standard triangularization, then $\mathcal{C}$ has a standard triangularization.*

Proof. Again, we offer two proofs: the first one uses our methodology, while the second one does not.

(*First proof.*) Let the property $\mathcal{P}$ be that of having a standard triangularization, so that $\mathcal{P}$ clearly passes to semigroups and non-empty subsets. Hence we can invoke our Key Lemma and assume without loss of generality that $\mathcal{C}$ is a semigroup.

Suppose that $A, B \in \mathcal{C}$. As before we can conclude that there are sequences (C_n) and (D_n) such that

- C_n and D_n have a common standard triangularization for each n;
- $\|A - C_n\| \longrightarrow 0$,
- $\|B - D_n\| \longrightarrow 0$.

Since it is clear that (C_n) and (D_n) must be bounded, it follows that

$$\mathcal{D}(AB) = \mathcal{D}(A)\mathcal{D}(B),$$

just as it did in the proof of Theorem 4.5, and so the proof is complete by Theorem 4.3.

(*Second proof.*) Select a finite basis $\Upsilon \subset \mathcal{C}$ for the *span* $\{\mathcal{C}\}$. It is sufficient to show that the elements of Υ have a common standard triangularization. By the hypothesis Υ is close to being a collection with a standard triangularization; for every $\epsilon > 0$ there is a permutation matrix P_ϵ and a set $\mathcal{G}_\epsilon$ of matrices such that $P_\epsilon^{-1}\mathcal{G}_\epsilon P_\epsilon$ is a set of upper-triangular matrices and

$$\Upsilon \subset \mathcal{N}_\epsilon\left(\mathcal{G}_\epsilon\right).$$

Since the set of $n \times n$ permutation matrices is finite, there is an increasing sequence $[n_k]$ of natural numbers and a permutation matrix P such that for every $k \in \mathbb{N}$: $P^{-1}\mathcal{G}_{\frac{1}{n_k}} P$ is a set of upper-triangular matrices. Note that

$$P^{-1}\Upsilon P \subset P^{-1}\mathcal{N}_{\frac{1}{n_k}}\left(\mathcal{G}_{\frac{1}{n_k}}\right)P = \mathcal{N}_{\frac{1}{n_k}}\left(P^{-1}\mathcal{G}_{\frac{1}{n_k}}P\right),$$

which shows that the modulus of every sub-diagonal entry of every matrix in $P^{-1}\Upsilon P$ is at most $\frac{1}{n_k}$. Thus all matrices in $P^{-1}\Upsilon P$ are upper-triangular, and this completes the proof. □

5. Some Topological Considerations

In studying the concepts of "closeness" involved in the previous sections we had become curious as to whether there might be a way to place these in a true topological setting. While the answer in general is in the negative (see Example 5.17 below), in the process of enquiry we were led to consider a corresponding natural type of topology on power sets of metric spaces.

Notation 5.1. Given a subset $\mathcal{T}$ of a metric space $(\mathcal{M}, \mu)$, the **ϵ-neighbourhood of $\mathcal{T}$** is the set

$$\mathcal{N}_\epsilon(\mathcal{T}) \stackrel{\text{def}}{=} \{\ A \in \mathcal{M} \mid \mu(A, T) < \epsilon, \quad \text{for some } T \in \mathcal{T}\ \} = \bigcup_{T \in \mathcal{T}} \mathcal{B}_\epsilon(T).$$

As is common, we write $\mathcal{B}_\epsilon(a)$ instead of $\mathcal{N}_\epsilon(\{a\})$.

Notation 5.2. When f is a function and Δ is a subset of its domain, we shall write $f[\Delta]$ for $\{\ f(x) \mid x \in \Delta\ \}$, with the understanding that $f[\emptyset] = \emptyset$.

Definition 5.3. Suppose that $\mathcal{S}$ is a unital collection of functions on the power set $\mathcal{P}(\mathcal{M})$ of a metric space $(\mathcal{M}, \mu)$, where the term "unital" refers to the fact that $\mathcal{S}$ contains the identity function $id_{\mathcal{P}(\mathcal{M})}$.

We define $\Phi_{\mathcal{S}} : \mathcal{P}\big(\mathcal{P}(\mathcal{M})\big) \longrightarrow \mathcal{P}\big(\mathcal{P}(\mathcal{M})\big)$ by:

$$\Phi_{\mathcal{S}}(\Omega) \stackrel{\text{def}}{=} \{\ K \subset \mathcal{M} \mid \forall \epsilon > 0 \ \ \exists (f, T) \in \mathcal{S} \times \Omega \quad \text{such that} \quad f[K] \subset \mathcal{N}_\epsilon(T)\ \}.$$

In other words, $\Phi_{\mathcal{S}}(\Omega)$ is the collection of those subsets of $\mathcal{M}$, which can be mapped by (functions in) $\mathcal{S}$ into arbitrarily-small-ϵ-neighbourhoods of sets in Ω.

We shall refer to $\Phi_{\mathcal{S}}(\Omega)$ as the **thickening of Ω via $\mathcal{S}$**. In the case $\mathcal{S} = \{id_{\mathcal{P}(\mathcal{M})}\}$, we omit an explicit reference to $\mathcal{S}$.

One reason for our terminology is that $\Omega \subset \Phi_{\mathcal{S}}(\Omega)$, and

$$K \in \Phi_{\mathcal{S}}(\Omega) \Longrightarrow \mathcal{P}(K) \subset \Phi_{\mathcal{S}}(\Omega).$$

Observation 5.4. If we write

$$\Phi^\epsilon_{\mathcal{S}}(\Omega) \stackrel{\text{def}}{=} \{\ K \subset \mathcal{M} \mid \exists (f, T) \in \mathcal{S} \times \Omega \quad \text{such that} \quad f[K] \subset \mathcal{N}_\epsilon(T)\ \},$$

then

$$\Phi_{\mathcal{S}}^{\epsilon_1}(\Omega) \subset \Phi_{\mathcal{S}}^{\epsilon_2}(\Omega), \quad \text{whenever} \;\; \epsilon_1 \le \epsilon_2,$$

and

$$\Phi_{\mathcal{S}}(\Omega) = \bigcap_{\epsilon>0} \Phi_{\mathcal{S}}^{\epsilon}(\Omega) = \bigcap_{n\in\mathbb{N}} \Phi_{\mathcal{S}}^{\alpha_n}(\Omega),$$

for any positive sequence (α_n) convergent to zero.

Let us connect these new concepts to those we have discussed in previous sections.

Observation 5.5. Suppose that $\mathcal{P}$ is a property pertaining to collections of operators in $(B(X), \| \, \|)$, and let us write

$$\Omega_{\mathcal{P}} \stackrel{\text{def}}{=} \left\{ \; \mathcal{Z} \subset B(X) \mid \mathcal{Z} \;\; \text{has property } \mathcal{P} \; \right\},$$

and

$$\Omega_{\mathcal{P}}^n \stackrel{\text{def}}{=} \left\{ \; \mathcal{Z} \subset \mathcal{B}_n \mid \mathcal{Z} \;\; \text{has property } \mathcal{P} \; \right\},$$

where $n \in \mathbb{N}$, and $\mathcal{B}_n$ is the open ball of $\| \, \|$-radius n, centered at the origin in $B(X)$. Obviously

$$\Omega_{\mathcal{P}} = \bigcup_{n\in\mathbb{N}} \Omega_{\mathcal{P}}^n.$$

Each of the following is now clear.

(1) A subset $\mathcal{C}$ of $B(X)$ is close to having property $\mathcal{P}$ if and only if it belongs to the thickening of $\Omega_{\mathcal{P}}$.
(2) A subset $\mathcal{C}$ of $B(X)$ is close to having property $\mathcal{P}$ similarity-wise if and only if $\mathcal{C}$ belongs to the thickening of $\Omega_{\mathcal{P}}$ via the group of the similarities induced by the invertible operators in $B(X)$.
(3) A subset $\mathcal{C}$ of $B(X)$ is close to having property $\mathcal{P}$ in a bounded fashion similarity-wise if and only if it belongs to the thickening of some $\Omega_{\mathcal{P}}^n$ via the group of the similarities induced by the invertible operators in $B(X)$.

Observation 5.6. Every unital collection $\mathcal{S}$ of functions on $\mathcal{M}$ induces a unital collection $\hat{\mathcal{S}}$ of functions on $\mathcal{P}(\mathcal{M})$, if for each $f \in \mathcal{S}$ we define $\hat{f} : \mathcal{P}(\mathcal{M}) \longrightarrow \mathcal{P}(\mathcal{M})$ by

$$\hat{f}(K) \stackrel{\text{def}}{=} f[K].$$

In order to simplify the notation in such a case we shall write $\Phi_{\mathcal{S}}$, rather than $\Phi_{\hat{\mathcal{S}}}$, and rely on the context to keep the concepts straight.

Terminology 5.7. A function $F : \mathcal{P}(\mathcal{Z}) \longrightarrow \mathcal{P}(\mathcal{Z})$ is said be *a topological closure operation on* $\mathcal{Z}$ if there is a topology $\mathcal{T}$ on $\mathcal{Z}$ such that for each $U \subset \mathcal{Z}$: $F(U)$ is the $\mathcal{T}$-closure of U.

Theorem 5.8. *If $\mathcal{S}$ is a unital uniformly equicontinuous semigroup of functions on a metric space $(\mathcal{M}, \mu)$, then $\Phi_{\mathcal{S}}$ is a topological closure operation on $\mathcal{P}(\mathcal{M})$.*

Proof. It is well-known (see for example Theorem 3.7 in [9]) that to settle the claim it is enough to demonstrate that $\Phi_{\mathcal{S}} : \mathcal{P}(\mathcal{P}(\mathcal{M})) \longrightarrow \mathcal{P}(\mathcal{P}(\mathcal{M}))$ satisfies the following criteria:

1: $\Phi_{\mathcal{S}}(\emptyset) = \emptyset$;
2: $\Omega \subset \Phi_{\mathcal{S}}(\Omega)$;
3: $\Phi_{\mathcal{S}}(\Phi_{\mathcal{S}}(\Omega)) = \Phi_{\mathcal{S}}(\Omega)$;
4: $\Phi_{\mathcal{S}}(\Omega \cup \Gamma) = \Phi_{\mathcal{S}}(\Omega) \cup \Phi_{\mathcal{S}}(\Gamma)$.

It is not difficult to show that the following is an equivalent set of criteria:

1': $\Phi_{\mathcal{S}}(\emptyset) = \emptyset$;
2': $\Omega \subset \Phi_{\mathcal{S}}(\Omega)$;
3': $\Gamma \subset \Phi_{\mathcal{S}}(\Omega) \Longrightarrow \Phi_{\mathcal{S}}(\Gamma) \subset \Phi_{\mathcal{S}}(\Omega)$;
4': $\Phi_{\mathcal{S}}(\Omega \cup \Gamma) \subset \Phi_{\mathcal{S}}(\Omega) \cup \Phi_{\mathcal{S}}(\Gamma)$.

The claims in $1'$ and $2'$ are obviously true.

The next easiest claim to verify is that of $4'$. Suppose that $K \in \Phi_{\mathcal{S}}(\Omega \cup \Gamma)$. Then for each $n \in \mathbb{N}$ there exist $f_n \in \mathcal{S}$ and $T_n \in \Omega \cup \Gamma$ such that

$$f_n[K] \subset \mathcal{N}_{\frac{1}{n}}(T_n).$$

Since each T_n is an element of either Ω or Γ, without any loss of generality, we can assume that there is a subsequence $\left(\frac{1}{n_k}\right)$ of $\left(\frac{1}{n}\right)$ such that

$$T_{n_k} \in \Omega, \quad \text{for all} \quad k \in \mathbb{N}.$$

It follows that

$$K \in \bigcap_{k \in \mathbb{N}} \Phi_{\mathcal{S}}^{\frac{1}{n_k}}(\Omega) = \Phi_{\mathcal{S}}(\Omega),$$

and thus the claim of $4'$ holds true under our hypotheses.

Finally, let us verify the condition $3'$. Suppose that $\Gamma \subset \Phi_{\mathcal{S}}(\Omega)$ and $K \in \Phi_{\mathcal{S}}(\Gamma)$. We shall demonstrate that for every $\epsilon > 0$: $K \in \Phi_{\mathcal{S}}^{\epsilon}(\Omega)$, which is enough for our purposes by Observation 5.4.

Suppose that a positive ϵ is given. Then by the uniform equicontinuity of $\mathcal{S}$ there is a positive δ such that for all $f \in \mathcal{S}$:

$$\mu(u, w) < \delta \Longrightarrow \mu(f(u), f(w)) < \frac{\epsilon}{2}.$$

From this it follows that for any subset L of $\mathcal{M}$ and any $f \in \mathcal{S}$:

$$f\left[\mathcal{N}_{\delta}(L)\right] \subset \mathcal{N}_{\frac{\epsilon}{2}}\left(f[L]\right). \tag{5.1}$$

Since $K \in \Phi_{\mathcal{S}}^{\delta}(\Gamma)$, there is some $f \in \mathcal{S}$ and $T \in \Gamma$, such that

$$f[K] \subset \mathcal{N}_{\delta}(T).$$

Since $T \in \Gamma \subset \Phi_{\mathcal{S}}(\Omega)$, there exist $g \in \mathcal{S}$ and $U \in \Omega$ such that

$$g[T] \subset \mathcal{N}_{\frac{\epsilon}{2}}(U).$$

Then

$$T \subset g^{-1}\left[\mathcal{N}_{\frac{\epsilon}{2}}(U)\right] \quad (\stackrel{\text{def}}{=} W).$$

Thus

$$\begin{aligned} f[K] \subset \mathcal{N}_{\delta}(T) &\subset \mathcal{N}_{\delta}(W) \\ &\subset g^{-1}\left[g\left[\mathcal{N}_{\delta}(W)\right]\right] \\ &\subset g^{-1}\left[\mathcal{N}_{\frac{\epsilon}{2}}(g[W])\right] \\ &\subset g^{-1}\left[\mathcal{N}_{\frac{\epsilon}{2}}\left(\mathcal{N}_{\frac{\epsilon}{2}}(U)\right)\right] \\ &\subset g^{-1}\left[\mathcal{N}_{\epsilon}(U)\right]. \end{aligned}$$

This shows that

$$(g \circ f)[K] \subset \mathcal{N}_{\epsilon}(U),$$

which, since $g \circ f \in \mathcal{S}$, demonstrates that $K \in \Phi^{\epsilon}_{\mathcal{S}}(\Omega)$, and the proof is complete. □

A trivial application of Theorem 5.8 to the subject of our Section 3 is that the concept of "being close to having property $\mathcal{P}$" is that of a topological closure. (As we shall see soon, the same is true for "being Tukey-close to having property $\mathcal{P}$".) Indeed, one simply takes $\{id_{B(X)}\}$ to be the unital uniformly equicontinuous semigroup in Theorem 5.8.

Unfortunately, it turns out that the uniformly equicontinuous semigroups of similarities on $\mathbb{M}_n$ are rather limited, and so Theorem 5.8 does not offer any additional insight into the concepts of "being close to having property $\mathcal{P}$ b.f./v.s." even in finite dimensions. We will demonstrate in Example 5.17 that when $\mathcal{P}$ is the property of being traingularizable, already in $\mathbb{M}_2(\mathcal{C})$ "being close to having property $\mathcal{P}$ b.f./v.s." does not correspond to a topological closure.

Terminology 5.9. The similarity ν on $\mathbb{M}_n$ is said to be a *unitary similarity* if there exists a unitary $U \in \mathbb{M}_n$ such that

$$\nu(A) = U^*AU.$$

Theorem 5.10. *A semigroup $\mathcal{C}$ of similarities on $\mathbb{M}_n$ is uniformly equicontinuous on $\mathbb{M}_n$ if and only if there exists a similarity μ on $\mathbb{M}_n$ such that $\mu^{-1} \circ \mathcal{C} \circ \mu$ is a semigroup of unitary similarities.*

Proof. We demonstrate the forward implication only, and leave the reverse implication as an easy exercise for the reader. Our proof follows the path of the proof of Lemma 3.1.6 in [6], and is presented here for completeness and for the convenience of the reader.

A semigroup $\mathcal{C}$ of similarities on $\mathbb{M}_n$ is generated by a sub-semigroup $\mathcal{S}$ of GL_n, and such an $\mathcal{S}$ can be assumed to be closed under multiplication by non-zero scalars.

A family of (bounded) operators on a normed space is uniformly equicontinuous if and only if it is operator-norm-bounded. Thus $\mathcal{C}$ is uniformly equicontinuous on $\mathbb{M}_n$ exactly when the set

$$\left\{ \ \|T^{-1}\| \cdot \|T\| \ \middle| \ T \in \mathcal{S} \ \right\}$$

is bounded.

Assuming this is the case, for each $T \in \mathcal{S}$: $\{ \ \|T^{-m}\| \cdot \|T^m\| \mid m \in \mathbb{N} \ \}$ is a bounded set. On the other hand,

$$\|T^m\| \geq \max\{ \ |\lambda|^m \mid \lambda \in \sigma(T) \ \} = \max\{ \ |\lambda| \mid \lambda \in \sigma(T) \ \}^m,$$

and similarly

$$\|T^{-m}\| \geq \max\left\{ \ |\alpha|^m \mid \alpha \in \sigma(T^{-1}) \ \right\} = \frac{1}{\min\{ \ |\lambda| \mid \lambda \in \sigma(T) \ \}^m},$$

from where we see that the sequence

$$\left(\frac{\max\{ \ |\lambda| \mid \lambda \in \sigma(T) \ \}}{\min\{ \ |\lambda| \mid \lambda \in \sigma(T) \ \}}\right)^m$$

is bounded, and so it must be that

$$\max\{ \ |\lambda| \mid \lambda \in \sigma(T) \ \} = \min\{ \ |\lambda| \mid \lambda \in \sigma(T) \ \}.$$

Hence it must be that the spectrum of each element T of $\mathcal{S}$ lies on a circle of positive radius $\rho(T)$, i.e. the spectral radius of T.

Thus

$$\rho(T) = \sqrt[n]{|\det(T)|},$$

and therefore ρ is multiplicative on $\mathcal{S}$, i.e.,

$$\rho(AB) = \rho(A)\rho(B), \quad \text{for all} \quad A, B \in \mathcal{S}.$$

In particular,

$$\mathcal{S} = \mathbb{R}^+\mathfrak{S} \stackrel{\text{def}}{=} \{ \ tA \mid t > 0, \ A \in \mathfrak{S} \ \},$$

where $\mathfrak{S}$ is the semigroup

$$\left\{ \ \frac{T}{\rho(T)} \ \middle| \ T \in \mathcal{S} \ \right\}.$$

Clearly $\mathfrak{S}$ generates the same semigroup $\mathcal{C}$ of similarities on $\mathbb{M}_n$ as does $\mathcal{S}$. Furthermore,

(5.2)
$$\left\{ \ \|T^{-1}\| \cdot \|T\| \ \middle| \ T \in \mathfrak{S} \ \right\} = \left\{ \ \|T^{-1}\| \cdot \|T\| \ \middle| \ T \in \mathcal{S} \ \right\} = \text{a bounded set},$$

and the spectrum of every element of $\mathfrak{S}$ belongs to the unit circle. It follows that the spectrum of the inverse of each element of $\mathfrak{S}$ belongs to the unit circle, and therefore the norm of the inverse of each element of $\mathfrak{S}$ is at least 1, and so it follows from (5.2) that $\mathfrak{S}$ is bounded. By the continuity of the spectrum, every matrix in the norm-closure $\overline{\mathfrak{S}}$ has its spectrum on the unit circle as well.

In particular, for each $T \in \overline{\mathfrak{S}}$ the sequence $\|T^m\|$ is bounded, and so the same must be true for every Jordan block in the canonical Jordan form of T, which, since the spectrum of T belongs to the unit circle, can only happen if each Jordan block of T is 1×1. This shows that every T in $\overline{\mathfrak{S}}$ is similar to a unitary matrix.

Our next step is to show that the compact semigroup $\overline{\mathfrak{S}}$ is a group. To verify the claim we must show that $\overline{\mathfrak{S}}$ contains the inverse of each of its elements. Given $T = S^{-1}US \in \overline{\mathfrak{S}}$, where U is a unitary matrix, by the classical Dirichlet's Simultaneous Approximation Theorem, sequence $\left(U^{m}\right)_{m=1}^{\infty}$ has a subsequence $\left(U^{m_j}\right)_{j=1}^{\infty}$ that converges to the identity matrix. Thus

$$\left(T^{m_j}\right)_{j=1}^{\infty} \longrightarrow I_n,$$

which shows that

$$\left(T^{m_j-1}\right)_{j=2}^{\infty} \longrightarrow T^{-1},$$

and since $\overline{\mathfrak{S}}$ is closed, we have $T^{-1} \in \overline{\mathfrak{S}}$, as claimed.

Since it is well known (see for example Theorem 3.1.5 in [6]) that in GL_n every bounded group is simultaneously similar to a group of unitary matrices, we conclude that for some $S \in GL_n$, $S^{-1}\mathfrak{S}S$ is a sub-semigroup of a group $S^{-1}\overline{\mathfrak{S}}S$ of unitary matrices.

The desired conclusion follows using the similarity

$$\mu(A) = S^{-1}AS.$$

□

Notation 5.11. Let us denote by $\{\mathcal{M}\}_{\mathfrak{F}}$ the set of all finite subsets of $\mathcal{M}$, and in general, for $\Gamma \subset \mathcal{P}(\mathcal{M})$, let $\Gamma_{\mathfrak{F}}$ denote the collection of all finite subsets of the elements of Γ. Clearly $\Gamma \cap \{\mathcal{M}\}_{\mathfrak{F}}$ is the collection of all finite sets in Γ.

Definition 5.12. Given a unital collection $\mathcal{S}$ of functions on the power set $\mathcal{P}(\mathcal{M})$ of a metric space $(\mathcal{M}, \mu)$, let us define a function $\Psi_{\mathcal{S}} : \mathcal{P}\big(\mathcal{P}(\mathcal{M})\big) \longrightarrow \mathcal{P}\big(\mathcal{P}(\mathcal{M})\big)$ by:

$$\Psi_{\mathcal{S}}(\Omega) \stackrel{\text{def}}{=} \begin{cases} \left\{ K \subset \mathcal{M} \mid \{K\}_{\mathfrak{F}} \subset \Phi_{\mathcal{S}}(\Omega) \right\}, & \text{if } \Omega \neq \emptyset \\ \emptyset, & \text{if } \Omega = \emptyset. \end{cases}$$

We shall refer to $\Psi_{\mathcal{S}}(\Omega)$ as the **Tukey thickening of Ω via $\mathcal{S}$**, and note that when Ω is not empty, $\Psi_{\mathcal{S}}(\Omega)$ contains a subset K of $\mathcal{M}$ exactly when the thickening of Ω via $\mathcal{S}$ contains all finite subsets of K.

The reference to Tukey should be self-explanatory now, since it is clear that for any $\emptyset \neq \Omega \subset \mathcal{P}(\mathcal{M})$, $\Psi_{\mathcal{S}}(\Omega)$ is a set of finite character, and in particular, by Tukey's Lemma, contains a maximal element with respect to inclusion.

If we continue with the notation and terminology introduced in Observation 5.5, we see that a subset $\mathcal{C}$ of $B(X)$ is Tukey-close to having property $\mathcal{P}$ (v.s, b.f.v.s) if and only if it belongs to the Tukey thickening of $\Omega_{\mathcal{P}}$ via $\{id_{\mathcal{P}(\mathcal{M})}\}$ (resp. the group of (bounded) similarities and $\Omega^n_{\mathcal{P}}$).

Observation 5.13. We offer the following observations, the proofs of which are left as an exercise for the reader.

(1) $\Phi_{\mathcal{S}}(\emptyset) = \emptyset = \Psi_{\mathcal{S}}(\emptyset)$;
(2) $\Omega \subset \Phi_{\mathcal{S}}(\Omega) \subset \Psi_{\mathcal{S}}(\Omega)$; (the inclusions may be strict);
(3) $K \in \Psi_{\mathcal{S}}(\Omega) \Longrightarrow \mathcal{P}(K) \subset \Psi_{\mathcal{S}}(\Omega)$;
(4) If each element of Ω is a subset of an element of Γ, then $\Phi_{\mathcal{S}}(\Omega) \subset \Phi_{\mathcal{S}}(\Gamma)$ and $\Psi_{\mathcal{S}}(\Omega) \subset \Psi_{\mathcal{S}}(\Gamma)$;
(5) $\Psi_{\mathcal{S}}(\Omega_{\mathfrak{F}}) \cap \{\mathcal{M}\}_{\mathfrak{F}} = \Psi_{\mathcal{S}}(\Omega) \cap \{\mathcal{M}\}_{\mathfrak{F}} = \Phi_{\mathcal{S}}(\Omega) \cap \{\mathcal{M}\}_{\mathfrak{F}} = \Phi_{\mathcal{S}}(\Omega_{\mathfrak{F}}) \cap \{\mathcal{M}\}_{\mathfrak{F}} \subset \Phi_{\mathcal{S}}(\Omega_{\mathfrak{F}}) \subset \Phi_{\mathcal{S}}(\Omega) \subset \Psi_{\mathcal{S}}(\Omega) = \Psi_{\mathcal{S}}(\Omega_{\mathfrak{F}})$; (the inclusions may be strict).

Theorem 5.14. *If $\Phi_{\mathcal{S}}$ is a topological closure operation on the power set $\mathcal{P}(\mathcal{M})$, then so is $\Psi_{\mathcal{S}}$.*

Proof. At this point we only need to verify the following claims:

3”: $\Gamma \subset \Psi_{\mathcal{S}}(\Omega) \Longrightarrow \Psi_{\mathcal{S}}(\Gamma) \subset \Psi_{\mathcal{S}}(\Omega)$;
4”: $\Psi_{\mathcal{S}}(\Omega \cup \Gamma) \subset \Psi_{\mathcal{S}}(\Omega) \cup \Psi_{\mathcal{S}}(\Gamma)$.

[3”]: From the Observation 5.13 we know that

$$\begin{aligned} \Gamma \subset \Psi_{\mathcal{S}}(\Omega) &\Longrightarrow \Gamma_{\mathfrak{F}} \subset \Psi_{\mathcal{S}}(\Omega) \\ &\Longleftrightarrow \Gamma_{\mathfrak{F}} \subset \Psi_{\mathcal{S}}(\Omega) \cap \{\mathcal{M}\}_{\mathfrak{F}} = \Phi_{\mathcal{S}}(\Omega) \cap \{\mathcal{M}\}_{\mathfrak{F}} \\ &\Longleftrightarrow \Gamma_{\mathfrak{F}} \subset \Phi_{\mathcal{S}}(\Omega), \end{aligned}$$

and

$$\Psi_{\mathcal{S}}(\Gamma) \subset \Psi_{\mathcal{S}}(\Omega) \Longleftrightarrow \Psi_{\mathcal{S}}(\Gamma_{\mathfrak{F}}) \subset \Psi_{\mathcal{S}}(\Omega).$$

Hence it is sufficient to demonstrate that

$$\Gamma_{\mathfrak{F}} \subset \Phi_{\mathcal{S}}(\Omega) \Longrightarrow \Psi_{\mathcal{S}}(\Gamma_{\mathfrak{F}}) \subset \Psi_{\mathcal{S}}(\Omega).$$

To this end, suppose that $\Gamma_{\mathfrak{F}} \subset \Phi_{\mathcal{S}}(\Omega)$, and that $K \in \Psi_{\mathcal{S}}(\Gamma_{\mathfrak{F}})$. Then

$$\{K\}_{\mathfrak{F}} \subset \Phi_{\mathcal{S}}(\Gamma_{\mathfrak{F}}) \subset \Phi_{\mathcal{S}}\big(\Phi_{\mathcal{S}}(\Omega)\big) = \Phi_{\mathcal{S}}(\Omega),$$

and therefore

$$K \in \Psi_{\mathcal{S}}(\Omega),$$

as required.

[4”]: Suppose that $K \in \Psi_{\mathcal{S}}(\Omega \cup \Gamma)$. We aim to show that $K \in \Psi_{\mathcal{S}}(\Omega) \cup \Psi_{\mathcal{S}}(\Gamma)$. If $K \in \Psi_{\mathcal{S}}(\Gamma)$, we are done; so suppose that K has a finite subset P_0 such that $P_0 \notin \Phi_{\mathcal{S}}(\Gamma)$. Then $P_0 \in \Phi_{\mathcal{S}}(\Omega)$, because

$$\{K\}_{\mathfrak{F}} \subset \Phi_{\mathcal{S}}(\Omega \cup \Gamma) = \Phi_{\mathcal{S}}(\Omega) \cup \Phi_{\mathcal{S}}(\Gamma).$$

We claim that in such a case $\{K\}_{\mathfrak{F}} \subset \Phi_{\mathcal{S}}(\Omega)$; i.e. $K \in \Psi_{\mathcal{S}}(\Omega)$.

Suppose that P is a finite subset of K. Then $P \cup P_0$ is also a finite subset of K, and therefore

$$P \cup P_0 \in \Phi_{\mathcal{S}}(\Omega) \cup \Phi_{\mathcal{S}}(\Gamma).$$

If it were the case that $P \cup P_0 \in \Phi_{\mathcal{S}}(\Gamma)$, then every subset of $P \cup P_0$, including P_0, would be an element of $\Phi_{\mathcal{S}}(\Gamma)$, by Observation 5.13. Since $P_0 \notin \Phi_{\mathcal{S}}(\Gamma)$, we conclude that $P \cup P_0 \in \Phi_{\mathcal{S}}(\Omega)$, and thus every subset of $P \cup P_0$, including P, is an element of $\Phi_{\mathcal{S}}(\Omega)$. This completes the proof of the claim and of the whole theorem. $\square$

As we have mentioned at the start of this section, the concepts of "closeness" used in the preceding sections do not in general correspond to a closure with respect to a topology on $\mathcal{P}(\mathcal{P}(\mathbb{M}_n))$. We will demonstrate this in Example 5.17 below, but first we state two results that will aid us in that task.

In [1] it has been shown that if $\mathcal{C} \subset \mathbb{M}_n$ is Tukey-close v.s. to being diagonal, then it is Tukey-close b.f.v.s. to being diagonal as well. With this observation in hand the following result becomes an immediate corollary to Theorem 3.6.

Corollary 5.15 ([1])**.** *If $\mathcal{C} \subset \mathbb{M}_n$ is Tukey-close v.s. to being diagonal, then $\mathcal{C}$ is triangularizable.*

Corollary 5.15 is in fact an "if and only if" result, due to the following theorem of Perron, a proof of which can be found in [6] (Theorem 1.6.2).

Theorem 5.16 ([4])**.** *Let $\mathcal{U}$ the the algebra of all strictly upper-triangular matrices in $\mathbb{M}_n$. The operator norm of the similarity transformation $A \longrightarrow T_\alpha^{-1} A T_\alpha$ induced on $\mathcal{U}$ by a diagonal matrix $T_\alpha = diag(\alpha, \alpha^2, \ldots, \alpha^n)$ can be made as small as desired via a judicious choice of a positive α.*

In particular, for every $\epsilon > 0$ there is some $\alpha > 0$ such that

$$\|T_\alpha^{-1} A T_\alpha - \mathcal{D}(A)\| \leq \epsilon \|A\|$$

for all upper-triangular $A \in \mathbb{M}_n$.

It is pleasant to observe that a strong converse to the second claim of Perron's theorem holds true. If for every subset $\{A_1, A_2, A_3, \ldots, A_m\}$ of $\mathcal{C} \subset \mathbb{M}_n$ and every $\epsilon > 0$ there exists an invertible matrix T such that

$$\|T^{-1} A_i T - \mathcal{D}(A_i)\| \leq \epsilon \|A_i\|,$$

then $\mathcal{C}$ is triangularizable.

Indeed, in such a case $\left\{ \frac{A}{\|A\|} \;\middle|\; 0 \neq A \in \mathcal{C} \right\}$ is Tukey-close v.s. to being diagonal, and so is triangularizable by Corollary 5.15.

In particular, a bounded family $\mathcal{C} \subset \mathbb{M}_n$ is Tukey-close v.s. to being diagonal if and only if $\mathcal{C}$ (is triangularizable and) is close v.s. to being diagonal.

Example 5.17. Consider the metric space $(\mathbb{M}_2(\mathcal{C}), \|\ \|)$, and let $\mathcal{S}$ be the group of all similarities on $\mathbb{M}_2(\mathcal{C})$. Let us demonstrate that $\Psi_{\mathcal{S}}$ (and consequently $\Phi_{\mathcal{S}}$) is not a topological closure operation. It shall be sufficient to demonstrate that

$$\Psi_{\mathcal{S}}(\Psi_{\mathcal{S}}(\{\{0\}\})) \neq \Psi_{\mathcal{S}}(\{\{0\}\}).$$

We shall accomplish this by showing that

$$\left\{\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right\} \in \Psi_{\mathcal{S}}(\Psi_{\mathcal{S}}(\{\{0\}\})) \setminus \Psi_{\mathcal{S}}(\{\{0\}\}).$$

To this end, let us first argue that $\Psi_{\mathcal{S}}(\{\{0\}\})$ is the collection of all triangularizable sets of nilpotent matrices.

By Corollary 5.15, every set in $\Psi_{\mathcal{S}}(\{\{0\}\})$ is triangularizable, and by an observation we had made in the proof of Proposition 3.2, every set in $\Psi_{\mathcal{S}}(\{\{0\}\})$ consists of nilpotents. This shows the inclusion in one direction. The reverse inclusion is a consequence of Perron's theorem (Theorem 5.16).

Now, let $\mathcal{C}_\epsilon = \left\{\begin{bmatrix}0 & 0\\ \frac{2}{\epsilon} & 0\end{bmatrix}\right\}$. Then clearly $\mathcal{C}_\epsilon \in \Psi_{\mathcal{S}}(\{\{0\}\})$, for every positive ϵ, and

$$\begin{bmatrix}1 & \frac{\epsilon}{2}\\ 1 & -\frac{\epsilon}{2}\end{bmatrix}^{-1}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & \frac{\epsilon}{2}\\ 1 & -\frac{\epsilon}{2}\end{bmatrix} = \begin{bmatrix}0 & \frac{\epsilon}{2}\\ \frac{2}{\epsilon} & 0\end{bmatrix} \in \mathcal{N}_{\frac{\epsilon}{2}}\left(\mathcal{C}_\epsilon\right).$$

This shows that

$$\left\{\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right\} \in \Psi_{\mathcal{S}}(\Psi_{\mathcal{S}}(\{\{0\}\})).$$

Since $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ is not nilpotent,

$$\left\{\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right\} \notin \Psi_{\mathcal{S}}(\{\{0\}\}).$$

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