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ADDENDUM TO: ”INFINITE-DIMENSIONAL VERSIONS OF THE PRIMARY, CYCLIC AND JORDAN DECOMPOSITIONS”, BY M. RADJABALIPOUR

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(Communicated by Peter Rosenthal)

Dedicated to Professor Heydar Radjavi on his 80th birthday, whose uniqueness is as precious as his existence.

Abstract. In his paper mentioned in the title, which appears in the same issue of this journal, Mehdi Radjabalipour derives the cyclic decomposition of an algebraic linear transformation. A more general structure theory for linear transformations appears in Irving Kaplansky's lovely 1954 book on infinite abelian groups. We present a translation of Kaplansky's results for abelian groups into the terminology of linear transformations. We add an additional translation of a ring-theoretic result to give a characterization of algebraically hyporeflexive transformations and the strict closure of the set of polynomials in a transformation T.
Keywords: Abelian group, PID, module, cyclic, torsion, locally algebraic, hyporeflexive, scalar-reflexive ring, strict topology.
MSC(2010): Primary 05C38, 15A15; Secondary 05A15, 15A18.

1. **Introduction**

In his paper [9], which appears in the same issue of this journal, Mehdi Radjabalipour gives a very short and clear elegant proof of the cyclic decomposition of an algebraic linear transformation on a vector space V over a field $\mathbb{F}$. This is part of a larger structure theory for linear transformations that first appeared in the framework of abelian groups. We know that abelian groups are precisely the modules over the ring $(\mathbb{Z}, +, \cdot)$ of integers. Over sixty years ago, in his lovely book [5] on infinite abelian groups, Irving Kaplansky described how most of the important structure theorems translate to analogues for modules over a principal ideal domain (PID). Kaplansky also pointed out, when $\mathbb{F}$ is a field, that there is a natural correspondence between the modules over the polynomial

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ring $\mathbb{F}[x]$ and pairs (T, V), where T is a linear transformation on the vector space V over $\mathbb{F}$. Moreover, isomorphisms of $\mathbb{F}[x]$-modules correspond to the similarity of the corresponding linear transformations, submodules correspond to invariant subspaces, and module homomorphisms correspond to linear transformations in the commutant. Kaplansky also provided a table that translates familiar properties of abelian groups into the corresponding properties of linear transformations. Radjabalipour's result is essentially the linear transformation version of the theorem that an abelian group in which there is a bound on the order of all the elements is a direct sum of cyclic groups (See Theorem 3.5).

Since it appears that the applications of Kaplansky's results to linear transformations are not widely known, in this mostly expository paper, we completely describe the results on the structure theory of linear transformations based on Kaplansky's book [5]. We include other ring-theoretic applications [3], [4] to the notion of hyporeflexive transformations.

We first describe the correspondence between $\mathbb{F}[x]$-modules and transformations. Suppose V is an $\mathbb{F}[x]$-module. Then, since $\mathbb{F} \subseteq \mathbb{F}[x]$, it follows that V is a vector space over $\mathbb{F}$. We can define a linear transformation T on V as "multiplication by x", i.e., if $v \in V$, we define

$$Tv = x \cdot v.$$

It easily follows that if $f(x) \in \mathbb{F}[x]$ and $v \in V$, then

$$(*) \qquad f(T)\, v = f(x) \cdot v.$$

Hence the scalar multiplication is completely encoded in T. Hence, if we are given a vector space V and a linear transformation T, the formula $(*)$ above makes V into an $\mathbb{F}[x]$-module, which we will denote by $\mathcal{M}_T$. If $\mathcal{M}_S$ and $\mathcal{M}_T$ are isomorphic as $\mathbb{F}[x]$-modules, then there is a module isomorphism $W : \mathcal{M}_S \to \mathcal{M}_T$ and we must have that W is linear and $WSv = W(x \cdot v) = x \cdot (Wv) = TWv$. This means $WSW^{-1} = T$, i.e., S and T are *similar*. It is easy to see that if $W : \mathcal{M}_S \to \mathcal{M}_T$ is merely linear over $\mathbb{F}$ and $WSW^{-1} = T$, then W would be an $\mathbb{F}[x]$-module isomorphism. An $\mathbb{F}[x]$-submodule $\mathcal{N}$ of $\mathcal{M}_T$ is a vector subspace such that, for every polynomial f, $f \cdot \mathcal{N} \subseteq \mathcal{N}$. This is clearly that same as saying $T(\mathcal{N}) \subseteq \mathcal{N}$, i.e., $\mathcal{N}$ is a T-invariant linear subspace of $\mathcal{M}_T$. We can view the quotient module $\mathcal{M}_T/\mathcal{N}$ as the transformation $\hat{T}_{\mathcal{N}} : \mathcal{M}_T/\mathcal{N} \to \mathcal{M}_T/\mathcal{N}$ defined by

$$\hat{T}_{\mathcal{N}}(v + \mathcal{N}) = Tv + \mathcal{N},$$

which is well defined exactly when $T(\mathcal{N}) \subseteq \mathcal{N}$. In the special case that there is another submodule $\mathcal{N}_1$ such that $\mathcal{M}_T = \mathcal{N} \oplus \mathcal{N}_1$, then $T = T|_{\mathcal{N}} \oplus T|_{\mathcal{N}_1}$ and $T|_{\mathcal{N}_1}$ is similar to $\hat{T}_{\mathcal{N}}$. More generally, writing $\mathcal{M}_T$ as a direct sum

$$\mathcal{M}_T = \sum\nolimits_{\lambda \in \Lambda}^{\oplus} \mathcal{N}_\lambda$$

is the same as writing T as a direct sum

$$T=\sum\nolimits_{\lambda\in\Lambda}^{\oplus}T|_{\mathcal{N}_\lambda}.$$

2. Cyclic transformations

We all know what cyclic groups are. If $\mathcal{R}$ is a commutative ring with identity 1 and $\mathcal{M}$ is an $\mathcal{R}$-module, we say that $\mathcal{M}$ is a *cyclic* $\mathcal{R}$-module if there is a $v\in\mathcal{M}$ such that $\mathcal{M}=\mathcal{R}\cdot v$. This says that the map $L:\mathcal{R}\to\mathcal{M}$ defined by

$$L(r)=r\cdot v$$

is surjective. If we view $\mathcal{R}$ as an $\mathcal{R}$-module, it is clear that L is a module homomorphism and thus we have

$$\mathcal{M}\backsimeq\mathcal{R}/\ker L.$$

In the case where $\mathcal{R}=\mathbb{F}[x]$ and $\mathcal{M}=\mathcal{M}_T$, we see that $\mathcal{M}_T$ is a cyclic $\mathbb{F}[x]$-module if and only if T is a *cyclic transformation*, i.e., there is a $v\in\mathcal{M}_T$ such that

$$\{p(T)v:p\in\mathbb{F}[x]\}=\mathcal{M}_T.$$

This means that T is similar to multiplication by x on $\mathbb{F}[x]/\mathcal{J}$ for some ideal $\mathcal{J}$ of $\mathbb{F}[x]$. But $\mathbb{F}[x]$ is a PID, so $\mathcal{J}=\{0\}$ or there is a monic polynomial $f\in\mathbb{F}[x]$ such that $\mathcal{J}=f\cdot\mathbb{F}[x]$. First suppose $\deg f=n\ge 1$, and $f(x)=a_0+a_1x+\cdots+a_{n-1}x^{n-1}+x^n$; then $e_k=x^k+f\cdot\mathbb{F}[x]$, $0\le k<n$ defines a basis $\{e_0,e_1,\ldots,e_{n-1}\}$ for $\mathbb{F}[x]/\mathcal{J}$ and $Te_k=e_{k+1}$ if $0\le k<n-1$, and $Te_{n-1}=x^n+\mathcal{J}=-\sum_{k=0}^{n-1}a_kx^k+\mathcal{J}=\sum_{k=0}^{n-1}-a_ke_k$. Hence the matrix for T is the *companion matrix* for f, i.e.,

$$C_f=\begin{pmatrix}0&0&\cdots&0&-a_0\\1&0&\cdots&0&-a_1\\0&1&\cdots&0&-a_2\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\cdots&1&-a_{n-1}\end{pmatrix}.$$

If $\mathcal{J}=0$, then $\{x^n:n\ge 0\}$ is a basis for $\mathbb{F}[x]/\mathcal{J}=\mathbb{F}[x]$, and the matrix for T is

$$U=\begin{pmatrix}0&0&0&\cdots\\1&0&0&\cdots\\0&1&0&\cdots\\0&0&1&\ddots\\\vdots&\vdots&\vdots&\ddots\end{pmatrix},$$

which we call the (algebraic) unilateral shift matrix.

We summarize our remarks in the following statement.

Proposition 2.1. *The module $\mathcal{M}_T$ is cyclic if and only either*

(1) T is is similar to C_f for some polynomial $f \in \mathbb{F}[x]$, or
(2) T is similar to the unilateral shift matrix U.

Lemma 2.2. *Suppose $f = p^m$ with p a prime (i.e., irreducible) monic polynomial in $\mathbb{F}[x]$ with $\deg p = d$. Suppose $p'(x) \neq 0$. Then the matrix C_f is similar to the $m \times m$ operator matrix*

$$J_{m,p} = \begin{pmatrix} C_p & 0 & \cdots & 0 & 0 \\ I_d & C_p & \cdots & 0 & 0 \\ 0 & I_d & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & C_p & \vdots \\ 0 & 0 & \cdots & I_d & C_p \end{pmatrix}.$$

Proof. Since $p' \neq 0$ and p is prime, we know that p and p' are relatively prime. Thus there are polynomials $u(x)$ and $v(x)$ such that

$$up + vp' = 1.$$

Thus

$$u(C_p)\, p(C_p) + v(C_p)\, p'(C_p) = I_d.$$

However, since C_p is the companion matrix for p, we know $p(C_p) = 0$. It follows that $p'(C_p)$ is invertible. Moreover,

$$p(J_{m,p}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ p'(C_p) & 0 & \cdots & 0 & 0 \\ * & p'(C_p) & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ * & * & \cdots & p'(C_p) & 0 \end{pmatrix}.$$

It follows that $f(J_{m,p}) = (p(J_{m,p}))^m = 0$ but $(p(J_{m,p}))^{m-1} \neq 0$. Hence the minimal polynomial of $J_{m,p}$ is $f = p^m$. Hence there is a vector x such that $(p(J_{m,p}))^{m-1} x \neq 0$. If M is the cyclic invariant subspace for $J_{m,p}$ generated by x, the minimal polynomial for the restriction $J_{m,p}|_M$ must divide $f = p^m$ and since $p^{m-1}(\, J_{m,p}|_M) \neq 0$, $\dim M = mp$. It follows that x is a cyclic vector for $J_{m,p}$, so $J_{m,p}$ must be similar to C_f. □

Remark 2.3. (1) If the characteristic of $\mathbb{F}$ is zero or $\mathbb{F}$ is finite, then every irreducible polynomial has nonzero derivative [1]. However, if $\mathbb{F} = \mathbb{Z}_2(t)$, the field of rational functions over the integers modulo 2, then t has no square root. Hence $p(x) = x^2 - t$ is irreducible, but $p'(x) = 0$. In this case we see that

$$p(J_{2,p}) = 0,$$

so $J_{2,p}$ cannot be similar to C_{p^2}, whose minimal polynomial is p^2.

(2) Note that when $p(x) = x - \lambda$, then C_p is the 1×1 matrix λ and

$$J_{n,p} = \begin{pmatrix} \lambda & 0 & \cdots & 0 & 0 \\ 1 & \lambda & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \lambda & \vdots \\ 0 & 0 & \cdots & 1 & \lambda \end{pmatrix}$$

is the familiar $n \times n$ Jordan block matrix. Note that if $\mathbb{F}[x]$ is *algebraically closed*, i.e., every polynomial splits into linear factors, then all of the monic irreducible polynomials are of the form $p(x) = x - \lambda$.

3. **Cyclic decompositions**

The fundamental theorem for finitely generated abelian groups says that every finitely generated abelian group is a finite direct sum of cyclic groups so that the finite summands are cyclic groups of prime power order. For a PID $\mathcal{R}$, the analogue says that every finitely generated $\mathcal{R}$-module is a direct sum of cyclic $\mathcal{R}$-modules some of which may be isomorphic to $\mathcal{R}$ and others isomorphic to $\mathcal{R}/p^n\mathcal{R}$ where $p \in \mathcal{R}$ is prime and $n \geq 1$. When $\mathcal{R} = \mathbb{F}[x]$ and the module is $\mathcal{M}_T$, we get that $\mathcal{M}_T$ is finitely generated if and only if the transformation T is *finitely cyclic*, i.e., there are vectors $v_1, \ldots, v_n \in \mathcal{M}_T$ such that

$$\mathcal{M}_T = \{f_1(T) v_1 + \cdots + f_n(T) v_n : f_1, \ldots f_n \in \mathbb{F}[x]\}.$$

The translation to linear transformations follows.

Theorem 3.1. *Suppose T is a finitely cyclic linear transformation. Then T is similar to a finite direct sum of linear transformations some of which may be the unilateral shift matrix U and the others of the form C_{p^n} where p is a monic irreducible polynomial and n is a positive integer.*

Corollary 3.2. *If $0 \neq f \in \mathbb{F}[x]$ and $f = p_1^{n_1} \cdots p_k^{e_k}$ with $p_1, \ldots, p_k$ distinct monic primes and $n_1, \ldots, n_k \in \mathbb{N}$, then the matrix C_f is similar to the matrix*

$$C_{p_1^{n_1}} \oplus \cdots \oplus C_{p_k^{n_k}}.$$

In certain cases we can further decompose the matrices of the form C_{p^n} to look more like the Jordan form.

Lemma 3.3. *Suppose $p \in \mathbb{F}[x]$ is a prime monic polynomial in $\mathbb{F}[x]$ with $\deg p = d$. Suppose the derivative $p'(x) \neq 0$. Then the matrix C_f is similar to* *the $m \times m$ operator matrix*

$$J_{m,p} = \begin{pmatrix} C_p & 0 & \cdots & 0 & 0 \\ I_d & C_p & \cdots & 0 & 0 \\ 0 & I_d & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & C_p & \vdots \\ 0 & 0 & \cdots & I_d & C_p \end{pmatrix}.$$

Proof. Since $p' \neq 0$ and p is prime, we know p and p' are relatively prime. Thus there are polynomials $u(x)$ and $v(x)$ such that

$$up + vp' = 1.$$

Thus

$$u(C_p)\, p(C_p) + v(C_p)\, p'(C_p) = I_d.$$

However, since C_p is the companion matrix for p, we know $p(C_p) = 0$. It follows that $p'(C_p)$ is invertible. Moreover,

$$p(J_{m,p}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ p'(C_p) & 0 & \cdots & 0 & 0 \\ * & p'(C_p) & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ * & * & \cdots & p'(C_p) & 0 \end{pmatrix}.$$

It follows that $f(J_{m,p}) = (p(J_{m,p}))^m = 0$ but $(p(J_{m,p}))^{m-1} \neq 0$. Hence the minimal polynomial of $J_{m,p}$ is $f = p^m$. Hence there is a vector x such that $(p(J_{m,p}))^{m-1} x \neq 0$. If M is the cyclic invariant subspace for $J_{m,p}$ generated by x, the minimal polynomial for the restriction $J_{m,p}|_M$ must divide $f = p^m$ and since $p^{m-1}(\, J_{m,p}|_M) \neq 0$, $\dim M = mp$. It follows that x is a cyclic vector for $J_{m,p}$, so $J_{m,p}$ must be similar to C_f. $\square$

Corollary 3.4. *If T is a linear transformation on a finite-dimensional vector space, then T is similar to a finite direct sum of matrices of the form C_{p^n} where p is a monic irreducible polynomial and n is a positive integer. If the minimal polynomial for T splits over $\mathbb{F}$, this gives the usual Jordan canonical form for T.*

A related result concerns $\mathcal{R}$-modules $\mathcal{M}$ with *bounded torsion, i.e., there is a* nonzero $r \in \mathcal{R}$ such that $r \cdot \mathcal{M} = \{0\}$. The module $\mathcal{M}_T$ has bounded torsion if and only if T is *algebraic*, i.e., there is a nonzero polynomial f for which $f(T) = 0$. The theorem for abelian groups says that an abelian group with bounded torsion is a direct sum of cyclic groups of prime power order. Here is the analogue for linear transformations.

Theorem 3.5. *An algebraic linear transformation T is similar to a (possibly infinite) direct sum of matrices of the form C_{p^m} (p a monic prime in $\mathbb{F}[x]$ and $m \geq 1$), where $f = p^m$ and p is a monic irreducible polynomial and m is a positive integer. If $\mathbb{F}$ has characteristic 0 or $\mathbb{F}$ is finite, then we can replace the summands C_{p^m} with $J_{n,p}$. If the minimal polynomial of T splits over $\mathbb{F}$, then the summands have the form $J_{n,x-\lambda}$ ($\lambda \in \mathbb{F}$) and the decomposition is a Jordan canonical form for T (with possibly infinitely many blocks).*

4. **Torsion equals locally algebraic**

An abelian group G is called torsion if every element has finite order. A module M over a ring $\mathcal{R}$ is called *torsion* if for every $x \in M$ there is a $0 \neq r \in \mathcal{R}$ such that $r \cdot x = 0$. For the module $\mathcal{M}_T$ torsion means that, for every vector v there is a nonzero polynomial f such that $f(T)v = 0$. For a transformation T this means that T is locally algebraic. The set of all polynomials f such that $f(T)v = 0$ is an ideal, and since $\mathbb{F}[x]$ is a PID, there is a unique monic polynomial $p_{v,T}$ such that this ideal is $p_{v,T} \cdot \mathbb{F}[x]$. The polynomial $p_{v,T}$ is called the *minimal local polynomial for T at v*.

The module $\mathcal{M}_T$ is torsion if and only if

$$\mathcal{M}_T = \cup \{\ker f(T) : 0 \neq f \in \mathbb{F}[x]\}.$$

An example of a locally algebraic (in fact, locally nilpotent) transformation that is not algebraic is a *backward shift*, which has an infinite matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \vdots \\ 0 & 0 & 1 & 0 & \vdots \\ 0 & 0 & 0 & 1 & \vdots \\ \cdots & \cdots & \cdots & \ddots & \ddots \end{pmatrix},$$

i.e., there is a basis $\{e_0, e_1, \ldots\}$ such that $Te_0 = 0$ and $Te_n = e_{n-1}$ for $n \geq 1$. For example $T : \mathbb{F}[x] \to \mathbb{F}[x]$ defined by

$$(Tf)(x) = \frac{f(x) - f(0)}{x}$$

with the basis $e_n = x^n$ for $n \geq 0$. If the characteristic of $\mathbb{F}$ is 0 and we let $e_n = \frac{1}{n!}x^n$, the backward shift is merely the differentiation operator D, i.e.,

$$Df = f'.$$

Another example of a torsion module $\mathcal{M}_T$ is when T is multiplication by x on the vector space $\mathbb{F}(x)/\mathbb{F}[x]$ of rational functions modulo the polynomials. It is important to note that *none* of the above examples of locally algebraic transformations can be written as a direct sum of cyclic transformations.

For each monic prime $p \in \mathbb{F}[x]$, we define

$$E_p(T) = \cup_{n=0}^{\infty} \ker(p^n(T)).$$

It follows from that fact that torsion modules over a PID are direct sums of their p-modules that

$$\mathcal{M}_T = \sum\nolimits^{\oplus}_{p\in\mathbb{F}[x],\ p \text{ prime}} E_p(T).$$

If $T_p = T|_{E_p}$, then we have the following. Note that if $p(x) = x - \lambda$, then $T_p - \lambda$ is locally nilpotent. When $\mathbb{F}$ is algebraically closed, the next result shows why locally nilpotent transformations play a central role in the structure theory of locally algebraic transformations.

Proposition 4.1. *If T is locally algebraic, then $T = \sum^{\oplus}_{p\in\mathbb{F}[x], p\ prime} T_p$. Moreover, if all the local polynomials for T split over $\mathbb{F}$, then T is similar to a direct sum*

$$\sum\nolimits^{\oplus}_{\lambda\in\mathbb{F}} \lambda + A_\lambda$$

where each A_λ is locally nilpotent.

Even if T is not locally algebraic, we can define

$$\mathrm{Tor}(\mathcal{M}_T) = \cup\{\ker f(T) : 0 \neq f \in \mathbb{F}[x]\}.$$

It is clear that $\mathrm{Tor}(\mathcal{M}_T) = \sum^{\oplus}_{p\in\mathbb{F}[x],\ p \text{ prime}} E_p(T)$ is an invariant subspace for T and $T|_{\mathrm{Tor}(\mathcal{M}_T)}$ is locally algebraic. It is clear that $\mathrm{Tor}(\mathcal{M}_T)$ is the largest invariant subspace with this property. It is natural to ask if $\mathrm{Tor}(\mathcal{M}_T)$ is a direct summand of $\mathcal{M}_T$. This would mean that there is a T-invariant subspace $\mathcal{N} \subseteq \mathcal{M}_T$ such that $\mathcal{M}_T = \mathrm{Tor}(\mathcal{M}_T) \oplus \mathcal{N}$. This is the case when T is finitely cyclic. However, it is not true in general, as is shown in Example 6.6.

A module $\mathcal{M}$ over a ring $\mathcal{R}$ is *torsion free* if and only if $\mathrm{Tor}(\mathcal{M}) = 0$. Here is the translation for linear transformations.

Proposition 4.2. *The module $\mathcal{M}_T$ is torsion-free if and only if* $\ker f(T) = 0$ *for every polynomial $f \neq 0$. This means that $f(T)$ is injective for every nonzero polynomial f.*

If $\mathbb{F}$ is algebraically closed, $\mathcal{M}_T$ is torsion-free if and only if T has no eigenvalues.

Note $f(T)$ being injective for every nonzero polynomial f does not mean that $f(T)$ is invertible for every polynomial $f \neq 0$. For example if T has a unilateral shift matrix with respect to some basis, then $\mathcal{M}_T$ is torsion free, but if $f(x) = x$, then $f(T) = T$ is not surjective. We will see what every $f(T)$ being surjective means in Section 6.

Here is useful functional analysis result that Kaplansky proved about locally algebraic transformations. This result was generalized by various authors [7], [8], [6].

Proposition 4.3. *If T is a bounded linear transformation on a real or complex Banach space, then T is locally algebraic if and only if T is algebraic.*

5. **Free modules**

A module over a ring $\mathcal{R}$ is one that is isomorphic to a direct sum of m copies of $\mathcal{R}$ for some cardinal m (denoted by $\mathcal{R}^m$). In the case $\mathcal{R} = \mathbb{F}[x]$ we see that $\mathcal{M}_T$ is a free module if and only if T is similar to a (possibly infinite) direct sum of m copies of the unilateral shift matrix, which we denote by $U^{(m)}$. Well-known theorems about free modules over a PID is that the number of summands of $\mathcal{R}$ is an isomorphism invariant and every submodule of a free module is free. Recall that submodules of $\mathcal{M}_T$ correspond to T-invariant subspaces. Hence the results on free modules translates as follows.

Theorem 5.1. *Suppose m and n are cardinals. Then*

(1) $U^{(m)}$ is similar to $U^{(n)}$ if and only if $m = n$.
(2) The restriction of $U^{(m)}$ to any non-zero invariant subspace is similar to $U^{(k)}$ for some cardinal $k \leq n$.

Not every torsion free module is free, i.e., if T is multiplication by x on the vector space $\mathbb{F}(x)$ of rational functions, then $\mathbb{F}(x)$ is torsion free but not free. In fact T is not a direct sum of two transformations on $\mathbb{F}(x)$.

If $\mathcal{N}$ is a submodule of a module $\mathcal{M}$ over a PID $\mathcal{R}$ and if $\mathcal{M}/\mathcal{N}$ is free then there is a submodule $\mathcal{K}$ of $\mathcal{M}$ such that $\mathcal{M} = \mathcal{N} \oplus \mathcal{K}$, and (necessarily) $\mathcal{K}$ is isomorphic to $\mathcal{M}/\mathcal{N}$.

Recall that if T is a linear transformation on $\mathcal{M}_T$ and $\mathcal{N}$ is a T-invariant linear subspace of T, then the transformation $\hat{T}_{\mathcal{N}} : \mathcal{M}_T \to \mathcal{M}_T$ is defined by

$$\hat{T}_{\mathcal{N}}(v + \mathcal{N}) = Tv + \mathcal{N}$$

Proposition 5.2. *Suppose T is a linear transformation on $\mathcal{M}_T$ and $\mathcal{N}$ is a T-invariant subspace such that $\mathcal{M}_T/\mathcal{N}$ is free, i.e., $\hat{T}_{\mathcal{N}}$ is similar to $U^{(m)}$ for some cardinal m. Then T is similar to $T|_{\mathcal{N}} \oplus U^{(m)}$.*

It is well-known that every module is a free module. Here is the translation for linear transformations.

Proposition 5.3. *Suppose S is a linear transformation. Then there is a transformation $T = U^{(m)}$ for some cardinal m and a T-invariant subspace $\mathcal{N} \subseteq \mathcal{M}_T$ such that S is similar to $\hat{T}_{\mathcal{N}}$.*

6. **Divisible modules**

A module $\mathcal{M}$ over a PID $\mathcal{R}$ is *divisible* if and only if, for every $0 \neq v \in \mathcal{M}$ and every $0 \neq r \in \mathcal{R}$, there is a $w \in \mathcal{M}$ such that $rw = v$. This is something like dividing v by r to get w. If $\mathcal{M}$ is torsion free, then w is unique, but it is not unique in general since we could replace w with $w + w_1$ where $rw_1 = 0$. We will call a linear transformation T *polynomially divisible if and only if* $\mathcal{M}_T$ is divisible over $\mathbb{F}[x]$. Here is the simple translation for linear transformations. Note how this relates to torsion free in which $p(T)$ is injective for each nonzero polynomial p.

Proposition 6.1. *The transformation T is divisible if and only if, for every nonzero polynomial f, we have $f(T)$ is surjective.*

One example of a transformation T where $\mathcal{M}_T$ is divisible is when T is "ultiplication by x" on the vector space $\mathbb{F}(x)$ of rational functions over $\mathbb{F}$. This transformation is the linear transformation analogue of the group $(\mathbb{Q},+)$ of rational numbers, and we will denote it by $D_{\mathbb{Q}}$. When $\mathbb{F}$ is algebraically closed,

$$\left\{1,x,x^2,\ldots\right\}\cup\left\{\frac{1}{(x-\lambda)^n}:\lambda\in\mathbb{F}\text{ and }n\in\mathbb{N}\right\}$$

is a linear basis for $\mathbb{F}(x)$, and we can visualize a sparse, but infinite, matrix for $D_{\mathbb{Q}}$, given that

$$Tx^n=x^{n+1},\text{ and}$$
$$T\frac{1}{(x-\lambda)^n}=\frac{1}{(x-\lambda)^{n-1}}+\lambda\frac{1}{(x-\lambda)^n}.$$

An example of a torsion divisible $\mathcal{M}_T$ is obtained by taking a monic prime polynomial p, letting $\mathcal{M}=\left\{\frac{f}{p^n}:f\in\mathbb{F}[x],n\geq 0\right\}/\mathbb{F}[x]$, and letting T be "multiplication by x". Such a transformation is defined by a basis

$$\{v_{n,k}:n\geq 1,1\leq k\leq d=\deg(p)\}$$

such that

$$Tv_{n,k}=v_{n,k+1}\text{ if }n\geq 0,1\leq k<d,$$

and

$$p(T)\,v_{0,1}=0,\ p(T)\,v_{n+1,1}=v_{n,1}\text{ for }n\geq 0.$$

We denote this transformation by D_{p^∞}, and it corresponds to the divisible abelian group $\mathbb{Z}_{p^\infty}$, where p is a prime positive integer. Note when $p(x)=x$, D_{p^∞} is similar to backward shift with matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \vdots \\ 0 & 0 & 1 & 0 & \vdots \\ 0 & 0 & 0 & 1 & \vdots \\ \cdots & \cdots & \cdots & \ddots & \ddots \end{pmatrix}.$$

Proposition 6.2. *The following results are true for modules over a PID [5, Chap. 5]:*

1. *(1) A divisible submodule of a module is always a summand.*
2. *(2) If $\mathcal{M}$ is divisible, then* $\operatorname{Tor}(\mathcal{M})$ *is divisible.*

(3) A direct sum of modules is divisible if and only if each summand is divisible.
(4) The algebraic sum of all the divisible submodules of a module is divisible.
(5) Every module $\mathcal{M}$ is a direct sum $\mathcal{V} \oplus \mathcal{W}$ where $\mathcal{V}$ is divisible and $\mathcal{W}$ has no nonzero divisible submodules.

This gives us a complete description of the transformations T for which $\mathcal{M}_T$ is divisible. The number of each type of summands is unique. Note that the transformations $D_{\mathbb{Q}}$ and D_{p^∞} cannot be decomposed as nontrivial direct sums of transformations. This gives some insight to Problem 2 in [9].

Theorem 6.3. *Suppose T is a linear transformation and $\mathcal{M}_T$ is divisible. Then T is similar to a direct sum of transformations of the form $D_{\mathbb{Q}}$ and D_{p^∞} (p a monic prime in $\mathbb{F}[x]$).*

Proof. It follows from Proposition 6.2 that $\operatorname{Tor}(\mathcal{M}_T)$ is divisible and hence a summand of $\mathcal{M}_T$. Thus $\mathcal{M}_T = \operatorname{Tor}(\mathcal{M}_T) \oplus \mathcal{N}$, where $\mathcal{N}$ is a torsion free divisible $\mathbb{F}[x]$-module. Thus $T = A \oplus B$ relative to this decomposition. Since $\mathcal{N}$ is torsion free, we know that $f(B)$ is injective for each nonzero polynomial f and since $\mathcal{N}$ is divisible, we know that $f(B)$ is surjective for each nonzero polynomial f. Hence if $0 \neq f \in \mathbb{F}[x]$, then $f(B)$ is invertible. It follows that $\mathcal{N}$ is a module over the field $\mathbb{F}(x)$ and thus $\mathcal{N}$ is isomorphic to a direct sum of copies of $\mathbb{F}(x)$, and it follows that B is similar to a direct sum of copies of $D_{\mathbb{Q}}$.

We also know that $\operatorname{Tor}(\mathcal{M}_T)$ is a direct sum $\sum^{\oplus}_{p\in\mathbb{F}[x],\ p \text{ prime}} E_p$. Since every summand of a divisible module is divisible, we know that each $E_p(T)$ is divisible. Suppose $E_p(T) \neq 0$ for some monic prime $p \in \mathbb{F}[x]$. Using Zorn's lemma we can construct a maximal linearly independent collection $\mathcal{C}$ of submodules of $E_p(T)$ each having the property that the restriction of A to it is similar to D_{p^∞}. The direct sum E of these subspaces is divisible and is a summand. We can therefore write $E_p(T) = E \oplus F$ where $A|_E$ is similar to a direct sum of copies of D_{p^∞}. Assume, via contradiction that $F \neq 0$. Since $F \subset E_p$, it follows that there is a $0 \neq v_{0,1} \in F$ such that $p(A)\, v_{0,1} = 0$. Using the fact that F is divisible, we can find a sequence $\{v_{1,0}, v_{2,0}, \ldots\}$ so that $p(A)\, v_{n+1,0} = v_{n,0}$. If $d = \deg(p)$ and we define $v_{n,k} = A^k v_{n,0}$ for $1 \leq k < d$, we obtain a submodule F_0 of F so that $A|_{F_0}$ is similar to D_{p^∞}, which contradicts the maximality of $\mathcal{C}$. Hence $A = T|_{E_p}$ is similar to a direct sum of copies D_{p^∞}. □

Proposition 6.2 also give a decomposition of an arbitrary linear transformation into a "divisible" part and a "completely non-divisible" part.

Theorem 6.4. *Every linear transformation is the direct sum $D \oplus A$ where $\mathcal{M}_T$ is divisible and no submodule of $\mathcal{M}_A$ is divisible.*

Note that a submodule of $\mathcal{M}_T$ is just a T-invariant vector subspace. If $\mathcal{N}$ is a submodule of $\mathcal{M}_T$ and $A = T_{\mathcal{N}}$, then $\mathcal{N} = \mathcal{M}_A$. Hence $\mathcal{N}$ is divisible if and only if A is a direct sum as in Theorem 6.3. For example, if $\mathcal{N}$ is a T-invariant subspace and $T|_{\mathcal{N}}$ is similar to $D_{\mathbb{Q}}$ or D_{p^∞} (e.g., the backward shift matrix), then $\mathcal{N}$ is a direct summand of $\mathcal{M}_T$ and T is similar to $T|_{\mathcal{N}} \oplus B$ for some transformation B.

Kaplansky [5, p. 12, Problem 5] proves that if $\mathcal{R}$ is a PID, then every $\mathcal{R}$-module is a submodule of a divisible module. Here is the translation for linear transformations.

Proposition 6.5. *Every linear transformation is similar to the restriction to an invariant subspace of a transformation that is a direct sum of transformations of the form $D_{\mathbb{Q}}$ and D_{p^∞} (p a monic prime in $\mathbb{F}[x]$).*

Example 6.6. Here we present an example of a linear transformation T for which $\mathrm{Tor}(\mathcal{M}_T)$ is not a direct summand of $\mathcal{M}_T$. This example sheds light on Problem 1 in [9]. Euclid's proof that there are infinitely many prime integers works in $\mathbb{F}[x]$ for any field $\mathbb{F}$. Suppose $p_1, p_2, \ldots$ are distinct primes and let

$$\mathcal{M}_T = \prod_{n=1}^{\infty} \mathbb{F}[x] / p_n \mathbb{F}[x],$$

and let T be "multiplication by x" in every coordinate. It is clear that if $f \in \mathbb{F}[x]$, then f is the product of finitely many primes, so that multiplication by f is invertible on all but finitely many coordinates. It follows that

$$\mathrm{Tor}(\mathcal{M}_T) = \sum_{n=1}^{\infty} \mathbb{F}[x] / p_n \mathbb{F}[x].$$

Moreover, if $0 \neq f \in \mathbb{F}[x]$ and $e = (e_1, e_2, \ldots) \in \mathcal{M}_T$, then there is an $h = (h_1, h_2, \ldots)$ such that $f(x) \cdot h_n = e_n$ except for finitely many values of n. This implies $\mathcal{M}_T / \mathrm{Tor}(\mathcal{M}_T)$ is divisible. If $\mathcal{M}_T = \mathrm{Tor}(\mathcal{M}_T) \oplus \mathcal{N}$ for some submodule $\mathcal{N}$, then $\mathcal{N}$ would be isomorphic to $\mathcal{M}_T / \mathrm{Tor}(\mathcal{M}_T)$ and would therefore be divisible. However, if $0 \neq g = (g_1, g_2, \ldots) \in \mathcal{N}$, then there must be an $n \in \mathbb{N}$ such that $g_n \neq 0$. Thus there is no $y \in \mathcal{N}$ for which $p_n \cdot y = g$. This contradiction implies $\mathrm{Tor}(\mathcal{M}_T)$ is not a summand. We know that there is a vector subspace $\mathcal{N}$ of $\mathcal{M}_T$ so that $\mathcal{M}_T = \mathrm{Tor}(\mathcal{M}_T) \oplus \mathcal{N}$, but $\mathcal{N}$ cannot be an $\mathbb{F}[x]$-submodule, i.e., a T-invariant subspace. An interesting fact is that any finitely many summands of $\mathrm{Tor}(\mathcal{M}_T)$ is a summand of $\mathcal{M}_T$, since

$$\mathcal{M}_T = \sum_{n=1}^{N} \mathbb{F}[x] / p_n \mathbb{F}[x] \oplus \prod_{n=N+1}^{\infty} \mathbb{F}[x] / p_n \mathbb{F}[x].$$

Note that if we let $\mathcal{K}$ be any proper submodule containing $\mathrm{Tor}(\mathcal{M}_T)$, the same argument shows that $\mathcal{K}$ is not a direct summand of $\mathcal{M}_T$.

There is one case in which we know that $\mathrm{Tor}(\mathcal{M})$ is a direct summand of $\mathcal{M}$.

Proposition 6.7. *If the restriction of T to* $\mathrm{Tor}(\mathcal{M}_T)$ *is a direct sum of cyclic transformations, then* $\mathrm{Tor}(\mathcal{M}_T)$ *is a direct summand of* $\mathcal{M}_T$*. In this case $T = A \oplus B$ where A is locally algebraic and, for every nonzero polynomial $f \in \mathbb{F}[x]$,* $\ker f(B) = 0$.

7. Ulm's Theorem

In this section we continue the assumption that $\mathcal{M}_T = E_p(T)$ and $p \in \mathbb{F}[x]$ is prime. We also assume that T is reduced, i.e., T has no polynomially divisible summands. We will define cardinal-valued invariants for T, called the *Ulm invariants*. In the case where $\dim \mathcal{M}_T$ is countable, these are complete similarity invariants.

For each ordinal α we will define a submodule $\mathcal{K}_\alpha$ of $\mathcal{M}_T$ by transfinite construction as follows:

(1) $\mathcal{K}_0 = p(T)(\mathcal{M}_T)$.
(2) For every ordinal α, $\mathcal{K}_{\alpha+1} = p(T)(\mathcal{K}_\alpha)$.
(3) If $\alpha > 0$ is a limit ordinal, then
$$\mathcal{K}_\alpha = \cap_{\beta<\alpha}\mathcal{K}_\beta.$$

It is clear that $\alpha < \beta$ implies $\mathcal{K}_\beta \subset \mathcal{K}_\alpha$. Hence there must be the smallest ordinal γ such that $\mathcal{K}_\gamma = \mathcal{K}_{\gamma+1}$. This means $p(T)(\mathcal{K}_\gamma) = \mathcal{K}_\gamma$. Since $\mathcal{K}_\gamma \subset E_p(T)$, it follows that, for any polynomial f relatively prime to p, $f(T)|\mathcal{K}_\gamma$ is invertible on $\mathcal{K}_\gamma$. It follows that $T|\mathcal{K}_\gamma$ is polynomially divisible, and, by Proposition 6.2, is a summand. Since we are assuming that T is reduced, we conclude that $\mathcal{K}_\gamma = 0$.

Since $p \in \mathbb{F}[x]$ is prime, $\mathbb{F}[x]/p\mathbb{F}[x] = \mathbb{F}_p$ is a field. We see that $\ker p(T)$ is a vector space over $\mathbb{F}_p$. For each ordinal $\alpha < \gamma$ we define the Ulm invariant $u_\alpha(T)$ as follows
$$u_\alpha(T) = \dim_{\mathbb{F}_p}(\mathcal{K}_\alpha/\mathcal{K}_{\alpha+1}).$$

We are now ready to translate Ulm's theorem [5, Theorem 14].

Theorem 7.1. *Suppose $p \in \mathbb{F}[x]$ is prime, S and T are linear transformations such that*

(1) $\mathcal{M}_S = E_p(S)$, $\mathcal{M}_T = E_p(T)$,
(2) S and T are reduced,
(3) $\dim \mathcal{M}_S$ *and* $\dim \mathcal{M}_T$ *are countable, and*
(4) S and T have the same Ulm invariants.

Then S is similar to T.

Here is the version for locally algebraic transformations. Note that $\dim \mathcal{M}_T$ might be uncountable with each $E_p(T)$ (p prime) being countable.

Theorem 7.2. *Suppose S and T are locally algebraic linear transformations such that,* *(1) For every prime $p \in \mathbb{F}[x]$ $\dim E_p(S)$ and $\dim E_p(T)$ are countable,*

(2) For every prime $p \in \mathbb{F}[x]$, the Ulm invariants for $S|_{E_p(S)}$ and $T|_{E_p(T)}$ are the same.

Then S and T are similar.

It was pointed out by Kaplansky [5, p. 27] that the Ulm invariants prove that when a locally algebraic transformation is the direct sum of cyclic transformations, the decomposition is unique.

8. Summary

Suppose T is an arbitrary linear transformation. We can first use Theorem 6.4 to write $T = D \oplus A$ where $\mathcal{M}_D$ is divisible and $\mathcal{M}_A$ has no divisible summands. We know from Theorem 6.3 that D is (uniquely) similar to a direct sum of transformations of the form $D_{\mathbb{Q}}$ or D_{p^∞} (p a monic prime in $\mathbb{F}[x]$).

If we are lucky (we might not be), $\mathrm{Tor}(\mathcal{M}_A)$ is a summand of $\mathcal{M}_A$, and A can be written as a direct sum $B \oplus S$ where B is locally algebraic and S has the property that $\ker f(S) = 0$ for every nonzero polynomial $f \in \mathbb{F}[x]$. One such example is when $\mathcal{M}_T/\mathrm{Tor}(\mathcal{M}_T)$ is a free $\mathbb{F}[x]$-module, then $T = B \oplus U^{(m)}$ for some locally algebraic transformation S and some cardinal m. Another case is when $T|_{\mathrm{Tor}(\mathcal{M}_T)}$ is a direct sum of cyclic transformations, then the decomposition $T = B \oplus S$ exists.

We can decompose B as a direct sum $\sum_{p \in \mathbb{F}[x],\ p \text{ prime}} B_p$ relative to the direct sum $\mathcal{M}_B = \sum_{p \in \mathbb{F}[x],\ p \text{ prime}} E_p(B)$. If some B_p is algebraic, it can be uniquely written as a direct sum of cyclic transformations of the form C_{p^n} for some prime $p \in \mathbb{F}[x]$ and some $n \geq 1$. Otherwise, if $\dim E_p(B)$ is countable, we can use the Ulm invariants to characterize B_p. If $\dim E_p(B)$ is uncountable, the Ulm invariants are still invariants, but they may not completely characterize B_p up to similarity. If $\mathcal{M}_S$ is free, we can write S as a direct sum of copies of the unilateral shift matrix. In general, little seems to be known about S and B_p ($\dim E_p(B)$ uncountable). Whatever answers exist, they are probably written as theorems about abelian groups and not as theorems about linear transformations.

Since we seem to know the most about algebraic transformations, it is useful to note that every linear can be "approximated" by algebraic ones.

Proposition 8.1. *Suppose V is an infinite-dimensional vector space over the field $\mathbb{F}$. Then the set of algebraic linear transformations is strictly dense in the set of all linear transformations.*

Proof. Suppose T is a linear transformation on V and $E = \{x_1, \ldots, x_n\} \subset V$ is finite. Let W be the set of all vectors of the form

$$p_1(T)\,x_1 + \cdots + p_n(T)\,x_n.$$

If we view V as an $\mathbb{F}[x]$-module, then W is a finitely generated submodule. Hence the restriction $T|_W$ is a direct sum of cyclic transformations. This means $T|_W = A \oplus U^{(k)}$ relative to a decomposition $W = W_A \oplus H_1 \oplus \cdots \oplus H_s$ where A is algebraic and $0 \leq s < \infty$. We focus on the case where $s > 0$. In this case each H_m has a basis $\{e_{m,k} : k \in \mathbb{N}\}$ with $Te_{m,k} = e_{m,k+1}$ for $k \in \mathbb{N}$ and $1 \leq m \leq s$. Then there is an $N \in \mathbb{N}$ such that $x_1, \ldots, x_n$ is in the linear span X of

$$W_A \cup \{e_{m,k} : 1 \leq k < N, 1 \leq m \leq s\}.$$

We can choose a linear subspace Y of V containing $\{e_{m,k} : k \geq N, 1 \leq m \leq s\}$ such that $V = X \oplus Y$. We define a linear transformation S on V by $S|_X = T|_X$ and $S|_Y = 0$. Since $S^N e_{m,k} = 0$, we see that S is algebraic. More precisely, if f is the minimal polynomial for A, then $f(x)\, x^N$ kills S. Moreover, $S|_E = T_E$. Hence every strict neighborhood of T contains an algebraic transformation. □

9. **Hyporeflexivity**

Suppose $\mathcal{R}$ is a PID and $\mathcal{M}$ is an $\mathcal{R}$-module. Let $End_{\mathcal{R}}(\mathcal{M})$ denote the set of all $\mathcal{R}$-module homomorphisms from $\mathcal{M}$ to $\mathcal{M}$. There is a natural topology on $End_{\mathcal{R}}(\mathcal{M})$ called the *strict topology.* A basic strict neighborhood of an endomorphism T is given by a finite subset F of $\mathcal{M}$ and is defined by

$$U_F(T) = \{S \in End_{\mathcal{R}}(\mathcal{M}) : Sx = Tx \text{ for every } x \in F\}.$$

A net $\{T_\lambda\}$ converges *strictly* to T if and only if, for every $x \in \mathcal{M}$ there is a λ_0 such that, whenever $\lambda \geq \lambda_0$ we have $T_\lambda x = Tx$. In other words the strict topology is the topology of pointwise convergence with the discrete topology on $\mathcal{M}$.

It is an easy exercise to show that a linear transformation on a vector space leaves every linear subspace invariant if and only if it is a scalar multiple of the identity. (Hint: Consider a linear basis.) This is no longer true when the field is replaced with a ring $\mathcal{R}$ and the vector space is replaced with a module. Suppose $\mathcal{M}$ is a module over a PID $\mathcal{R}$ and $T \in End_{\mathcal{R}}(\mathcal{M})$ leaves invariant every submodule. Since $v \in \mathcal{R} \cdot v$ is a submodule, we see that there is an $r_v \in \mathcal{R}$ such that $Tv = r_v \cdot v$, i.e., every vector is an eigenvector. We call such an endomorphism *locally scalar.* We call an endomorphism *scalar* if there is an $r \in \mathcal{R}$ such that $Tv = r \cdot v$ for every $v \in \mathcal{M}$.

We call the module $\mathcal{M}$ *scalar reflexive* if every locally scalar $\mathcal{R}$-endomorphism on $\mathcal{M}$ is scalar. It is easily shown that the set of all locally scalar endomorphisms is strictly closed and thus contains the strict closure of the set of scalar endomorphisms. It follows that if the set of all scalar endomorphisms on $\mathcal{M}$ is not strictly closed, then $\mathcal{M}$ is not scalar reflexive. We call the ring $\mathcal{R}$ *scalar-reflexive* if every finitely generated $\mathcal{R}$-module is scalar-reflexive, and we call $\mathcal{R}$ *strongly scalar-reflexive* if every $\mathcal{R}$-module is scalar-reflexive. Scalar-reflexivity was studied in [3, 4, 10], and [11].

In [4] it was shown for a ring $\mathcal{R}$ the following are equivalent

(1) $\mathcal{R}$ is strongly scalar reflexive.
(2) $\mathcal{R}$ is a finite direct sum of maximal valuation rings.
(3) For every $\mathcal{R}$-module $\mathcal{M}$, the set of scalar endomorphisms is strictly closed.
(4) For every family $\{\mathcal{J}_\lambda : \lambda \in \Lambda\}$ of ideals in $\mathcal{R}$ and family $\{r_\lambda : \lambda \in \Lambda\}$ in $\mathcal{R}$, if for every finite nonempty subset $E \subset \Lambda$, the family of congruences

$$x = r_\lambda \bmod \mathcal{J}_\lambda,\ \lambda \in \Lambda$$

has a solution x, then there is a solution for the entire collection of congruences.

It is clear from (1) $\Leftrightarrow$ (4) above that if $\mathcal{R}$ has only finitely many ideals, then $\mathcal{R}$ is strongly scalar-reflexive. Hence if $\mathcal{R}$ is a PID and $0 \neq r$ is not a unit in $\mathcal{R}$, then $\mathcal{R}/r\mathcal{R}$ is a finite direct sum of rings of the form $\mathcal{R}/p^n\mathcal{R}$ with p prime and $n \geq 1$, whose only ideals are $p^k\mathcal{R}/p^n\mathcal{R}$ for $1 \leq k \leq n$. Hence $\mathcal{R}/r\mathcal{R}$ is strongly scalar-reflexive when $0 \neq r$ is not a unit.

Suppose $\mathcal{M}$ is an $\mathcal{R}$-module. We let $Ann_{\mathcal{R}}(\mathcal{M}) = \{r \in \mathcal{R} : r \cdot \mathcal{M} = 0\}$ be the annihilator of $\mathcal{M}$ in $\mathcal{R}$. Since $\mathcal{R}$ is commutative, $Ann_{\mathcal{R}}(\mathcal{M})$ is an ideal in $\mathcal{R}$. It is clear that $\mathcal{M}$ is a scalar-reflexive $\mathcal{R}$ module if and only if $\mathcal{M}$ is a scalar-reflexive $\mathcal{R}/Ann_{\mathcal{R}}(\mathcal{M})$ module. We say that a submodule $\mathcal{N}$ of $\mathcal{M}$ is *separating* if $Ann_{\mathcal{R}}(\mathcal{N}) = Ann_{\mathcal{R}}(\mathcal{M})$.

Proposition 9.1. *Suppose $\mathcal{R}$ is a scalar-reflexive ring and $\mathcal{M}$ is an $\mathcal{R}$-module. Then the set of all locally scalar endomorphisms is the strict closure of the set of all scalar endomorphisms.*

Proof. Suppose $T \in End_{\mathcal{R}}(\mathcal{M})$ is locally scalar and suppose U is a strict neighborhood of T. Then there is a finite subset $E \subset \mathcal{M}$ such that

$$\{A \in End_{\mathcal{R}}(\mathcal{M}) : A|_E = T|_E\} \subset U.$$

Since $\mathcal{R}$ is scalar-reflexive, the $\mathcal{R}$-module $\mathcal{N}$ generated by E is scalar-reflexive, and since T is locally scalar, we see that there is a scalar endomorphism S_E such that $S_E = T$ on E. Thus, $S_E \in U$. Since every strictly open neighborhood of T contains a scalar endomorphism, we see that T is the strict closure of the scalar endomorphisms.

Next suppose $\mathcal{M}$ has a torsion-free summand and suppose $T \in End_{\mathcal{R}}(\mathcal{M})$ is locally scalar. Hence we can write $\mathcal{M} = \mathcal{K} \oplus \mathcal{R}$. Let $e = 0 \oplus 1$. Then there is an $r \in \mathcal{R}$ such that $Te = re$. It follows for each $x \in \mathcal{K}$ that $T(x \oplus 1) = \gamma(x \oplus 1)$ for some $\gamma \in \mathcal{R}$ and $T(x \oplus 0) = \beta(x \oplus 0)$ for some $\beta \in \mathcal{R}$. However, we have

$$\gamma(x \oplus 1) = T(x \oplus 1) = T(x \oplus 0 + 0 \oplus 1) = \beta x \oplus r.$$

It follows that $\gamma = r$ and that

$$T(x \oplus 0) = \beta x \oplus 0 = rx \oplus 0 = r(x \oplus 0).$$

It follows that $Tv = rv$ for every $v \in \mathcal{M}$. $\square$

Suppose T is a linear transformation on a vector space $\mathcal{M}_T$ over $\mathbb{F}$. The set of all $\mathbb{F}[x]$-submodules of $\mathcal{M}_T$ is precisely $\mathrm{Lat}_0 T$, the set of all T-invariant linear subspaces of $\mathcal{M}_T$. The set of all $\mathbb{F}[x]$-endomorphisms is precisely the *commutant* $\{T\}'$ of T consisting of all linear transformations S on $\mathcal{M}_T$ such that $ST = TS$. The set of all linear transformations on $\mathcal{M}_T$ that leave invariant every T-invariant linear subspace is called $\mathrm{AlgLat}_0(T)$ (see [2]). Hence the locally scalar $\mathbb{F}[x]$-endomorphisms are precisely the linear transformations commuting with T and in $\mathrm{AlgLat}_0(T)$.

Proposition 9.2. *Suppose T is a linear transformation on $\mathcal{M}_T$. Then*

$$\{T\}' \cap AlgLat_0(T) = \{f(T) : f \in \mathbb{F}[x]\}^{-(strict)}.$$

We call a transformation *algebraically reflexive* if

$$AlgLat_0(T) = \{f(T) : f \in \mathbb{F}[x]\}.$$

and *algebraically hyporeflexive* if

$$\{T\}' \cap AlgLat_0(T) = \{f(T) : f \in \mathbb{F}[x]\}.$$

We see from Proposition 9.2 that T is algebraically hyporeflexive if and only if $\mathcal{M}_T$ is a scalar-reflexive $\mathbb{F}[x]$-module.

In [4, Theorem 8] the scalar-reflexive modules over a PID were completely characterized. In the case when $\mathcal{R} = \mathbb{F}[x]$, this characterization translates to the following characterization of algebraically hyporeflexive linear transformations.

Theorem 9.3. *A linear transformation T is algebraically hyporeflexive if and only if T is algebraic or not locally algebraic.*

Suppose the linear transformation T is locally algebraic but not algebraic. What is $\{f(T) : f \in \mathbb{F}[x]\}^{-(strict)}$? We know that $\mathcal{M}_T$ is the direct sum of $E_p(T)$, taken over all of the primes $p \in \mathbb{F}[x]$.

Claim 1: For each prime $p \in \mathbb{F}[x]$, the projection Q_p of $\mathcal{M}_T$ onto $E_p(T)$ is in $\{f(T) : f \in \mathbb{F}[x]\}^{-(strict)}$. To see this, suppose $W \subset \mathcal{M}_T$ is finite. Then there are finitely many distinct primes $p, p_1, \ldots, p_n$ such that $W \subset E_p(T) \oplus E_{p_1}(T) \oplus \cdots \oplus E_{p_n}(T)$. There is also a positive integer N such that if $f = p^N p_1^N \cdots p_n^N$, then $W \subset \ker f(T)$. Choose polynomials $u, v \in \mathbb{F}[x]$ such that $up^N + vp_1^N \cdots p_n^N = 1$. If $g = vp_1^N \cdots p_n^N$, then $g(T)x = Q_p x$ for every $x \in W$. Hence every strict neighborhood of Q_p intersects $\{f(T) : f \in \mathbb{F}[x]\}$, which shows $Q_p \in \{f(T) : f \in \mathbb{F}[x]\}^{-(strict)}$.

It easily follows from Claim 1 that

$$\{f(T) : f \in \mathbb{F}[x]\}^{-(strict)} = \prod_{p \in \mathbb{F}[x],\ p \text{ prime}} \left[\left\{h\left(T|_{E_p(T)}\right) : h \in \mathbb{F}[x]\right\}^{-(strict)}\right].$$

Claim 2: If $p \in \mathbb{F}[x]$ is prime and $T|_{E_p(T)}$ is algebraic, then

$$\left\{h\left(T|_{E_p(T)}\right) : h \in \mathbb{F}[x]\right\}^{-(strict)} = \left\{h\left(T|_{E_p(T)}\right) : h \in \mathbb{F}[x]\right\}.$$

This is because $T|_{E_p(T)}$ is hyporeflexive.

Claim 3: If $p \in \mathbb{F}[x]$ is prime, $d = \deg p$, and $T|_{E_p(T)}$ is not algebraic, then $\left\{h\left(T|_{E_p(T)}\right) : h \in \mathbb{F}[x]\right\}^{-(strict)} =$

(9.1)
$$\left\{\sum_{k=0}^{\infty} a_k\left(T|_{E_p(T)}\right) p^k\left(T|_{E_p(T)}\right) : a_0, a_1, \ldots \in \mathbb{F}[x],\ \deg a_k(x) < d \text{ for } k \geq 0\right\}.$$

It is clear that each such series converges in the strict topology, since $E_p(T) = \cup_{n=1}^{\infty} \ker p(T)^n$. On the other hand, $T|_{\ker p(T)^n}$ is algebraic, so $T|_{\ker p(T)^n}$ is hyporeflexive. This means that the strict closure of the polynomials in T, restricted to $\ker p(T)^n$ must be a polynomial in T, restricted to $\ker p(T)^n$. Every such polynomial can be written in the form $\sum_{k=0}^{n} a_k(T) p^k(T)$. This shows that any element in $\{f(T) : f \in \mathbb{F}[x]\}^{-(strict)}$, restricted to $E_p(T)$, must have the form as in (2) above.

Proposition 9.4. *If T is locally algebraic but not algebraic, then*

$$\{T\}' \cap AlgLat_0(T) = \{f(T) : f \in \mathbb{F}[x]\}^{-(strict)}$$

is the set of all linear transformations S on $\mathcal{M}_T$ such that, for each prime $p \in \mathbb{F}[x]$,

(1) $S|_{E_p(T)} = h(T)|_{E_p(T)}$ for some polynomial h if $T|_{E_p(T)}$ is algebraic, and

(2) $S|_{E_p(T)} = \sum_{k=0}^{\infty} a_k(T) p^k(T)|_{E_p(T)}$ for $a_0, a_1, \ldots \in \mathbb{F}[x]$, $\deg a_k(x) <$ d for $k \geq 0$ if $T|_{E_p(T)}$ is not algebraic.

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