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INFINITE-DIMENSIONAL VERSIONS OF THE PRIMARY, CYCLIC AND JORDAN DECOMPOSITIONS

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(Communicated by Peter Rosenthal)

Dedicated to the sincere, ever energetic Professor Heydar Radjavi on turning 40+40

Abstract. The famous primary and cyclic decomposition theorems along with the tightly related rational and Jordan canonical forms are extended to linear spaces of infinite dimensions with counterexamples showing the scope of extensions.

Keywords: Jordan canonical form, rational canonical form, splitting field.
MSC(2010): Primary; 15A21; Secondary: 12F05.

1. Introduction

In finite dimensions, polynomials in an operator are the simplest and yet the most important type of a functional calculus. It provides particular decompositions for the operator resulting a transparent structure of the operator revealing some of its important properties ([2,3,5]). Throughout the paper, the notation V is fixed for a linear space with the ground field $\mathbb{F}$, and $L(V)$ denotes the algebra of all linear operators on V. The parts of the domain appearing in the above mentioned decompositions are invariant subspaces of the following types:

$$\mathcal{S}(f,T) = \bigcup_{n=1}^{\infty} \mathbf{ker}(f^n(T)), \quad (\textbf{spectral subspace}) \tag{1.1}$$

$$Z(v,T) = \mathbf{span}_{\mathbb{F}}\{v, Tv, T^2v, \ldots\}, \; (\textbf{cyclic subspace}) \tag{1.2}$$

for some $f \in \mathbb{F}[x]$ and some $v \in V$; the linear span $\mathbf{span}_{\mathbb{F}}\, M$ of M is the smallest linear subspace of V containing a subset M which, if $M \neq \emptyset$, is equivalent to the family of all (finite) linear combinations of the elements of the set M with coefficients from the field $\mathbb{F}$. (Note that $f^n(T) = [f(T)]^n$ and $\mathbf{span}_{\mathbb{F}}\, \emptyset = \{0\}$.)

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We drop the subscript $\mathbb{F}$ in $\mathbf{span}_{\mathbb{F}}$ if no ambiguity arises. More generally, if $\Delta \subset V$, then $Z(\Delta, T)$ will denote $\mathbf{span}[\bigcup_{v\in\Delta} Z(v,T)]$. If M_j ($j \in \mathbb{J}$) is a family of linear subspaces of V, then $\mathbf{span}(\bigcup_{j\in\mathbb{J}} M_j)$, abbreviated as $\mathbf{span}_{j\in\mathbb{J}}M_j$, is the totality of all vectors represented as $v = \Sigma_{j\in\mathbb{J}}v_j$ in which $v_j \in M_j$ and all but finitely many v_j are 0. (The sum Σ_j means the finite sum of the nonzero terms only.) It is immediate that

$$\mathbf{span}_{j\in\mathbb{J}}M_j = \bigcup\{\sum_{j\in J} M_j : \ J \text{ finite subset of } \mathbb{J}\}. \tag{1.3}$$

The space $\mathbf{span}_{j\in\mathbb{J}}M_j$ is denoted by $\oplus_{j\in\mathbb{J}}M_j$ if the representation $v = \Sigma_{j\in\mathbb{J}}v_j$ is unique or, equivalently, $v_j = 0$ for all $j \in \mathbb{J}$ whenever $v = 0$. Again, it is immediate that

$$\oplus_{j\in\mathbb{J}}\, M_j = \bigcup\{\oplus_{j\in J}M_j : \ J \text{ finite subset of } \mathbb{J}\}. \tag{1.4}$$

As a consequence of (1.3)-(1.4), if $\{W_{ij} : \ i \in \mathcal{I},\ j \in \mathbb{J}_i\}$ is a family of linear subspaces of V for index sets $\mathbb{J}_i$ and some linearly ordered set $\mathcal{I}$ such that $J_i \subset J_k$ for $i \le k \in \mathcal{I}$, then

$$\mathbf{span}\{W_{ij} : \ i \in \mathcal{I},\ j \in \mathbb{J}_i\} = \bigcup_{i\in\mathcal{I}} \mathbf{span}\ \{W_{ij} : \ j \in \mathbb{J}_i\}. \tag{1.5}$$

Moreover, if $\mathbf{span}\ \{W_{ij} : \ j \in \mathbb{J}_i\} = \oplus\ \{W_{ij} : \ j \in \mathbb{J}_i\}$ for all $i \in \mathcal{I}$, then

$$\oplus\ \{W_{ij} : \ i \in \mathcal{I},\ j \in \mathbb{J}_i\} = \cup_{i\in\mathcal{I}}\ \oplus\ \{W_{ij} : \ j \in \mathbb{J}_i\}. \tag{1.6}$$

If V is finite-dimensional, then $V = \mathbf{ker}(f(T))$ for some nonzero $f \in \mathbb{F}[x]$; f is called an *annihilator* of T. If $f = p_1^{k_1}p_2^{k_2}\cdots p_h^{k_h}$ is the prime factorization of an annihilator f of T, it follows from the **primary decomposition theorem** that

$$V = \oplus_{i=1}^h \mathcal{S}(p_i, T), \text{ and } T = \oplus_{i=1}^h T|_{\mathcal{S}(p_i,T)}. \tag{1.7}$$

The collection of all annihilators of T form a principal ideal in $\mathbb{F}\,[x]$ generated by the so-called minimal polynomial of T. The minimal polynomial of $T|_{\mathcal{S}(p_i,T)}$ is equal to $p_i^{m_i}$, where $m_i \le k_i$ is the power of p_i appearing in the prime factorization of the minimal polynomial of T ($i = 1, 2, \dots, h$). If $m_i = 0$, then $\mathcal{S}(p_i, T) = \{0\}$ which is justified by the fact that the minimal polynomial of the zero operator is the constant polynomial $\mathbf{1}(x) \equiv 1$ or the identity polynomial $\mathbf{id}(x) \equiv x$ depending on whether or not $V = \{0\}$.

The family of all prime polynomials [3] (p. 135) in $\mathbb{F}[x]$ is denoted by $\mathcal{P}_{\mathbb{F}}$ or, simply, $\mathcal{P}$ if no ambiguity arises. In general, a minimal polynomial f means the (nonzero) monic polynomial [3] (p. 120) generating the ideal of all polynomials g satisfying a certain (nice) condition. The ideals that we are usually dealing with in this paper are of one of the following forms:

(1.8) $$\{g \in \mathbb{F}[x] : \ g(T) = 0\},$$

(1.9) $$\{g \in \mathbb{F}[x] : \ g(T)\omega = 0\},$$

(1.10) $$\{g \in \mathbb{F}[x] : \ g(T)\omega \in W\} \text{ and}$$

(1.11) $$\{g \in \mathbb{F}[x] : \ g(\lambda_i) = 0; i = 1, 2, \cdots, n\}$$

for some fixed T, W, ω, λ_i, where $T \in L(V)$, W is an invariant subspace of T, $\omega \in V$ and $\lambda_1, \ldots, \lambda_n$ are in an extension $\mathbb{G}$ of the field $\mathbb{F}$. The generator of the ideal (1.8) is called the *minimal polynomial of* T, that of (1.9) is called the local minimal polynomial of T at ω, that of (1.10) is called the *minimal* T*-conductor* of ω into W, and that of (1.11) is called the *minimal* $\mathbb{F}$*-vanisher* at $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{G}$. Finally, the notation

(1.12) $$x \oplus y \in X \oplus Y \text{ means } x \in X \text{ and } y \in Y;$$

i.e., besides expressing $x + y \in X \oplus Y$, it also asserts that $x \in X$ and $y \in Y$; the mere notation $x + y \in X \oplus Y$ does not reveal any information on the locations of x and y.

The **cyclic decomposition theorem** (for T on a finite-dimensional space V) employs the primary decomposition theorem to yield a different decomposition

(1.13) $$V = \oplus_{j=1}^{r} \oplus_{\alpha \in \Lambda_j} Z(\alpha, T),$$

in which

(1) r is a positive integer,
(2) for $1 \le j \le r$, the operator $T|_{Z(\alpha,T)}$ has a fixed minimal polynomial f_j as α runs in a given nonempty finite subset Λ_j of V, and
(3) if $1 \le j \le r-1$, then $f_j \ne f_{j+1}|f_j$.

(In case $V = \{0\}$, the equation (1.13) reduces to $V = Z(0, T)$.)

However, the cyclic decomposition theorem has the following simple version.

Theorem 1.1. ***(Simplified cyclic decomposition theorem)*** *For* $T \in L(V)$ *with a minimal polynomial* $f \in \mathbb{F}[x]$*, the following equivalent assertions are true.*

(1) *There exists a subset* Λ *of* V *such that* $V = \oplus\{Z(\alpha, T) : \ \alpha \in \Lambda\}$.
(2) *There exists a subset* Λ *of* $\cup_{p \in \mathcal{P}_{\mathbb{F}}} \mathcal{S}(p, T)$ *such that* $V = \oplus\{Z(\alpha, T) : \ \alpha \in \Lambda\}$.
(3) *There exist subsets* $\Lambda_1, \ldots, \Lambda_r$ *of* V *for which (1.13) holds.*

Proof. Assume (1) holds and let $\alpha \in \Lambda$ be arbitrary. Let $g = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$ be the prime factorization of the minimal polynomial of $S = T|_{Z(\alpha,T)}$. In view of the primary decomposition theorem $Z(\alpha, T) = \oplus_{j=1}^{t} \mathcal{S}(p_j, S)$ and $\alpha = \oplus_{j=1}^{t} \beta_j$, where $\beta_j \in \mathcal{S}(p_j, S) = \{x \in Z(\alpha, T) : \ p_j^{\ell}(S)x = p_j^{\ell}(T)x = 0; \ell \in \mathbb{N}\}$. Then

$Z(\alpha, T) \supset \oplus_{j=1}^{t} Z(\beta_j, S) = \oplus_{j=1}^{t} Z(\beta_j, T) \supset Z(\alpha, T)$. Now, replacing each α by its components β_1, β_2, etc., yields a new set Λ satisfying (2).

Next, assume (2) holds and let $p_1, p_2, \ldots, p_s$ be the distinct prime factors of f. Rearrange the prime factors in such a way that the cardinality of the set $\Theta_k := \Lambda \cap \mathcal{S}(p_k, T)$ increases as k increases. Let Θ_k be indexed as $\Theta_k = \{\alpha_{ki} : \ i \in \mathcal{I}_k\}$ and assume without loss of generality that $\mathcal{I}_k \subset \mathcal{I}_{k+1}$ for $i = 1, 2, \ldots, s-1$. Let $k_1 = 1, k_2, \ldots, k_r$ be the indices at which the cardinalities of $\mathcal{I}_{k_j}$ and $\mathcal{I}_{k_j-1}$ are different. (Set $\mathcal{I}_0 = \emptyset$.) Now, define

$$\Lambda_j = \{\oplus_{k \geq k_j} \ \alpha_{ki} : \ i \in \mathcal{I}_{k_j} \backslash \mathcal{I}_{k_j-1}\}, \ (j = 1, 2, \ldots, r).$$

Again, by letting $\alpha = \oplus_{k \geq k_j} \ \alpha_{ki}$, $f_j = \Pi_{k \geq k_j} \ p_k$ and applying the primary decomposition theorem to $S = T|_{Z(\alpha,T)}$, one can conclude that

$$Z(\alpha, T) = \oplus_{k \geq k_j} \ \mathcal{S}(p_k, S) \supset \oplus_{k \geq k_j} Z(\alpha_{ki}, T) \supset Z(\alpha, T),$$

which implies that $Z(\alpha, T) = \oplus_{k \geq k_j} Z(\alpha_{ki}, T)$ and that T has a local minimal polynomial f_j at $\alpha \in \Lambda_j$. This proves (3) and the proof of (3) $\Longrightarrow$ (1) is clear.

The proof that T satisfies (3) is the classical cyclic decomposition theorem in finite-dimensional case. The proof in the infinite dimension follows from Theorem 1.5 of the present paper. $\square$

Let Λ be as in Parts (1) or (2) of Theorem 1.1; for Part (2), define $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_r$. Any of the **cyclic decompositions** given in Theorem 1.1 specifies a **rational canonical form** $T = \oplus_{\alpha \in \Lambda} T_\alpha$ in which

$$[T_\alpha]_{\mathcal{A}_\alpha} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & 0 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -c_{\ell-2} \\ 0 & 0 & 0 & \cdots & 1 & -c_{\ell-1} \end{bmatrix}, \ \alpha \in \Lambda \tag{1.14}$$

with respect to the ordered basis $\mathcal{A}_\alpha := \{\alpha, T\alpha, T^2\alpha, \ldots, T^{\ell-1}\alpha\}$, where ℓ and c_i's are the parameters appearing in the minimal polynomial $f_\alpha(x) = x^\ell + c_{\ell-1}x^{\ell-1} + \cdots + c_1 x + c_0$ of T at α.

(Note that the 1×1 matrix $[0]$ represents the matrix of the zero operator on a 1-dimensional space, and that the matrix of the zero operator on the space $\{0\}$ is a vacuous matrix.)

Furthermore, if the minimal polynomial f of T splits in $\mathbb{F}$ as

$$f(x) = (x - \lambda_1)^{m_1}(x - \lambda_2)^{m_2} \cdots (x - \lambda_n)^{m_n}, \tag{1.15}$$

it yields a primary decomposition $V = \oplus_{i=1}^{n} \mathcal{S}(\mathbf{id} - \lambda_i \mathbf{1}, T)$. Also, applying the cyclic decomposition theorem to each operator $S = T|_{\mathcal{S}(\mathbf{id}-\lambda_i\mathbf{1},T)}$ yields the so-called **Jordan decomposition**

$$\mathcal{S}(\mathbf{id} - \lambda_i \mathbf{1}, T) = \oplus_{j=1}^{r_i} \oplus_{\alpha \in \Lambda_{ij}} Z(\alpha, T), \tag{1.16}$$

where T has a minimal polynomial $(x-\lambda_i\mathbf{1})^{k_{ij}}$ for all $\alpha \in \Lambda_{ij}$ and some finite sequence of positive integers $k_{i1} > k_{i2} > \cdots > k_{ir_i}$. For $\alpha \in \Lambda_{ij}$, the restriction of the operator $T-\lambda_i$ to $Z(\alpha, T)$ is a nilpotent operator N_{ij} and, hence, the matrix of $T|_{Z(\alpha,T)}$ with respect to the basis

$$\mathcal{C} = \{\alpha, (T-\lambda_i I)\alpha, (T-\lambda_i I)^2\alpha, \ldots, (T-\lambda_i I)^{k_{ij}-1}\alpha\},$$

is the so-called **Jordan block**

$$[\lambda_i I + N_{ij}]_{\mathcal{C}} = \begin{bmatrix} \lambda_i & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda_i & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 0 \\ 0 & 0 & 0 & \cdots & 1 & \lambda_i \end{bmatrix}. \tag{1.17}$$

The direct sum of all Jordan blocks related to a given operator T is called its **Jordan canonical form**.

The aim of the present paper is to see the extent to which the primary decomposition (1.7), the cyclic decomposition (1.13), the rational canonical form (1.14), the Jordan decomposition (1.16) and the Jordan canonical form (1.17) can be extended when the underlying space is infinite-dimensional; this is done in Section 2. (See also [1,4,5].) The uniqueness of the decompositions is discussed in Section 3. The proofs in Section 2 are simplification as well as adaptation to the infinite dimension of those given in [4]. The paper is concluded with a section on open problems.

The following lemma constitutes some essential facts needed in the proof of our main results. The proof is left to the interested reader.

Lemma 1.2. *Let $p \in \mathcal{P}_{\mathbb{F}}$ and let $f = p^m g$ for some nonnegative integer m and some polynomial $g \in \mathbb{F}[x]$ not divisible by p. Let $T \in L(V)$, let $v \in V$ and let W be an invariant subspace of T. The following assertions are true.*

(1) *If f is the minimal polynomial of T, then p^m is the minimal polynomial of $T|_{\mathcal{S}(p,T)}$. Moreover, if $m \geq 1$, then there exists $u \in V$ such that $p^{m-1}(T)g(T)u \neq 0$.*
(2) *If f is the local minimal polynomial of T at v and if $p^{m-1}(T)g(T)y = 0$ for some $y \in Z(v,T)$, then $y = p(T)u$ for some $u \in Z(v,T)$.*
(3) *If f is the minimal polynomial of T and h is the minimal T-conductor of v into W, then $h|f$.*
(4) *If f is the minimal T-conductor of v into W, then p is the minimal T-conductor of $u := p^{m-1}(T)g(T)v$ into W.*

The following examples reveal the motivation for the main result of the paper stated in Theorem 1.5.

Example 1.3. Let $\mathcal{A} = \mathbb{N}$ (resp. $\mathcal{A} = \mathbb{Z}$) and define $T \in L(V_{\mathbb{F}})$ to be the forward shift defined by $Te_n = e_{n+1}$ $(n \in \mathcal{A})$, where e_n is the nth element of the standard basis of $V_{\mathbb{F}}$. Fix $p(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 \in \mathcal{P}$ and $v = e_n + b_{n-1}e_{n-1} + \cdots + b_{n-k}e_{n-k} \in V\backslash\{0\}$ with $a_i, b_j \in \mathbb{F}$, $m \in \mathbb{N}$, $n \in \mathcal{A}$ and $k \geq 0$. Then $p^r(T)v = y + e_{n+rm} \neq 0$ for all positive integers r, where $y \in \mathbf{span}\{e_j : j \leq n+rm-1\}$. This shows that $\mathbf{span}_{p\in\mathcal{P}}\mathcal{S}(p,T) = \oplus_{p\in\mathcal{P}}\mathcal{S}(p,T) = \{0\}$ and $V_{\mathbb{F}} = Z(e_1, T)$ (resp. $V_{\mathbb{F}} = Z(\tilde{e}, T)$), where $\tilde{e} = \{e_n : n \in \mathbb{Z}\}$. (See Section 4 for the definition of T-invariant sequences $\tilde{v}$.)

Example 1.4. With the notation of Example 1.3, assume $\mathcal{A} = \mathbb{N}$. Let $S \in L(V_{\mathbb{F}})$ be the backward shift defined by $Se_n = e_{n-1}$ where we set $e_0 = 0$. Then $S^n v = 0$, which shows that $V_{\mathbb{F}} = \mathcal{S}(\mathbf{id}, S)$, where $\mathbf{id}(x) \equiv x$. Moreover, if $\mathbf{id} \neq p \in \mathcal{P}$, then $p^r(S)v = z + a_0^r e_n \neq 0$, where $z \in \mathbf{span}\{e_1, e_2, \ldots, e_{n-1}\}$. Thus, $\mathcal{S}(p, S) = \{0\}$ for $\mathbf{id} \neq p \in \mathcal{P}$ and that $V_{\mathbb{F}} = \oplus_{p\in\mathcal{P}}\mathcal{S}(p,T)$.

The two extreme Examples 1.3 and 1.4 point out the following infinite dimensional versions of the primary and cyclic decomposition theorems combined in one theorem; the proof will be given in the next section.

Theorem 1.5. ***(The primary-cyclic decomposition theorem)*** *Let $T \in L(U)$ be an arbitrary linear operator on a general nonzero vector space U such that T has a minimal polynomial f. Then there exists a subset Λ of $[\cup_{p\in\mathcal{P}}\ \mathcal{S}(p,T)]$ such that*

$$\mathbf{span}_{g\in\mathbb{F}[x]}\mathcal{S}(g,T) = \oplus_{p\in\mathcal{P}}\ \mathcal{S}(p,T), \text{ and} \tag{1.18}$$

$$U = \oplus_{p\in\mathcal{P}}\ \mathcal{S}(p,T) = \oplus_{\alpha\in\Lambda}\ Z(\alpha,T). \tag{1.19}$$

Moreover, if $W \subset U$ is such that $Z(W,T) = \oplus_{w\in W}\ Z(w,T)$ and $U = Z(W,T) + Z(\Theta(W),T)$, then Λ can be chosen to satisfy $W \subset \Lambda$, where

$$\Theta(W) := \{\theta \in U : \mathbf{deg}(f_\theta) \leq \mathbf{deg}(f_w)\ \forall w \in W\} \tag{1.20}$$

$$f_\zeta = \textit{the minimal polynomial of } T \text{ at } \zeta \textit{ for all } \zeta \in U. \tag{1.21}$$

Note. Some summands in (1.18) may be trivial.

2. Proof of Theorem 1.5

Proof. We break the proof into several steps.

Step 1. *Prove that* $\mathbf{span}_{g\in\mathbb{F}[x]}\ \mathcal{S}(g,T) = \oplus_{p\in\mathcal{P}}\ \mathcal{S}(p,T)$.

Fix $g \in \mathbb{F}[x]$ and $y \in \mathcal{S}(g,T)$ with the prime factorization $g = p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$. Apply the primary decomposition theorem to the restriction S of T to the finite-dimensional invariant subspace $Z(y,T)$, to conclude that $y = y_1 + y_2 + \cdots + y_k$ with $y_j \in \mathcal{S}(p_j,T)$ $(j = 1, 2, \ldots, k)$. Thus $\mathbf{span}_{g\in\mathbb{F}[x]}\ \mathcal{S}(g,T) =$

$\mathbf{span}_{p\in\mathcal{P}}\ \mathcal{S}(p,T)$. In view of (1.3) and (1.4), it follows from the primary decomposition theorem for the finite-dimensional spaces that

$$\begin{aligned}\mathbf{span}_{p\in\mathcal{P}}\ \mathcal{S}(p,T) &= \bigcup\{\sum_{p\in\mathcal{P}} \mathcal{S}(p,T):\ P \text{ finite subset of } \mathcal{P}\}\\ &= \bigcup\{\oplus_{p\in\mathcal{P}}\mathcal{S}(p,T):\ P \text{ finite subset of } \mathcal{P}\}\\ &= \oplus_{p\in\mathcal{P}}\mathcal{S}(p,T).\end{aligned}$$

Step 2. *Prove that* $U = \oplus_{p\in\mathcal{P}}\ [U\cap\mathcal{S}(p,T)]$.

It follows from Step 1 that $U = \oplus_{p\in\mathcal{P}}\ \mathcal{S}(p,T) = \oplus_{p\in\mathcal{P}}\ \mathcal{S}(p,T)$. But $\mathcal{S}(p,T)\neq\{0\}$ if and only if $p|f$ and thus $U = \oplus_{p\in\mathcal{P}}\ \mathcal{S}(p,T)$.

To proceed to the next step, we retreat without loss of generality to a simpler case of the problem. We may and shall assume without loss of generality in the remainder of the proof that $U=\mathcal{S}(p,T)$ for some $p\in\mathcal{P}_{\mathbb{F}}$; i.e., T has a minimal polynomial $f(x)=p^k$ for some $p\in\mathcal{P}_{\mathbb{F}}$.

Step 3. *For* W *as in the last conclusion of the theorem, show that there exists a maximal subset* Λ *of* $V(=\mathcal{S}(p,T))$ *such that* $W\subset\Lambda$ *and*

$$V = Z(\Lambda,T) = \oplus_{\omega\in\Lambda}Z(\omega,T). \tag{2.1}$$

Let $\mathcal{X}$ be the collection of all pairs (Ω,Θ) such that $W\subset\Omega\subset V$, $Z(\Omega,T) = \oplus_{\omega\in\Omega}Z(\omega,T)$ and $V = Z(\Theta,T)+Z(\Omega,T)$, where $\Theta=\Theta(\Omega)$ as defined in the theorem. Define $(\Omega,\Theta)\preceq(\Omega',\Theta')$ in $\mathcal{X}$ if $\Omega\subset\Omega'$. Note that, in view of Lemma 1.2, there exists $\omega\in\mathcal{S}(p,T)$ such that $p^{m-1}(T)\omega\neq 0$. Therefore, $(\{\omega\},V)\in\mathcal{X}\neq\emptyset$. We prove that every monotone increasing chain $\{(\Omega_j,\Theta)_j\}_{j\in\mathbb{J}}$ of elements of $\mathcal{X}$ has an upper bound in $\mathcal{X}$. Here, the index set $\mathbb{J}$ is linearly ordered. Let $\Omega=\cup_{j\in\mathbb{J}}\Omega_j$ and let $\Theta=\Theta(\Omega)$. By (1.3)-(1.6),

$$Z(\Omega,T) = \cup_{j\in\mathbb{J}}Z(\Omega_j,T) = \cup_{j\in\mathbb{J}}\oplus_{\omega\in\Omega_j} Z(\omega,T) = \oplus_{\omega\in\Omega}Z(\omega,T).$$

Let $m=m(\Omega):=\min\{\mathbf{deg}(f_\omega):\ \omega\in\Omega\}$ and let $n_i=n(\Theta_i)\ \ (i\in\mathbb{J})$, where $n(\Theta):=\max\{\mathbf{deg}(f_\theta):\ \theta\in\Theta\}$. Since $\{n_i\}_{i\in\mathbb{J}}$ is decreasing, it follows that $n_i\equiv n$ for some constant integer n and for all i greater than or equal to a fixed $\kappa\in\mathbb{J}$. Let $\Theta=\Theta_\kappa$. Then $V=Z(\Theta_\kappa,T)+Z(\Omega_\kappa,T)\subset Z(\Theta,T)+Z(\Omega,T)=V$ and, hence, $(\Omega,\Theta)\in\mathcal{X}$.

Now, by the Hausdorff maximal principle, $\mathcal{X}$ contains a maximal element (Λ,Θ) with the corresponding parameters $m=m(\Omega)$ and $n=n(\Theta)$. Let $\theta\in\Theta$ be arbitrary and assume p^ℓ is its minimal T-conductor to $Z(\Lambda,T)$. Then $\nu=\nu(\theta):=\mathbf{deg}(p^\ell)\leq n$ and, thus, $p^\ell(T)\theta=p^h(T)\xi$ where $\xi=g_1(T)\omega_1\oplus g_2(T)\omega_2\oplus\cdots\oplus g_\mu(T)\omega_\mu$ for some integers $\mu\geq 1$ and $h\geq 0$, some $\omega_i\in\Omega$, and some $g_i\in\mathbb{F}[x]$ $(i=1,2,\ldots,\mu)$ such that $p\nmid g_1$. Let t be the smallest nonnegative integer such that $p^{\ell+t}(T)\theta=0$. Then $p^{h+t}(T)\omega_1=0$, which implies that $h\geq\ell$. Let $\theta'^{h-\ell}(T)\xi$ and observe that the minimal polynomial of T at θ' is p^ℓ. Choose θ with a maximal ℓ and call it ω_0. Let $\Omega'=\Omega\cup\{\omega_0\}$, let

$\Theta' = \Theta(\Omega)$ and observe that $(\Omega, \Theta) \preceq (\Omega', \Theta') \in \mathcal{X}$. Thus, $\omega_0 = 0$ and, hence, $\omega_0 = 0$. This means that $V = Z(\Lambda, T)$. The complete proof follows from the special case that $W = \{0\}$. □

The formula (1.19) is the cyclic decomposition of the underlying space of T and the corresponding rational canonical form is formulated as

$$T = \oplus_{\alpha\in\Lambda} T_\alpha, \tag{2.2}$$

where each T_α is an operator satisfying (1.14).

We conclude the section with a result which was crossed by a couple of times in the above arguments.

Corollary 2.1. *If $p_1, p_2, \dots, p_r \in \mathcal{P}$ are distinct and if $\alpha_i \in \mathcal{S}(p_i, T)$ for $i = 1, 2, \dots, r$, then $\oplus_{i=1}^r Z(\alpha_i, T) = Z(\Sigma_{i=1}^r \alpha_i, T)$.*

The proof follows from the fact that the local minimal polynomial of T at each α_i is of the form $p_i^{s_i}$ $(i = 1, 2, \dots, r)$, and that of T at $\sum_{i=1}^r \alpha_i$ is $q := p_1^{s_1} \cdots p_r^{s_r}$. Hence,

$$\mathbf{dim} \oplus_{i=1}^r \ Z(\alpha_i, T) = \sum_{i=1}^r \ \mathbf{deg}\ p_i^{s_i} = \mathbf{deg}\ q = \mathbf{dim}\ Z(\sum_{i=1}^r \alpha_i, T).$$

Since $Z(\sum_{i=1}^r \alpha_i, T) \subset \oplus_{i=1}^r Z(\alpha_i, T)$, it follows that $Z(\sum_{i=1}^r \alpha_i, T) = \oplus_{i=1}^r Z(\alpha_i, T)$.

3. Uniqueness results

One of the main features of the cyclic decomposition (1.13) for the operators on a finite-dimensional subspace is the uniqueness of the cardinalities $\mathbf{card}(\Lambda_j)$, $j = 1, 2, \dots, r$. Whether the result remains true in the infinite dimension is left as an open problem. Here, we show that the polynomials appearing in (1.19) are unique. In fact, more is revealed by the following theorem.

Theorem 3.1. *Fix T and U as in Theorem 1.5 and let Λ and Γ be two different collections of nonzero vectors establishing the cyclic decompositions of the form (1.19). Fix $p \in \mathcal{P}$ and $j \in \mathbb{N}$. Define $\Lambda(p, j) = \{\alpha \in \Lambda :\ p^j$ is the minimal polynomial of T at $\alpha\}$ and define $\Gamma(p, j)$, accordingly. Then*

$$\mathbf{card}(\Lambda(p, j)) = \mathbf{card}(\Gamma(p, j)) \text{ if } \ 0 \le \mathbf{card}(\Lambda(p, j)) < \infty.$$

Proof. Assume $\mathbf{card}(\Lambda(p, j)) \ge n$ for some $n \in \mathbb{N}$. Fix a subset $\Lambda_0 = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ of $\Lambda(p, j)$. There exists a finite subset $\Gamma_0 = \{\beta_1, \beta_2, \dots, \beta_m\}$ of Γ such that $\Lambda_0 \subset W := \oplus_{i=1}^m \ Z(\beta_i, T)$. Obviously, W is finite-dimensional and Γ_0 yields a cyclic decomposition for $T|_W$. On the other hand, $W = W_0 \oplus W_1$, where $W_0 = \oplus_{i=1}^n Z(\alpha_i, T)$ and $W_1 = W \cap \oplus_{\alpha\in\Theta} Z(\alpha, T)$ with $\Theta = \Lambda \backslash \Lambda_0$. Find $\Lambda_1 \subset W_1$ such that $W_1 = \oplus_{\alpha\in\Lambda_1} Z(\alpha, T)$ and define $\Omega = \Lambda_0 \cup \Lambda_1$. By Theorem 1.5, $W = \oplus_{\omega\in\Omega} Z(\omega, T)$ and, by the uniqueness of the finite-dimensional case, $\Gamma_0 \cap \Gamma(p, j)$ has at least n elements. □

4. Open problems

For a general operator $T \in L(V)$ the following problems remain unsettled.

Problem 1. When is there an invariant subspace W of T such that

$$V = W \oplus [\oplus_{p\in\mathcal{P}}\mathcal{S}(p,T)]?$$

In the following, $\tilde{v} = (v_n)_{n\in\mathbb{K}} \subset V$ is called an invariant sequence of $T \in L(V)$, if $\mathbb{K} = \mathbb{N}$ or $\mathbb{Z}$ and $v_{n+1} = Tv_n$ for all $n \in \mathbb{K}$. For an invariant sequence $\tilde{v} = (v_n)_{n\in\mathbb{K}}$, we define $Z(\tilde{v},T) = \cup_{n\in\mathbb{K}}Z(v_n,T)$ and observe that, if $\mathbb{K} = \mathbb{N}$, then $Z(\tilde{v},T) = Z(v_1,T)$.

Problem 2. If $V = W \oplus [\oplus_{p\in\mathcal{P}}\mathcal{S}(p,T)]$ for some invariant subspace W of T, when is it true that

$$W = \oplus_{\tilde{v}\in\mathfrak{S}}Z(\tilde{v},T)$$

for some collection $\mathfrak{S}$ of invariant sequences of T?

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References

[1] J. D. Botha, Alternative proofs of the rational canonical form theorem, *Internat. J. Math. Ed. Sci. Tech.* **25** (1994), no. 5, 745–749.

[2] I.N. Herstein, Topics in Algebra, Blaisdell Publishing Co. Ginn and Co., New York-Toronto-London, 1964.

[3] K. Hoffman and R. Kunze, Linear Algebra, Second Edition, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1971.

[4] M. Radjabalipour, The rational canonical form via the splitting field, *Linear Algebra Appl.* **439** (2013), no. 8, 2250–02255.

[5] S.H. Weintraub, Jordan Canonical Form: Theory and Practice, *Synthesis Lectures on Mathematics and Statistics*, **2** (2009), no. 1, 1–108.

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