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SUBMAJORIZATION INEQUALITIES ASSOCIATED WITH τ-MEASURABLE OPERATORS

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(Communicated by Laurent Marcoux)

Dedicated to Professor Heydar Radjavi on his 80th birthday

Abstract. The aim of this note is to study the submajorization inequalities for τ-measurable operators in a semi-finite von Neumann algebra on a Hilbert space with a normal faithful semi-finite trace τ. The submajorization inequalities generalize some results due to Zhang, Furuichi and Lin, etc..
Keywords: Submajorization, von Neumann algebra, τ-measurable operators.
MSC(2010): Primary: 46L10; Secondary: 47A63, 47C15.

1. Introduction

First we recall the definition of majorization. Given a real vector $x = (x_1, x_2, \dots, x_n) \in R^n$, we rearrange its components as $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$. For $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in R^n$, if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]},\ k = 1, 2, \dots, n$$

then we say that x is weakly majorized by y and write $x \prec_w y$. If $x \prec_w y$ and $\sum\limits_{i=1}^{n} x_i = \sum\limits_{i=1}^{n} y_i$, then we say that x is majorized by y and write $x \prec y$.

Denote by M_n the usual von Neumann algebra consisting of $n \times n$ complex matrices. Let $A \in M_n$. We always denote the singular values of A by $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$, and denote $s(A) := (s_1(A), s_2(A), \dots, s_n(A))$. If A is a Hermitian matrix, we denote the eigenvalues of A in decreasing order by

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$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ and denote $\lambda(A) := (\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))$. Recently, Zhang [15] proved the following result:

Theorem 1.1. *Let $H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ be a positive semidefinite block matrix with square matrices A and B of the same order. Then for any complex number z with $|z| = 1$,*

$$\lambda(H) \prec \frac{1}{2}\lambda([A + B + 2Re(zX)] \oplus 0) + \frac{1}{2}\lambda([A + B - 2Re(zX)] \oplus 0).$$

If, in addition, X is Hermitian, then for any real number $r \in [-2, 2]$

$$\lambda(H) \prec \frac{1}{2}\lambda((A + B + rX) \oplus 0) + \frac{1}{2}\lambda((A + B - rX) \oplus 0),$$

while if X is skew-Hermitian, then for any real number $r \in [-2, 2]$

$$\lambda(H) \prec \frac{1}{2}\lambda((A + B + irX) \oplus 0) + \frac{1}{2}\lambda((A + B - irX) \oplus 0),$$

where $Re(X) = \frac{X+X^*}{2}$, $i = \sqrt{-1}$ and 0 is the zero matrix of compatible size.

It is worth pointing that in the majorization theory, the submajorization (see definition in Section 2), is one of the most important orderings. Many authors have studied it. We refer to [1, 3, 4, 7–13] for more related results on this topic.

The purpose of this note is to extend Theorem 1.1 to the case that $H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in M_2(\mathcal{M})$ is $\hat{\tau}$-measurable positive operator in a semifinite von Neumann algebra. In addition, we also study the submajorization among $(A + B) \oplus (A + B)$, $A \oplus B$ and $(|A| + |B|) \oplus 0$, where A, $B \in \mathcal{M}$ (see below) are τ-measurable operators.

2. **Preliminaries**

Denote by $\mathcal{M}$ a semi-finite von Neumann algebra on the Hilbert space $\mathcal{H}$ with a normal faithful semi-finite trace τ. The closed densely defined linear operator T in $\mathcal{H}$ with domain $D(T)$ is said to be affiliated with $\mathcal{M}$ if $U^*TU = T$ for all unitary U that belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If T is affiliated with $\mathcal{M}$, then T is said to be τ-measurable if for every $\varepsilon > 0$ there exists a projection $P \in \mathcal{M}$ such that $P(\mathcal{H}) \subseteq D(T)$ and $\tau(P^\perp) < \varepsilon$ (where for any projection P we let $P^\perp = I - P$, I here denotes the identity operator). The set of all τ-measurable operators will be denoted by $\overline{\mathcal{M}}$ which is the closure of $\mathcal{M}$ with respect to the measure topology, which is the linear (Hausdorff) topology whose fundamental system of neighborhoods around 0 is given by

$$V(\varepsilon, \delta) = \{T \in \mathcal{M} \mid \|TP\| \leq \varepsilon \text{for some projection} P \in \mathcal{M} \text{ with} \tau(I - P) \leq \delta\}$$

Here, ε, δ run over all strictly positive numbers and $\|\cdot\|$ is the operator norm.

For a positive self-adjoint operator $T=\int_0^\infty \lambda dE_\lambda$ affiliated with $\mathcal{M}$, we set

$$\tau(T)=\sup_n \tau\big(\int_0^n \lambda dE_\lambda\big)=\int_0^\infty \lambda d\tau(E_\lambda).$$

For $0<p<\infty$, $L^p(\mathcal{M};\tau)$ is defined as the set of all densely defined closed operators T affiliated with $\mathcal{M}$ such that

$$\|T\|_p=\tau(|T|^p)^{\frac{1}{p}}<\infty,$$

where $|T|=(T^*T)^{\frac{1}{2}}$. In addition, we put $L^\infty(\mathcal{M};\tau)=\mathcal{M}$ and denote by $\|\cdot\|_\infty$ $(=\|\cdot\|)$ the usual operator norm. It is well known that $L^p(\mathcal{M};\tau)$ is a Banach space under $\|\cdot\|_p$ $(1\le p\le\infty)$ satisfying all the expected properties such as duality.
Let T be a τ-measurable operator and $t>0$. The t-th singular number (generalized s-number) of T, denoted by $\mu_t(T)$, is

$$\mu_t(T)=\inf\{\|TP\| \mid \text{P a projection in } \mathcal{M}/mboxwith\tau(I-P)\le t\}.$$

Then $\mu_t(T)$ has the following properties (see [5, Lemma 2.5]).

Lemma 2.1. *Let T, S, R be τ-measurable operators.*

(i) *The map: $t\in(0,\infty)\to\mu_t(T)$ is non-increasing and continuous from the right. Moreover*

$$\lim_{t\to 0^+}\mu_t(T)=\|T\|\in[0,\infty].$$

(ii) $\mu_t(T)=\mu_t(|T|)=\mu_t(T^*)$ *and* $\mu_t(\alpha T)=|\alpha|\mu_t(T)$ *for* $t>0$ *and* $\alpha\in C$.
(iii) $\mu_t(T)\le\mu_t(S)$, $t>0$, *if* $0\le T\le S$.
(iv) $\mu_t(f(|T|))=f(\mu_t(|T|))$, $t>0$ *for any continuous increasing function* f *on* $[0,\infty]$ *with* $f(0)\ge 0$.
(v) $\mu_{t+s}(T+S)\le\mu_t(T)+\mu_s(S)$, $t,s>0$.
(vi) $\mu_t(STR)\le\|S\|\|R\|\mu_t(T)$, $t>0$.
(vii) $\mu_{t+s}(TS)\le\mu_t(T)\mu_s(S)$, $t,s>0$.

From (vi) of Lemma 2.1, we immediately know that $\mu_t(\cdot)$ has unitary invariance property, i.e.,

$$\mu_t(U^*TU)=\mu_t(T)$$

for τ measurable operators T, U with U unitary.

Next, we will give the definition of submajorization. The submajorization of operators is a generalization of the notion of submajorization for functions introduced by Hardy, Littlewood and Polya.

Definition 2.2. If f and g are measurable positive decreasing functions on $(0,\infty)$, then we say that g is submajorized by f and write $g \prec\prec f$ if

$$\int_0^t g(s)ds \le \int_0^t f(s)ds$$

for all $t > 0$. Given $A,\ B \in \overline{\mathcal{M}}$, we say that B is submajorized by A and write $B \prec\prec A$ if $\mu(B) \prec\prec \mu(A)$.

For example, it has been established that $\mu(A+B) \prec\prec \mu(A)+\mu(B)$ and $\mu(AB) \prec\prec \mu(A)\mu(B)$ for $A,\ B \in \overline{\mathcal{M}}$ (see Theorems 4.4 and 4.2 in [5]).

We remark that if $\mathcal{M} = M_n$ and τ is the standard trace, i.e., $\tau(A) = tr(A) = \sum_{i=1}^{n} a_{ii}$ for $A = (a_{ij}) \in M_n$, then

$$\mu_t(A) = s_j(A),\ t \in [j-1, j),\ j = 1, 2, \ldots, n,$$

and if $A, B \in \mathcal{M}$, then $A \prec\prec B$ is equivalent to

$$\sum_{i=1}^{k} s_i(A) \le \sum_{i=1}^{k} s_i(B),\ k = 1, 2, \ldots, n.$$

3. **Submajorization inequalities**

Let $M_2(\mathcal{M})$ be the von Neumann algebra

$$M_2(\mathcal{M}) = \left\{ \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \Big| A_{ij} \in \mathcal{M},\ i,j = 1,2 \right\}$$

on the Hilbert space $\mathcal{H} \oplus \mathcal{H}$ with trace $\hat{\tau} = tr \otimes \tau$, i.e., $\hat{\tau}(A) = \tau(A_{11} + A_{22})$ for $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in M_2(\mathcal{M})$. For $A \in M_2(\mathcal{M})$, let

$$\mathcal{C}(A) = \mathcal{C}\left(\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \right) = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} = \frac{1}{2}\sum_{k=1}^{2} (U^*)^k A U^k, \tag{3.1}$$

where $U = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. Then $\mathcal{C} : M_2(\mathcal{M}) \to M_2(\mathcal{M})$ is a trace-preserving positive contraction.

In this section, we mainly study submajorization inequalities. To achieve our goal, we need the following lemma. Its matrix version was given by Bourin and Lee in [2].

Lemma 3.1. *Let* $H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in M_2(\mathcal{M})$ *be* $\hat{\tau}$*-measurable positive operator. Then there exist partial isometries* $U, V \in M_2(\mathcal{M})$ *such that*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = U \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} V^*. \tag{3.2}$$

Proof. Since H is positive, then there exists a positive $\hat{\tau}$-measurable operator $\begin{pmatrix} C & Y \\ Y^* & D \end{pmatrix} \in M_2(\mathcal{M})$ such that

$$\begin{aligned} \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} &= \begin{pmatrix} C & Y \\ Y^* & D \end{pmatrix} \begin{pmatrix} C & Y \\ Y^* & D \end{pmatrix} \\ &= \begin{pmatrix} C & 0 \\ Y^* & 0 \end{pmatrix} \begin{pmatrix} C & Y \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & Y \\ 0 & D \end{pmatrix} \begin{pmatrix} 0 & 0 \\ Y^* & D \end{pmatrix} \\ &= T^*T + S^*S \end{aligned}$$

where $T = \begin{pmatrix} C & Y \\ 0 & 0 \end{pmatrix}$, $S = \begin{pmatrix} 0 & 0 \\ Y^* & D \end{pmatrix}$.

Let $T^* = U|T^*|$, $S^* = V|S^*|$ be the polar decompositions, respectively. It is easy to check that

$$T^*T = U \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} U^*$$

and

$$S^*S = V \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} V^*.$$

□

Based on Lemma 3.1, we obtain the main result.

Theorem 3.2. *Let* $H = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in M_2(\mathcal{M})$ *be a* $\hat{\tau}$*-measurable positive operator. Then for any complex number* z *with* $|z| = 1$,

$$\begin{aligned} &\frac{1}{2}\mu([A+B-Y] \oplus [A+B+Y]) \prec\prec \\ &\mu(H) \prec\prec \frac{1}{2}\mu([A+B-Y] \oplus 0) + \frac{1}{2}\mu([A+B+Y] \oplus 0) \end{aligned} \tag{3.3}$$

where $Y = 2Re(zX)$, *and* 0 *is the zero operator.*

Proof. By Lemma 2.1, 3.1 and [5, Theorem 4.4], we have

$$\begin{aligned}
&\int_0^t \mu_s\left(\begin{pmatrix} A & X \\ X^* & B \end{pmatrix}\right) ds \\
&\leq \int_0^t \mu_s\left(\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\right) ds + \int_0^t \mu_s\left(\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}\right) ds \\
&= \int_0^t \mu_s\left(\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\right) ds \\
&+ \int_0^t \mu_s\left(\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right) ds \\
&= \int_0^t \mu_s\left(\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\right) ds + \int_0^t \mu_s\left(\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}\right) ds \qquad (3.4)
\end{aligned}$$

for $t > 0$.

Taking $J_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}$ and $J_z = \begin{pmatrix} I & 0 \\ 0 & \overline{z}I \end{pmatrix}$, then J_1 and J_z are unitaries and

$$J_1 J_z H J_z^* J_1^* = \begin{pmatrix} \frac{A+B-Y}{2} & K \\ K^* & \frac{A+B+Y}{2} \end{pmatrix} \qquad (3.5)$$

where $K = \frac{A-B+(zX-\overline{z}X^*)}{2}$.

Noting that $\mu_s(H) = \mu_s(J_1 J_z H J_z^* J_1^*)$ for $s > 0$, then the second inequality in (3.3) follows from (3.4).

On the other hand, by Lemma 2.1, equality (3.1), [5, Theorem 4.4], we have

$$\int_0^t \mu_s(\mathcal{C}(H)) ds \leq \int_0^t \mu_s(H) ds,$$

for $t > 0$, which is equivalent to

$$\mu(A \oplus B) \prec\prec \mu(H). \qquad (3.6)$$

According to (3.5) and $\mu_s(H) = \mu_s(J_1 J_z H J_z^* J_1^*)$ for $s > 0$, the relation (3.6) yields

$$\frac{1}{2}\mu([A+B-Y] \oplus [A+B+Y]) \prec\prec \mu(H).$$

This completes the proof of the first inequality in (3.3). □

Corollary 3.3. *Let X, $Y \in \mathcal{M}$ be τ-measurable operators. Then for any complex number z with $|z| = 1$,*

$$2\mu(XX^* + YY^*) \prec\prec \mu(X^*X + Y^*Y - Z) + \mu(X^*X + Y^*Y + Z) \qquad (3.7)$$

where $Z = \overline{z}X^*Y + zY^*X$.

Proof. Let $T_1 = \begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix}$. Then

$$T_1T_1^* = \begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X^* & 0 \\ Y^* & 0 \end{pmatrix} = \begin{pmatrix} XX^*+YY^* & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$T_1^*T_1 = \begin{pmatrix} X^* & 0 \\ Y^* & 0 \end{pmatrix}\begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X^*X & X^*Y \\ Y^*X & Y^*Y \end{pmatrix}.$$

for $t > 0$.

On the other hand, let $T_1 = U_1|T_1|$ be a polar decomposition. Then

$$T_1T_1^* = U_1(T_1^*T_1)U_1^*.$$

Thus,

$$\int_0^t \mu_s(T_1T_1^*)ds = \int_0^t \mu_s(U_1T_1^*T_1U_1^*)ds \le \int_0^t \mu_s(T_1^*T_1)ds.$$

By Theorem 3.2, we get the desired results. □

Putting $z = 1$ in Z. Then inequality (3.7) gives

$$\begin{aligned} 2\mu(XX^*+YY^*) \prec\prec\, &\mu(X^*X+Y^*Y-(X^*Y+Y^*X)) \\ &+\mu(X^*X+Y^*Y+(X^*Y+Y^*X)). \end{aligned}$$

This is a generalization of a matricial result obtained by Turkman, Paksoy and Zhang in [14].

Corollary 3.4. *Let $S, T \in \mathcal{M}$ be τ-measurable positive operators. Then for any real number $r \in [-2, 2]$,*

$$2\mu(T^2+ST^2S) \prec\prec \mu(T^2+TS^2T+rTST)+\mu(T^2+TS^2T-rTST). \tag{3.8}$$

Proof. Let $X = T$ and $Y = ST$. Then $zX^*Y + \overline{z}Y^*X = 2Re(z)TST$. Since $2Re(z)$ ranges over [-2,2] as we vary z over all complex number of modulus 1, the result follows from Corollary 3.6. □

Remark 3.5. The matrix form of 3.8 when $r = 1$ was obtained by Furuichi and Lin in [6].

Corollary 3.6. *Let $A, B \in \mathcal{M}$ be τ-measurable operators. Then for any real number $r \in [-2, 2]$, the following inequality holds,*

$$\begin{aligned} 2\mu\left(\begin{pmatrix} A^*A+B^*B & A^*B+B^*A \\ A^*B+B^*A & A^*A+B^*B \end{pmatrix}\right) \prec\prec\, &\mu([A^*A+B^*B-rC]\oplus 0) \\ &+\mu([A^*A+B^*B+rC]\oplus 0) \end{aligned} \tag{3.9}$$

where $C = A^*B + B^*A$.

Proof. Since both $\begin{pmatrix} A^*A & A^*B \\ B^*A & B^*B \end{pmatrix}$ and $\begin{pmatrix} B^*B & B^*A \\ A^*B & A^*A \end{pmatrix}$ are $\hat{\tau}$-measurable positive operators, so is $\begin{pmatrix} A^*A+B^*B & A^*B+B^*A \\ A^*B+B^*A & A^*A+B^*B \end{pmatrix}$. Then by Theorem 3.2, we obtain the result (3.9). □

4. **Further results**

In this section, we mainly study the submajorization among $(A+B)\oplus(A+B)$, $A\oplus B$ and $(|A|+|B|)\oplus 0$, where A, $B \in \mathcal{M}$ are τ-measurable operators. We have the following results.

Theorem 4.1. *Let A, $B \in \mathcal{M}$ be τ-measurable operators. Then the following holds:*

$$\frac{1}{2}\mu([A+B]\oplus[A+B]) \prec\prec \mu(A\oplus B) \prec\prec \mu([|A|+|B|]\oplus 0). \tag{4.1}$$

Proof. Since

$$\begin{pmatrix} A+B & 0 \\ 0 & A+B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{4.2}$$

by [5, Theorem 4.4] and Lemma 2.1, the first submajorization of (4.1) holds at once.

On the other hand, let $A = U_1|A|$, $B = U_2|B|$ be polar decompositions. Then

$$\begin{pmatrix} |A|+|B| & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} |A|^{\frac{1}{2}} & |B|^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} |A|^{\frac{1}{2}} & 0 \\ |B|^{\frac{1}{2}} & 0 \end{pmatrix} \tag{4.3}$$

and

$$\begin{pmatrix} |A|^{\frac{1}{2}} & 0 \\ |B|^{\frac{1}{2}} & 0 \end{pmatrix}\begin{pmatrix} |A|^{\frac{1}{2}} & |B|^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} |A| & |A|^{\frac{1}{2}}|B|^{\frac{1}{2}} \\ |B|^{\frac{1}{2}}|A|^{\frac{1}{2}} & |B| \end{pmatrix}. \tag{4.4}$$

There exists a partial isometry U_3 such that

$$\begin{pmatrix} |A| & |A|^{\frac{1}{2}}|B|^{\frac{1}{2}} \\ |B|^{\frac{1}{2}}|A|^{\frac{1}{2}} & |B| \end{pmatrix} = U_3\begin{pmatrix} |A|^{\frac{1}{2}} & |B|^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} |A|^{\frac{1}{2}} & 0 \\ |B|^{\frac{1}{2}} & 0 \end{pmatrix}U_3^*. \tag{4.5}$$

Then by Lemma 2.1, (3.1), (4.2), (4.4), (4.5) and [5, Theorem 4.4], we have

$$\begin{aligned}
&\int_0^t \mu_s\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right) ds \\
&\quad \le \int_0^t \mu_s\left(\begin{pmatrix} |A| & 0 \\ 0 & |B| \end{pmatrix}\right) ds \\
&\quad \le \int_0^t \mu_s\left(\begin{pmatrix} |A| & |A|^{\frac{1}{2}}|B|^{\frac{1}{2}} \\ |B|^{\frac{1}{2}}|A|^{\frac{1}{2}} & |B| \end{pmatrix}\right) ds \\
&\quad = \int_0^t \mu_s\left(U_3\begin{pmatrix} |A|^{\frac{1}{2}} & |B|^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} |A|^{\frac{1}{2}} & 0 \\ |B|^{\frac{1}{2}} & 0 \end{pmatrix}U_3^*\right) ds \\
&\quad \le \int_0^t \mu_s\left(\begin{pmatrix} |A|+|B| & 0 \\ 0 & 0 \end{pmatrix}\right) ds
\end{aligned}$$

for $t > 0$.

By Definition 2.2, the second submajorization of (4.1) also holds. □

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