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EQUIVALENT A POSTERIORI ERROR ESTIMATES FOR SPECTRAL ELEMENT SOLUTIONS OF CONSTRAINED OPTIMAL CONTROL PROBLEM IN ONE DIMENSION

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Abstract. In this paper, we study spectral element approximation for a constrained optimal control problem in one dimension. The equivalent a posteriori error estimators are derived for the control, the state and the adjoint state approximation. Such estimators can be used to construct adaptive spectral elements for the control problems.
Keywords: Optimal control problem, spectral element method, a posteriori error estimates, control constraint, polynomial inverse estimates.
MSC(2010): Primary: 65N15; Secondary: 49M25.

1. Introduction

Optimal control problems can be found in many scientific and engineering applications, and it has become a very active and successful research area in recent years. The literature on this field is huge, and it is impossible to give even a very brief review here. Beginning with the papers by Babuška and Rheinboldt in 1978 [2,3], the study of a posteriori error estimates for the finite element solution of partial differential equations has attracted great interest and resulted in an enormous body of literature on the subject. We refer the interested reader in posteriori error estimates to see [1, 4] for an overview, and [5, 14, 18, 23] for some recent works in the subject. In the recent years, there has been intensive research in adaptive finite element methods (FEMs) for optimal control problems, see, for example, [6,19,21,22,25], and the references cited therein.

To the best of the authors' knowledge, most research concerning adaptive finite element methods for optimal control problems are all related to low order FEM, i.e., h-FEM, there are not many published works related to the use of high order methods, such as the p and hp-version FEMs, spectral methods,

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and spectral element methods, which have been applied to many practical problems. In recent years, the spectral method has been extended to approximate unconstrained optimal control problems and a posteriori error estimate was obtained in [16]. Later, Chen *et al* in [11, 12] respectively derived a priori and a posteriori error estimates for the spectral approximation of optimal control problems governed by elliptic equations and Stokes equations. Very recently, a posteriori error estimates of the *hp* FEM for optimal control problems was investigated in [13, 17], both reliability and efficiency of the estimators were analyzed. In [11], the spectral element method was also applied in the analysis of the optimal control problems, but only a posteriori upper error estimates was obtained.

The purpose of this paper is to derive sharp a posteriori error estimates for the spectral element approximation of an optimal control problem in one dimension to partially improve the above mentioned result in [11]. In our work, we use some techniques that have been used for a posteriori error estimates of the *hp*-version FEM for optimal control problems (see, e.g., [13, 17] for more details). We also use some polynomial inverse estimates and the weighted techniques in [9], where a posteriori error estimates was obtained for spectral element method of the elliptic partial differential equations.

The outline of the paper is as follows: In the next section we formulate the optimal control problem under consideration and give the spectral element discretization of the control problem. In Section 3, some technical lemmas are introduced, which are used for the later a posteriori error analysis. Section 4 is devoted to deriving sharp a posteriori error estimators for the control problem. We carry out, in Section 5, some numerical tests to verify the theoretical results. Some concluding remarks are given at the end of the paper.

2. Optimization problem and spectral element approximation

2.1. Notations and problem description. We first introduce some notations that will be used throughout the paper. Let c or C be a generic positive constant independent of any functions and of any discretization parameters. We use the expression $A \lesssim B$ to mean that $A \leq cB$, and use the expression $A \cong B$ to mean that $A \lesssim B \lesssim A$. Let $\Lambda = (a, b)$, we use $L^2(\Lambda), H^1(\Lambda)$, and $H_0^1(\Lambda)$ to denote the usual Sobolev spaces, equipped with the norms $\|\cdot\|_{0,\Lambda}$ and $\|\cdot\|_{1,\Lambda}$ respectively. Hereafter, in cases where no confusion would arise, the domain symbols Λ may be dropped from the notations.

Throughout this paper, we mainly concentrate on the following one-dimensional optimal control problem for the state variable u and the control variable q with an integral constraint:

$$\min_q \left\{ \frac{1}{2} \int_\Lambda (u - \bar{u})^2 \mathrm{d}x + \frac{\lambda}{2} \int_\Lambda q^2 \mathrm{d}x \right\}, \tag{2.1}$$

where λ and $\bar{u}$ are given, u is governed by the state equation

$$\begin{aligned} -u'' &= f+q \quad \text{in} \quad \Lambda, \\ u &= 0 \quad \text{on} \quad \partial\Lambda, \end{aligned} \tag{2.2}$$

and q satisfying

$$\int_\Lambda q \mathrm{d}x \geq 0. \tag{2.3}$$

We take the state space $V = H_0^1(\Lambda)$, then the standard weak formulation of the state equation (2.2) reads: given $q, f \in L^2(\Lambda)$, find $u \in V$ such that

$$a(u,v) = (f+q, v), \quad \forall v \in V, \tag{2.4}$$

where the bilinear form $a(\cdot,\cdot)$ is defined by

$$a(u,v) = \int_\Lambda u'v' \mathrm{d}x. \tag{2.5}$$

It is well known that the following continuity and coercivity hold

$$a(u,v) \lesssim \|u\|_1 \|v\|_1, \; a(u,u) \gtrsim \|u\|_1^2, \quad \forall u,v \in V,$$

and the problem (2.4) is well-posed.

To formulate the optimal control problem we introduce the admissible set $\mathcal{K}$ associated to the constraints (2.3) as

$$\mathcal{K} := \left\{ q \in L^2(\Lambda) : \int_\Lambda q \mathrm{d}x \geq 0 \right\},$$

and define the cost functional

$$\mathcal{J}(q,u) := \frac{1}{2}\|u-\bar{u}\|_{0,\Lambda}^2 + \frac{\lambda}{2}\|q\|_{0,\Lambda}^2, \; (q,u) \in \mathcal{K} \times V, \tag{2.6}$$

where the given desired state $\bar{u} \in L^2(\Lambda)$.

Then the optimal control problem reads: find $(q^*, u(q^*)) \in \mathcal{K} \times V$, such that

$$\mathcal{J}(q^*, u(q^*)) = \min_{(q,u)\in\mathcal{K}\times V} \mathcal{J}(q,u) \quad \text{subject to (2.4)}. \tag{2.7}$$

Proposition 2.1. *For given $f, \bar{u} \in L^2(\Lambda)$ and $\lambda > 0$, the optimal control problem (2.7) admits a unique solution $(q^*, u(q^*)) \in \mathcal{K} \times V$.*

Proof. The existence follows from weak sequential limit arguments, see e.g. [20]. The uniqueness relies on the convexity of Λ and on the strict convexity of the cost functional $\mathcal{J}(q,u)$. □

The above proposition ensures the unique existence of a control-to-state mapping $q \mapsto u = u(q)$ defined through (2.4). By means of this mapping we introduce the reduced cost functional $J : L^2(\Lambda) \to \mathbb{R}$:

$$J(q) := \mathcal{J}(q, u(q)).$$

Then the optimal control problem (2.7) can be equivalently reformulated as: find $q^* \in \mathcal{K}$, such that

$$J(q^*) = \min_{q \in \mathcal{K}} J(q). \tag{2.8}$$

The first order necessary optimality condition for (2.8) reads as

$$J'(q^*)(\delta q - q^*) \geq 0, \quad \forall \delta q \in \mathcal{K}. \tag{2.9}$$

It is known that the convexity of the quadratic functional implies that (2.9) is also sufficient for optimality.

Utilizing the adjoint state equation for $z \in V$ given by

$$a(\varphi, z) = (u - \bar{u}, \varphi), \ \forall \varphi \in V, \tag{2.10}$$

and the mapping $q \to u(q) \to z(q)$, where for any given q, $u(q)$ is defined by (2.4), and once $u(q)$ is known $z(q)$ is defined by (2.10), then the first derivative of the reduced cost functional can be expressed as

$$J'(q)(\delta q) = (\lambda q + z(q), \delta q). \tag{2.11}$$

2.2. **Spectral element discretization.** To proceed with the spectral element discretization of the proposed optimal control problem, we partition the domain Λ into a set of $K \geq 2$ disjoint subintervals so that

$$\bar{\Lambda} = \bigcup_{k=1}^{K} \bar{\Lambda}_k, \quad \Lambda_k \cap \Lambda_l = \emptyset, \text{ if } k \neq l,$$

where $\Lambda_k = (x_{k-1}, x_k), k = 1, \cdots, K$, with $a = x_0 < x_1 < \cdots < x_{K-1} < x_K = b$. Let $h_k = x_k - x_{k-1}$ be length of the k-th interval. Let $\hat{\Lambda} = (-1, 1)$ be the reference element, then there exists the affine map $F_k : \hat{\Lambda} \to \Lambda_k$

$$x = F_k(\hat{x}) = \frac{h_k}{2}\hat{x} + \frac{x_k + x_{k-1}}{2}, \ \hat{x} \in \hat{\Lambda} \tag{2.12}$$

that maps $(-1, 1)$ into Λ_k.

We then introduce the piecewise polynomial spaces as follows:

$$\begin{aligned} V_{\mathcal{M}} &= \left\{ v \in C^0(\Lambda) : v|_{\Lambda_k} \in P_{M_k}(\Lambda_k), k = 1, \cdots, K \right\}, \\ Q_{\mathcal{N}} &= \left\{ v \in L^2(\Lambda) : v|_{\Lambda_k} \in P_{N_k}(\Lambda_k), k = 1, \cdots, K \right\}, \end{aligned}$$

where $P_{M_k}(\Lambda_k)$ (respectively, $P_{N_k}(\Lambda_k)$) denotes the space of all polynomials on Λ_k of degree less than or equal to M_k (resp.N_k), $\mathcal{M}$ and $\mathcal{N}$ respectively collect the positive integers M_k $(k = 1, \cdots, K)$ and N_k $(k = 1, \cdots, K)$.

Let $Q_{\mathcal{N}}$ be the spectral element space for the control variable, and $V_{\mathcal{M}}$ be the spectral element space for the state and costate, then a spectral element approximation of the state equation (2.4) reads: find a state $u_{\mathcal{M}}(q_{\mathcal{N}}) \in V_{\mathcal{M}}$ such that

$$a(u_{\mathcal{M}}(q_{\mathcal{N}}), v_{\mathcal{M}}) = (f + q_{\mathcal{N}}, v_{\mathcal{M}}), \quad \forall v_{\mathcal{M}} \in V_{\mathcal{M}}, \tag{2.13}$$

where $q_{\mathcal{N}} \in Q_{\mathcal{N}}$.

Similar to the continuous case, we introduce the discrete reduced cost functional $J_{\mathcal{M},\mathcal{N}} : Q_{\mathcal{N}} \to \mathbb{R}$:

$$J_{\mathcal{M},\mathcal{N}}(q_{\mathcal{N}}) := \mathcal{J}(q_{\mathcal{N}}, u_{\mathcal{M}}(q_{\mathcal{N}})),$$

where $u_{\mathcal{M}}(q_{\mathcal{N}})$ is given by (2.13). Let $\mathcal{K}_{\mathcal{N}} = \mathcal{K} \cap Q_{\mathcal{N}}$, then the spectral element approximation for the optimal control problem (2.8) reads as: find $q^*_{\mathcal{N}} \in \mathcal{K}_{\mathcal{N}}$, such that

$$J_{\mathcal{M},\mathcal{N}}(q^*_{\mathcal{N}}) = \min_{q_{\mathcal{N}} \in \mathcal{K}_{\mathcal{N}}} J_{\mathcal{M},\mathcal{N}}(q_{\mathcal{N}}). \tag{2.14}$$

The unique solution of (2.14), $q^*_{\mathcal{N}}$, satisfies the following optimality condition:

$$J'_{\mathcal{M},\mathcal{N}}(q^*_{\mathcal{N}})(\delta q - q^*_{\mathcal{N}}) \geq 0, \quad \forall \delta q \in \mathcal{K}_{\mathcal{N}}, \tag{2.15}$$

where

$$J'_{\mathcal{M},\mathcal{N}}(q_{\mathcal{N}})(\delta q) = (\lambda q_{\mathcal{N}} + z_{\mathcal{M}}(q_{\mathcal{N}}), \delta q), \ \forall q_{\mathcal{N}}, \delta q \in \mathcal{K}_{\mathcal{N}}, \tag{2.16}$$

with $z_{\mathcal{M}}(q_{\mathcal{N}}) \in V_{\mathcal{M}}$ being the solution of the discrete adjoint state equation

$$a(\varphi_{\mathcal{M}}, z_{\mathcal{M}}(q_{\mathcal{N}})) = (u_{\mathcal{M}}(q_{\mathcal{N}}) - \bar{u}, \varphi_{\mathcal{M}}), \quad \forall \varphi_{\mathcal{M}} \in V_{\mathcal{M}}. \tag{2.17}$$

3. Some preliminary properties

In this section, we recall some results which will be used in what follows. The first lemma shows that the functional $J(\cdot)$ defined in previous section is uniformly convex.

Lemma 3.1. *[17] For all $p, q \in L^2(\Lambda)$, it holds*

$$J'(p)(p-q) - J'(q)(p-q) \geq \lambda \|p - q\|^2 .$$

In the following, we state two lemmas, which give a relationship between the control variable and the adjoint state variable. The readers can refer to [11] and [15] for the details.

Lemma 3.2. *[11] Let $(q^*, u(q^*))$ be the solution of the continuous optimal control problem (2.7) and $z(q^*)$ be the corresponding adjoint state. Then we have*

$$q^* = \frac{1}{\lambda} \max\{0, \overline{z(q^*)}\} - \frac{1}{\lambda} z(q^*),$$

where $\overline{z(q^*)} = \int_\Lambda z(q^*) / \int_\Lambda 1$.

Lemma 3.3. *[15] Let $q^*_{\mathcal{N}}$ be the solution of the discrete optimal control problem (2.14), and $z_{\mathcal{M}}(q^*_{\mathcal{N}})$ be the corresponding adjoint state. Then it holds*

$$q^*_{\mathcal{N}} = \frac{1}{\lambda} \Pi_{\mathcal{N}} \left(-z_{\mathcal{M}}(q^*_{\mathcal{N}}) + \max\{0, \overline{z_{\mathcal{M}}(q^*_{\mathcal{N}})}\} \right), \tag{3.1}$$

where $\Pi_{\mathcal{N}}$ is L^2-projection from $L^2(\Lambda)$ onto $Q_{\mathcal{N}}$.

The following polynomial inverse estimates, which can be found in [7, 8], play an important part in the analysis of the a posteriori error estimators.

Lemma 3.4. *For all polynomials* $\phi_N \in P_N(-1,1)$, *it holds*

$$\int_{-1}^{1} \phi_N'^2(x)(1-x^2)^2 \mathrm{d}x \le cN^2 \int_{-1}^{1} \phi_N^2(x)(1-x^2)\mathrm{d}x, \tag{3.2}$$

$$\int_{-1}^{1} \phi_N^2(x)\mathrm{d}x \le cN^2 \int_{-1}^{1} \phi_N^2(x)(1-x^2)\mathrm{d}x. \tag{3.3}$$

4. **Equivalent a posteriori error estimators**

We aim in this section at deriving a sharp estimates of the error between continuous solution and its spectral element approximation in terms of known and computable quantities, i.e, a posteriori error estimates. We will confine ourselves to the so-called residual-based estimates [10].

Throughout this section, for arbitrary variable $\mathcal{O}$, we shall denote by $\mathcal{O}^{(k)}$ the restriction of $\mathcal{O}$ to the k-th element Λ_k. In order to define a posteriori error indicator, let us introduce the L^2 orthogonal projection of $\bar{u}^{(k)}$ upon the space $P_{M_k}(\Lambda_k)$, which we denote by $\bar{u}_{M_k}$, i.e. $\bar{u}_{M_k} = \Pi_{M_k}\bar{u}^{(k)}$. Let f_{M_k} be defined similarly. Next, we introduce the weight function

$$w_k(x) = (x_k - x)(x - x_{k-1})$$

vanishing at the endpoints of the interval, and the associated weighted L^2-norm

$$\|g\|_{0,w_k} = \left(\int_{\Lambda_k} g^2(x) w_k(x)\mathrm{d}x\right)^{\frac{1}{2}}.$$

We now define some a posteriori error estimators:

$$\zeta^2 = \sum_{k=1}^{K} \zeta_k^2, \tag{4.1}$$

$$\xi^2 = \sum_{k=1}^{K} \left(\xi_k^2 + \frac{1}{M_k(M_k+1)} \left\|\bar{u}^{(k)} - \bar{u}_{M_k}^{(k)}\right\|_{0,w_k}^2\right), \tag{4.2}$$

$$\eta^2 = \sum_{k=1}^{K} \left(\eta_k^2 + \frac{1}{M_k(M_k+1)} \left\|f^{(k)} - f_{M_k}^{(k)}\right\|_{0,w_k}^2\right), \tag{4.3}$$

where

$$\zeta_k^2 = \left\| z_{\mathcal{M}}^{(k)}(q_{\mathcal{N}}^*) - \Pi_{N_k} z_{\mathcal{M}}^{(k)}(q_{\mathcal{N}}^*) \right\|_{0,\Lambda_k}^2, \tag{4.4}$$

$$\xi_k^2 = \frac{1}{M_k(M_k+1)} \left\| \left(z_{\mathcal{M}}^{(k)}(q_{\mathcal{N}}^*) \right)'' + u_{\mathcal{M}}^{(k)}(q_{\mathcal{N}}^*) - \bar{u}_{M_k}^{(k)} \right\|_{0,w_k}^2, \tag{4.5}$$

$$\eta_k^2 = \frac{1}{M_k(M_k+1)} \left\| \left(u_{\mathcal{M}}^{(k)}(q_{\mathcal{N}}^*) \right)'' + q_{\mathcal{N}}^{*(k)} + f_{M_k}^{(k)} \right\|_{0,w_k}^2, \tag{4.6}$$

with Π_{N_k} being the standard L^2-projection onto $P_{N_k}(\Lambda_k)$.

In the following two subsections, we will prove the efficiency and reliability of these a posteriori error indicators. In addition, just for simplicity of presentation, the symbols (k) in $\mathcal{O}^{(k)}$ may be dropped if there is no confusion.

4.1. **A posteriori upper error estimates.** We are now in a position to discuss the reliability of the above mentioned error indicators, which gives the upper bound of the error between the exact solution and its spectral element approximation. To this end, we need to introduce some auxiliary problems:

$$a(u(q_{\mathcal{N}}), v) = (f + q_{\mathcal{N}}, v), \quad \forall v \in V, \tag{4.7}$$

$$a(\varphi, z(q_{\mathcal{N}})) = (u(q_{\mathcal{N}}) - \bar{u}, \varphi), \quad \forall \varphi \in V. \tag{4.8}$$

Lemma 4.1. *Let q^* be the solution of the continuous optimal control problem (2.8), $q_{\mathcal{N}}^*$ be the solution of the discrete optimization problem (2.14), then it holds*

$$\|q^* - q_{\mathcal{N}}^*\|_0 \le \frac{1}{\lambda} \|z_{\mathcal{M}}(q_{\mathcal{N}}^*) - \Pi_{\mathcal{N}} z_{\mathcal{M}}(q_{\mathcal{N}}^*)\|_0 + \frac{1}{\lambda} \|z_{\mathcal{M}}(q_{\mathcal{N}}^*) - z(q_{\mathcal{N}}^*)\|_0,$$

where $z_{\mathcal{M}}(q_{\mathcal{N}}^*)$ and $z(q_{\mathcal{N}}^*)$ are respectively the solutions of (2.17) and (4.8) associated to $q_{\mathcal{N}}^*$.

Proof. It follows from Lemma 3.1, (2.9), (2.11), (2.15) and (2.16) that for arbitrary $p_{\mathcal{N}} \in \mathcal{K}_{\mathcal{N}}$,

(4.9)
$$\begin{aligned}
\lambda \|q^* - q_{\mathcal{N}}^*\|_0^2 \le & J'(q^*)(q^* - q_{\mathcal{N}}^*) - J'(q_{\mathcal{N}}^*)(q^* - q_{\mathcal{N}}^*) \\
\le & - J'(q_{\mathcal{N}}^*)(q^* - q_{\mathcal{N}}^*) \\
= & J'_{\mathcal{M},\mathcal{N}}(q_{\mathcal{N}}^*)(q_{\mathcal{N}}^* - q^*) - J'(q_{\mathcal{N}}^*)(q^* - q_{\mathcal{N}}^*) + J'_{\mathcal{M},\mathcal{N}}(q_{\mathcal{N}}^*)(q^* - q_{\mathcal{N}}^*) \\
= & J'_{\mathcal{M},\mathcal{N}}(q_{\mathcal{N}}^*)(q_{\mathcal{N}}^* - p_{\mathcal{N}}) + J'_{\mathcal{M},\mathcal{N}}(q_{\mathcal{N}}^*)(p_{\mathcal{N}} - q^*) \\
& + (z_{\mathcal{M}}(q_{\mathcal{N}}^*) - z(q_{\mathcal{N}}^*), q^* - q_{\mathcal{N}}^*) \\
\le & J'_{\mathcal{M},\mathcal{N}}(q_{\mathcal{N}}^*)(p_{\mathcal{N}} - q^*) + (z_{\mathcal{M}}(q_{\mathcal{N}}^*) - z(q_{\mathcal{N}}^*), q^* - q_{\mathcal{N}}^*) \\
= & (\lambda q_{\mathcal{N}}^* + z_{\mathcal{M}}(q_{\mathcal{N}}^*), p_{\mathcal{N}} - q^*) + (z_{\mathcal{M}}(q_{\mathcal{N}}^*) - z(q_{\mathcal{N}}^*), q^* - q_{\mathcal{N}}^*).
\end{aligned}$$

Let $\Pi_{\mathcal{N}}$ be the L^2 orthogonal projection operator defined in Lemma 3.3, then it holds

$$\int_\Lambda (q^* - \Pi_{\mathcal{N}} q^*) r_{\mathcal{N}} \mathrm{d}x = 0, \ \forall r_{\mathcal{N}} \in Q_{\mathcal{N}},$$

and in particular

$$\int_\Lambda (q^* - \Pi_{\mathcal{N}} q^*) \mathrm{d}x = 0,$$

that is

$$\int_\Lambda \Pi_{\mathcal{N}} q^* \mathrm{d}x = \int_\Lambda q^* \mathrm{d}x \geq 0.$$

This means $\Pi_{\mathcal{N}} q^* \in \mathcal{K}_{\mathcal{N}}$. Thus, by taking $p_{\mathcal{N}} = \Pi_{\mathcal{N}} q^*$ in the first term of the estimate (4.9), we get

(4.10)

$$\begin{aligned}
&(\lambda q^*_{\mathcal{N}} + z_{\mathcal{M}}(q^*_{\mathcal{N}}), p_{\mathcal{N}} - q^*) \\
=&(\lambda q^*_{\mathcal{N}} + z_{\mathcal{M}}(q^*_{\mathcal{N}}) - \Pi_{\mathcal{N}} z_{\mathcal{M}}(q^*_{\mathcal{N}}) + \Pi_{\mathcal{N}} z_{\mathcal{M}}(q^*_{\mathcal{N}}), \Pi_{\mathcal{N}} q^* - q^*) \\
=&(\lambda q^*_{\mathcal{N}} + \Pi_{\mathcal{N}} z_{\mathcal{M}}(q^*_{\mathcal{N}}), \Pi_{\mathcal{N}} q^* - q^*) + (z_{\mathcal{M}}(q^*_{\mathcal{N}}) - \Pi_{\mathcal{N}} z_{\mathcal{M}}(q^*_{\mathcal{N}}), \Pi_{\mathcal{N}} q^* - q^*) \\
=&(z_{\mathcal{M}}(q^*_{\mathcal{N}}) - \Pi_{\mathcal{N}} z_{\mathcal{M}}(q^*_{\mathcal{N}}), \Pi_{\mathcal{N}} q^* - q^*).
\end{aligned}$$

Now let I_d denote the identity operator, combining (4.9) and (4.10), and noting $\Pi_{\mathcal{N}} q^*_{\mathcal{N}} = q^*_{\mathcal{N}}$, we claim that

$$\begin{aligned}
& \lambda \|q^* - q^*_{\mathcal{N}}\|_0^2 \\
\leq\ & (z_{\mathcal{M}}(q^*_{\mathcal{N}}) - \Pi_{\mathcal{N}} z_{\mathcal{M}}(q^*_{\mathcal{N}}), \Pi_{\mathcal{N}} q^* - q^*) + (z_{\mathcal{M}}(q^*_{\mathcal{N}}) - z(q^*_{\mathcal{N}}), q^* - q^*_{\mathcal{N}}) \\
=\ & (z_{\mathcal{M}}(q^*_{\mathcal{N}}) - \Pi_{\mathcal{N}} z_{\mathcal{M}}(q^*_{\mathcal{N}}), (\Pi_{\mathcal{N}} - I_d)(q^* - q^*_{\mathcal{N}})) \\
& + (z_{\mathcal{M}}(q^*_{\mathcal{N}}) - z(q^*_{\mathcal{N}}), q^* - q^*_{\mathcal{N}}) \\
\leq\ & \|z_{\mathcal{M}}(q^*_{\mathcal{N}}) - \Pi_{\mathcal{N}} z_{\mathcal{M}}(q^*_{\mathcal{N}})\|_0 \|(\Pi_{\mathcal{N}} - I_d)(q^* - q^*_{\mathcal{N}})\|_0 \\
& + \|z_{\mathcal{M}}(q^*_{\mathcal{N}}) - z(q^*_{\mathcal{N}})\|_0 \|q^* - q^*_{\mathcal{N}}\|_0 \\
\leq\ & \|z_{\mathcal{M}}(q^*_{\mathcal{N}}) - \Pi_{\mathcal{N}} z_{\mathcal{M}}(q^*_{\mathcal{N}})\|_0 \|q^* - q^*_{\mathcal{N}}\|_0 \\
& + \|z_{\mathcal{M}}(q^*_{\mathcal{N}}) - z(q^*_{\mathcal{N}})\|_0 \|q^* - q^*_{\mathcal{N}}\|_0,
\end{aligned}$$

which immediately leads to the desired result. □

Lemma 4.2. *Suppose q^* and $q^*_{\mathcal{N}}$ are respectively the solutions of the continuous optimal control problem (2.8) and its discrete counterpart (2.14), $u(q^*)$ and $u_{\mathcal{M}}(q^*_{\mathcal{N}})$ are the state solutions of (2.4) and (2.13) associated to q^* and $q^*_{\mathcal{N}}$, $z(q^*)$ and $z_{\mathcal{M}}(q^*_{\mathcal{N}})$ are the associated solutions of (2.10) and (2.17), respectively. Then, it holds:*

(4.11) $$\|u_{\mathcal{M}}(q^*_{\mathcal{N}}) - u(q^*_{\mathcal{N}})\|_1 \lesssim \eta, \quad \|z_{\mathcal{M}}(q^*_{\mathcal{N}}) - z(q^*_{\mathcal{N}})\|_1 \lesssim \xi + \eta,$$

where ξ, η are defined by (4.1)-(4.3).

Proof. Let $e_z = z_{\mathcal{M}}(q_{\mathcal{N}}^*) - z(q_{\mathcal{N}}^*)$, by (2.17), (4.8) and the coercivity of the bilinear form a defined in (2.5), we have for all $\phi_{\mathcal{M}} \in V_{\mathcal{M}}$

$$\begin{aligned} &\|z_{\mathcal{M}}(q_{\mathcal{N}}^*) - z(q_{\mathcal{N}}^*)\|_1^2 \\ \lesssim & a(e_z, e_z) = a(e_z - \phi_{\mathcal{M}}, e_z) + a(\phi_{\mathcal{M}}, e_z) \\ = & a(e_z - \phi_{\mathcal{M}}, e_z) + (u_{\mathcal{M}}(q_{\mathcal{N}}^*) - u(q_{\mathcal{N}}^*), \phi_{\mathcal{M}}). \end{aligned} \tag{4.12}$$

Assume that $\phi_{\mathcal{M}}$ is chosen to satisfy $\phi_{\mathcal{M}}(x_k) = e_z(x_k)$ for $k = 1, \cdots, K-1$. Counter-integrating by parts in each element and using (4.8) we have

$$\begin{aligned} & a(e_z - \phi_{\mathcal{M}}, e_z) + (u_{\mathcal{M}}(q_{\mathcal{N}}^*) - u(q_{\mathcal{N}}^*), \phi_{\mathcal{M}}) \\ = & \sum_{k=1}^K \int_{\Lambda_k} \left(-(z_{\mathcal{M}}^{(k)}(q_{\mathcal{N}}^*))'' - u_{\mathcal{M}}^{(k)}(q_{\mathcal{N}}^*) + \bar{u}^{(k)}\right)\left(e_z^{(k)} - \phi_{\mathcal{M}}^{(k)}\right) \mathrm{d}x \\ & + (u_{\mathcal{M}}(q_{\mathcal{N}}^*) - u(q_{\mathcal{N}}^*), e_z) \\ = & \sum_{k=1}^K \int_{\Lambda_k} \left(-(z_{\mathcal{M}}^{(k)}(q_{\mathcal{N}}^*))'' - u_{\mathcal{M}}^{(k)}(q_{\mathcal{N}}^*) + \bar{u}_{M_k}^{(k)}\right) \sqrt{w_k} \frac{e_z^{(k)} - \phi_{\mathcal{M}}^{(k)}}{\sqrt{w_k}} \mathrm{d}x \\ & + \sum_{k=1}^K \int_{\Lambda_k} \left(\bar{u}^{(k)} - \bar{u}_{M_k}^{(k)}\right) \sqrt{w_k} \frac{e_z^{(k)} - \phi_{\mathcal{M}}^{(k)}}{\sqrt{w_k}} \mathrm{d}x + (u_{\mathcal{M}}(q_{\mathcal{N}}^*) - u(q_{\mathcal{N}}^*), e_z) \\ \lesssim & \sum_{k=1}^K \left(\left\| -(z_{\mathcal{M}}^{(k)}(q_{\mathcal{N}}^*))'' - u_{\mathcal{M}}^{(k)}(q_{\mathcal{N}}^*) + \bar{u}_{M_k}^{(k)} \right\|_{0,w_k} + \left\| \bar{u}^{(k)} - \bar{u}_{M_k}^{(k)} \right\|_{0,w_k} \right) \left\| \frac{e_z^{(k)} - \phi_{\mathcal{M}}^{(k)}}{\sqrt{w_k}} \right\|_{0,\Lambda_k} \\ & + \|u_{\mathcal{M}}(q_{\mathcal{N}}^*) - u(q_{\mathcal{N}}^*)\|_{0,\Lambda} \|e_z\|_{0,\Lambda}. \end{aligned} \tag{4.13}$$

Let us now define $\phi_{\mathcal{M}}^{(k)}$. Given a function $\hat{v} \in H^1(-1,1)$, define

$$\hat{v}_M(\hat{x}) = \hat{v}(-1) + \int_{-1}^{\hat{x}} (\Pi_{M-1}\hat{v}')(s)\mathrm{d}s,$$

where Π_{M-1} denotes the L^2- orthogonal projection upon $P_{M-1}(-1,1)$. It is easy to check that $\hat{v}_M(\pm 1) = \hat{v}(\pm 1)$. Moreover, it has been proved by Schwab [24] that

$$\int_{-1}^{1} \frac{(\hat{v} - \hat{v}_M)^2(\hat{x})}{1 - \hat{x}^2} \mathrm{d}\hat{x} \le \frac{1}{M(M+1)} \int_{-1}^{\hat{x}} (\hat{v}')^2(\hat{x})\mathrm{d}\hat{x}. \tag{4.14}$$

Set $\hat{v}(\hat{x}) = e_z^{(k)}(F_k(\hat{x}))$, where F_k is the affine mapping (2.12), and define

$$\phi_{\mathcal{M}}^{(k)} = \hat{v}_{M_k}(F_k^{-1}(x)).$$

Then, the previous inequality (4.14) yields

$$\left\| \frac{e_z^{(k)} - \phi_{\mathcal{M}}^{(k)}}{\sqrt{w_k}} \right\|_{0,\Lambda_k} \le \frac{1}{\sqrt{M_k(M_k+1)}} \left\| e_z^{(k)} \right\|_{1,\Lambda_k}. \tag{4.15}$$

Thus, plugging (4.15) into (4.13), and by the Cauchy-Schwarz inequality we get

$$\begin{aligned} &a(e_z-\phi_{\mathcal{M}},e_z)+(u_{\mathcal{M}}(q_{\mathcal{N}}^*)-u(q_{\mathcal{N}}^*),\phi_{\mathcal{M}}) \\ &\lesssim \left(\sum_{k=1}^{K}\xi_k^2\right)^{\frac{1}{2}}\|e_z\|_1+\left(\sum_{k=1}^{K}\frac{1}{M_k(M_k+1)}\left\|\bar{u}^{(k)}-\bar{u}_{M_k}^{(k)}\right\|_{0,w_k}^2\right)^{\frac{1}{2}}\|e_z\|_1 \\ &\quad+\|u_{\mathcal{M}}(q_{\mathcal{N}}^*)-u(q_{\mathcal{N}}^*)\|_0\|e_z\|_1\,, \end{aligned} \tag{4.16}$$

where ξ_k is defined in (4.5). Thus, (4.16) together with (4.12) leads to

$$\|z_{\mathcal{M}}(q_{\mathcal{N}}^*)-z(q_{\mathcal{N}}^*)\|_1\lesssim\xi+\|u_{\mathcal{M}}(q_{\mathcal{N}}^*)-u(q_{\mathcal{N}}^*)\|_0\,. \tag{4.17}$$

Similarly, let $e_u=u_{\mathcal{M}}(q_{\mathcal{N}}^*)-u(q_{\mathcal{N}}^*)$, and $\psi_{\mathcal{M}}\in V_{\mathcal{M}}$ be defined similar to $\phi_{\mathcal{M}}$ except that $\psi_{\mathcal{M}}$ is chosen to satisfy $\psi_{\mathcal{M}}(x_k)=e_u(x_k)$ for $k=1,\cdots,K-1$. Then it follows from (2.4) and (2.13) that $a(e_u,\psi_{\mathcal{M}})=0$, and thus

$$\begin{aligned} &\|u_{\mathcal{M}}(q_{\mathcal{N}}^*)-u(q_{\mathcal{N}}^*)\|_1^2 \\ &\lesssim a(e_u,e_u)=a(e_u,e_u-\psi_{\mathcal{M}}) \\ &=\sum_{k=1}^{K}\int_{\Lambda_k}(u_{\mathcal{M}}^{(k)}(q_{\mathcal{N}}^*))'(e_u^{(k)}-\psi_{\mathcal{M}}^{(k)})'\mathrm{d}x-(f+q_{\mathcal{N}}^*,e_u-\psi_{\mathcal{M}}) \\ &=\sum_{k=1}^{K}\int_{\Lambda_k}\left(-(u_{\mathcal{M}}^{(k)}(q_{\mathcal{N}}^*))''-q_{\mathcal{N}}^{*(k)}-f_{M_k}^{(k)}\right)\sqrt{w_k}\frac{e_u^{(k)}-\psi_{\mathcal{M}}^{(k)}}{\sqrt{w_k}}\mathrm{d}x \\ &\quad+\sum_{k=1}^{K}\int_{\Lambda_k}\left(f_{M_k}^{(k)}-f^{(k)}\right)\sqrt{w_k}\frac{e_u^{(k)}-\psi_{\mathcal{M}}^{(k)}}{\sqrt{w_k}}\mathrm{d}x \\ &\lesssim\left(\sum_{k=1}^{K}\eta_k^2\right)^{\frac{1}{2}}\|e_u\|_1+\left(\sum_{k=1}^{K}\frac{1}{M_k(M_k+1)}\left\|f^{(k)}-f_{M_k}^{(k)}\right\|_{0,w_k}^2\right)^{\frac{1}{2}}\|e_u\|_1\,, \end{aligned} \tag{4.18}$$

where η_k is defined by (4.6). This leads to

$$\|u_{\mathcal{M}}(q_{\mathcal{N}}^*)-u(q_{\mathcal{N}}^*)\|_1\lesssim\eta. \tag{4.19}$$

Consequently, (4.17) and (4.19) imply that

$$\|z_{\mathcal{M}}(q_{\mathcal{N}}^*)-z(q_{\mathcal{N}}^*)\|_1\lesssim\xi+\eta. \tag{4.20}$$

□

Theorem 4.3. *Suppose q^* and $q_{\mathcal{N}}^*$ are respectively the solutions of the continuous optimal control problem (2.8) and its discrete counterpart (2.14), $u(q^*)$ and $u_{\mathcal{M}}(q_{\mathcal{N}}^*)$ are the state solutions of (2.4) and (2.13) associated to q^* and $q_{\mathcal{N}}^*$, $z(q^*)$ and $z_{\mathcal{M}}(q_{\mathcal{N}}^*)$ are the associated solutions of (2.10) and (2.17), respectively. Then, the following estimate holds:*

$$\|q^*-q_{\mathcal{N}}^*\|_0+\|u(q^*)-u_{\mathcal{M}}(q_{\mathcal{N}}^*)\|_1+\|z(q^*)-z_{\mathcal{M}}(q_{\mathcal{N}}^*)\|_1\lesssim\zeta+\xi+\eta, \tag{4.21}$$

where ζ, ξ, η are defined by (4.1)-(4.3).

Proof. In virtue of Lemmas 4.1 and 4.2, we get

$$\|q^* - q^*_{\mathcal{N}}\|_0 \lesssim \xi + \eta + \zeta. \tag{4.22}$$

Furthermore, using the triangle inequalities

$$\begin{aligned} \|z_{\mathcal{M}}(q^*_{\mathcal{N}}) - z(q^*)\|_1 &\leq \|z_{\mathcal{M}}(q^*_{\mathcal{N}}) - z(q^*_{\mathcal{N}})\|_1 + \|z(q^*_{\mathcal{N}}) - z(q^*)\|_1 \,, \\ \|u_{\mathcal{M}}(q^*_{\mathcal{N}}) - u(q^*)\|_1 &\leq \|u_{\mathcal{M}}(q^*_{\mathcal{N}}) - u(q^*_{\mathcal{N}})\|_1 + \|u(q^*_{\mathcal{N}}) - u(q^*)\|_1 \,, \end{aligned}$$

and the following obvious estimates

$$\|u(q^*_{\mathcal{N}}) - u(q^*)\|_1 \lesssim \|q^*_{\mathcal{N}} - q^*\|_0 \,,$$

$$\|z(q^*_{\mathcal{N}}) - z(q^*)\|_1 \lesssim \|u(q^*_{\mathcal{N}}) - u(q^*)\|_0 \lesssim \|q^*_{\mathcal{N}} - q^*\|_0 \,,$$

we obtain

$$\|u_{\mathcal{M}}(q^*_{\mathcal{N}}) - u(q^*)\|_1 + \|z_{\mathcal{M}}(q^*_{\mathcal{N}}) - z(q^*)\|_1 \lesssim \xi + \eta + \zeta. \tag{4.23}$$

Thus (4.21) follows from (4.22) and (4.23). □

4.2. **A posteriori lower error estimates.** We now turn our attention towards lower a posteriori bounds, i.e., the efficiency of the error estimators provided in Theorem 4.3. The proof of the main result in this subsection will be accomplished with a series of lemmas for the estimating ζ, ξ and η, which we present below.

Lemma 4.4. *Let q^* be the solution of the continuous optimal control problem (2.8), $z(q^*)$ be the corresponding adjoint state. Let $q^*_{\mathcal{N}}$ be the solution of the discrete optimal control problem (2.14) with the corresponding discrete adjoint state $z_{\mathcal{M}}(q^*_{\mathcal{N}})$. Then, the following estimate holds:*

$$\zeta^2 \lesssim \|q^* - q^*_{\mathcal{N}}\|_0^2 + \|z(q^*) - z_{\mathcal{M}}(q^*_{\mathcal{N}})\|_0^2 \,,$$

where ζ is defined by (4.1).

Proof. It is clear that

$$
\begin{aligned}
&\sum_{k=1}^{K}\zeta_k^2=\sum_{k=1}^{K}\int_{\Lambda_k}\left(z_{\mathcal{M}}(q_{\mathcal{N}}^*)-\Pi_{N_k}z_{\mathcal{M}}(q_{\mathcal{N}}^*)\right)^2\mathrm{d}x\\
=\;&\sum_{k=1}^{K}\int_{\Lambda_k}\left(z_{\mathcal{M}}(q_{\mathcal{N}}^*)-\Pi_{N_k}z_{\mathcal{M}}(q_{\mathcal{N}}^*)\right)\\
&\left(z_{\mathcal{M}}(q_{\mathcal{N}}^*)-z(q^*)+z(q^*)-\Pi_{N_k}z(q^*)+\Pi_{N_k}z(q^*)-\Pi_{N_k}z_{\mathcal{M}}(q_{\mathcal{N}}^*)\right)\mathrm{d}x\\
\le\;&\frac{1}{2}\sum_{k=1}^{K}\int_{\Lambda_k}\left(z_{\mathcal{M}}(q_{\mathcal{N}}^*)-\Pi_{N_k}z_{\mathcal{M}}(q_{\mathcal{N}}^*)\right)^2\mathrm{d}x\\
&+C\left\{\sum_{k=1}^{K}\|z_{\mathcal{M}}(q_{\mathcal{N}}^*)-z(q^*)\|_0^2+\sum_{k=1}^{K}\|\Pi_{N_k}\left(z(q^*)-z_{\mathcal{M}}(q_{\mathcal{N}}^*)\right)\|_0^2\right\}\\
&+\sum_{k=1}^{K}\int_{\Lambda_k}\left(z_{\mathcal{M}}(q_{\mathcal{N}}^*)-\Pi_{N_k}z_{\mathcal{M}}(q_{\mathcal{N}}^*)\right)\left(z(q^*)-\Pi_{N_k}z(q^*)\right)\mathrm{d}x,
\end{aligned}
$$

such that

$$
\begin{aligned}
\sum_{k=1}^{K}\zeta_k^2\le&C\Big\{\|z_{\mathcal{M}}(q_{\mathcal{N}}^*)-z(q^*)\|_0^2\\
&+\sum_{k=1}^{K}\int_{\Lambda_k}\left(z_{\mathcal{M}}(q_{\mathcal{N}}^*)-\Pi_{N_k}z_{\mathcal{M}}(q_{\mathcal{N}}^*)\right)\left(z(q^*)-\Pi_{N_k}z(q^*)\right)\mathrm{d}x\Big\}.
\end{aligned}\tag{4.24}
$$

By Lemma 3.2, we know $\lambda q^*+z(q^*)=\mathit{const}$ such that

$$\Pi_{N_k}(\lambda q^*+z(q^*))=\lambda q^*+z(q^*).$$

Hence,

$$
\begin{aligned}
&\sum_{k=1}^{K}\int_{\Lambda_k}\left(z_{\mathcal{M}}(q_{\mathcal{N}}^*)-\Pi_{N_k}z_{\mathcal{M}}(q_{\mathcal{N}}^*)\right)\left(z(q^*)-\Pi_{N_k}z(q^*)\right)\mathrm{d}x\\
=&\sum_{k=1}^{K}\int_{\Lambda_k}\left(z_{\mathcal{M}}(q_{\mathcal{N}}^*)-\Pi_{N_k}z_{\mathcal{M}}(q_{\mathcal{N}}^*)\right)\\
&\left(z(q^*)+\lambda q^*-\Pi_{N_k}(z(q^*)+\lambda q^*)-\lambda q^*+\Pi_{N_k}(\lambda q^*)\right)\mathrm{d}x\\
=&\sum_{k=1}^{K}\int_{\Lambda_k}\left(z_{\mathcal{M}}(q_{\mathcal{N}}^*)-\Pi_{N_k}z_{\mathcal{M}}(q_{\mathcal{N}}^*)\right)\left(\Pi_{N_k}(\lambda q^*)-\lambda q^*\right)\mathrm{d}x\\
=&\sum_{k=1}^{K}\int_{\Lambda_k}\lambda\left(z_{\mathcal{M}}(q_{\mathcal{N}}^*)-\Pi_{N_k}z_{\mathcal{M}}(q_{\mathcal{N}}^*)\right)\left(\Pi_{N_k}(q^*-q_{\mathcal{N}}^*)-(q^*-q_{\mathcal{N}}^*)\right)\mathrm{d}x\\
\le&\frac{\lambda}{2}\sum_{k=1}^{K}\int_{\Lambda_k}\left(z_{\mathcal{M}}(q_{\mathcal{N}}^*)-\Pi_{N_k}z_{\mathcal{M}}(q_{\mathcal{N}}^*)\right)^2\mathrm{d}x+C\left\|q^*-q_{\mathcal{N}}^*\right\|_0^2.
\end{aligned}\tag{4.25}
$$

Therefore, by combining (4.24) and (4.25) we obtain

$$\sum_{k=1}^{K} \zeta_k^2 \lesssim \|q^* - q_{\mathcal{N}}^*\|_0^2 + \|z(q^*) - z_{\mathcal{M}}(q_{\mathcal{N}}^*)\|_0^2 \, .$$

The proof of Lemma 4.4 is completed. □

Lemma 4.5. *Let q^* be the solution of the continuous optimal control problem (2.8), $u(q^*)$ and $z(q^*)$ be the corresponding state and adjoint state, respectively. Let, moreover, $q_{\mathcal{N}}^*$ be the solution of the discrete optimal control problem (2.14) with the corresponding discrete state $u_{\mathcal{M}}(q_{\mathcal{N}}^*)$ and adjoint state $z_{\mathcal{M}}(q_{\mathcal{N}}^*)$. Then we have*

(4.26)

$$\xi^2 \lesssim \|z(q^*) - z_{\mathcal{M}}(q_{\mathcal{N}}^*)\|_1^2 + \|u(q^*) - u_{\mathcal{M}}(q_{\mathcal{N}}^*)\|_1^2 + \sum_{k=1}^{K} \frac{4}{M_k(M_k+1)} \left\| \bar{u} - \bar{u}_{M_k}^{(k)} \right\|_{0,w_k}^2 ,$$

where ξ is defined by (4.2).

Proof. Let $r_k = \left((z_{\mathcal{M}}^{(k)}(q_{\mathcal{N}}^*))'' + u_{\mathcal{M}}^{(k)}(q_{\mathcal{N}}^*) - \bar{u}_{M_k}^{(k)} \right) w_k$, $a_{\Lambda_k}(\cdot,\cdot)$ be the restriction of $a(\cdot,\cdot)$ upon Λ_k, then we have

$$\begin{aligned}
(4.27) \qquad & \left\| r_k \frac{1}{\sqrt{w_k}} \right\|_{0,\Lambda_k}^2 = \int_{\Lambda_k} r_k^2 \frac{1}{w_k} \mathrm{d}x \\
&= \int_{\Lambda_k} \left((z_{\mathcal{M}}(q_{\mathcal{N}}^*))'' + u_{\mathcal{M}}(q_{\mathcal{N}}^*) - \bar{u}_{M_k} \right) r_k \mathrm{d}x \\
&= -\int_{\Lambda_k} (z_{\mathcal{M}}(q_{\mathcal{N}}^*))' r_k' \mathrm{d}x + \int_{\Lambda_k} (u(q^*) - \bar{u}) r_k \mathrm{d}x + \int_{\Lambda_k} (\bar{u} - \bar{u}_{M_k}) r_k \mathrm{d}x \\
&\quad + \int_{\Lambda_k} (u_{\mathcal{M}}(q_{\mathcal{N}}^*) - u(q^*)) r_k \mathrm{d}x \\
&= a_{\Lambda_k}(r_k, z(q^*) - z_{\mathcal{M}}(q_{\mathcal{N}}^*)) + \int_{\Lambda_k} (\bar{u} - \bar{u}_{M_k}) w_k \frac{r_k}{w_k} \mathrm{d}x \\
&\quad + \int_{\Lambda_k} (u_{\mathcal{M}}(q_{\mathcal{N}}^*) - u(q^*)) r_k \mathrm{d}x \\
&\lesssim \|z(q^*) - z_{\mathcal{M}}(q_{\mathcal{N}}^*)\|_{1,\Lambda_k} \|r_k\|_{1,\Lambda_k} + \|u(q^*) - u_{\mathcal{M}}(q_{\mathcal{N}}^*)\|_{1,\Lambda_k} \|r_k\|_{1,\Lambda_k} \\
&\quad + \|\bar{u} - \bar{u}_{M_k}\|_{0,w_k} \left\| r_k \frac{1}{\sqrt{w_k}} \right\|_{0,\Lambda_k} .
\end{aligned}$$

Now we compute $\|r_k\|_1$. Denote $s_k = (z_{\mathcal{M}}^{(k)}(q_{\mathcal{N}}^*))'' + u_{\mathcal{M}}^{(k)}(q_{\mathcal{N}}^*) - \bar{u}_{M_k}^{(k)}$. We observe that

$$
\begin{aligned}
\|r_k\|_{1,\Lambda_k}^2 &= \|r_k\|_{0,\Lambda_k}^2 + \|r_k'\|_{1,\Lambda_k}^2 \\
&= \int_{\Lambda_k} s_k^2 w_k^2 \mathrm{d}x + \int_{\Lambda_k} (s_k' w_k + s_k w_k')^2 \mathrm{d}x \\
&\le \frac{h_k^2}{4} \int_{\Lambda_k} s_k^2 w_k \mathrm{d}x + 2\int_{\Lambda_k} (s_k')^2 w_k^2 \mathrm{d}x + 2h_k^2 \int_{\Lambda_k} s_k^2 \mathrm{d}x.
\end{aligned} \tag{4.28}
$$

We now invoke that the inverse inequalities (3.2) and (3.3) on the reference interval hold for all polynomials $r \in P_N$. Translating these inequalities on the element Λ_k via the affine mapping F_k defined by (2.12), we get

$$
\begin{aligned}
\int_{\Lambda_k} (r_k')^2 w_k^2 \mathrm{d}x &\le cM_k^2 \int_{\Lambda_k} r_k^2 w_k \mathrm{d}x, \\
\int_{\Lambda_k} r_k^2 \mathrm{d}x &\le CM_k^2 h_k^{-2} \int_{\Lambda_k} r_k^2 w_k \mathrm{d}x,
\end{aligned}
$$

whence,

$$
\begin{aligned}
\|r_k\|_{1,\Lambda_k}^2 \quad &\le \quad \frac{h_k^2}{4} \int_{\Lambda_k} s_k^2 w_k \mathrm{d}x + 2cM_k^2 \int_{\Lambda_k} s_k^2 w_k \mathrm{d}x + 2Ch_k^2 M_k^2 h_k^{-2} \int_{\Lambda_k} s_k^2 w_k \mathrm{d}x \\
&\lesssim \quad M_k^2 \int_{\Lambda_k} s_k^2 w_k \mathrm{d}x,
\end{aligned}
$$

which implies

$$
\|r_k\|_{1,\Lambda_k} \lesssim M_k \|s_k \sqrt{w_k}\|_{0,\Lambda_k} = M_k \left\| r_k \frac{1}{\sqrt{w_k}} \right\|_{0,\Lambda_k}. \tag{4.29}
$$

Now combining (4.27) and (4.29) we obtain

$$
\begin{aligned}
&\left\| r_k \frac{1}{\sqrt{w_k}} \right\|_{0,\Lambda_k} \\
\lesssim & M_k \|z(q^*) - z_{\mathcal{M}}(q_{\mathcal{N}}^*)\|_{1,\Lambda_k} + M_k \|u(q^*) - u_{\mathcal{M}}(q_{\mathcal{N}}^*)\|_{1,\Lambda_k} + \|\bar{u} - \bar{u}_{M_k}\|_{0,w_k}.
\end{aligned}
$$

As a result, we get

$$
\begin{aligned}
\xi_k^2 \quad &= \quad \frac{1}{M_k(M_k+1)} \left\| r_k \frac{1}{\sqrt{w_k}} \right\|_{0,\Lambda_k}^2 \\
&\lesssim \quad \|z(q^*) - z_{\mathcal{M}}(q_{\mathcal{N}}^*)\|_{1,\Lambda_k}^2 + \|u(q^*) - u_{\mathcal{M}}(q_{\mathcal{N}}^*)\|_{1,\Lambda_k}^2 \\
&\quad + \frac{3}{M_k(M_k+1)} \left\| \bar{u} - \bar{u}_{M_k}^{(k)} \right\|_{0,w_k}^2,
\end{aligned}
$$

which leads to (4.26). □

Similarly, we can have the following estimation for η.

Lemma 4.6. *Let q^* be the solution of the continuous optimal control problem (2.8), $u(q^*)$ and $z(q^*)$ be the corresponding state and adjoint state, respectively. Let, moreover, $q^*_{\mathcal{N}}$ be the solution of the discrete optimal control problem (2.14) with the corresponding discrete state $u_{\mathcal{M}}(q^*_{\mathcal{N}})$ and adjoint state $z_{\mathcal{M}}(q^*_{\mathcal{N}})$. Then it holds that*

$$\eta^2 \lesssim \|u(q^*) - u_{\mathcal{M}}(q^*_{\mathcal{N}})\|_1^2 + \|q^* - q^*_{\mathcal{N}}\|_0^2 + \sum_{k=1}^{K} \frac{4}{M_k(M_k+1)} \left\| f - f^{(k)}_{M_k} \right\|^2_{0,w_k},$$

where η is defined by (4.3).

Summing up, we can immediately arrive at the main result of this subsection by combining the foregoing results Lemmas 4.4-4.6 into the following theorem.

Theorem 4.7. *Let q^* be the solution of the continuous optimal control problem (2.8), $u(q^*)$ and $z(q^*)$ be the corresponding state and adjoint state respectively. Let, moreover, $q^*_{\mathcal{N}}$ be the solution of the discrete optimal control problem (2.14) with the corresponding discrete state $u_{\mathcal{M}}(q^*_{\mathcal{N}})$ and adjoint state $z_{\mathcal{M}}(q^*_{\mathcal{N}})$. Then we have*

$$\zeta^2 + \xi^2 + \eta^2 \lesssim \|q^* - q^*_{\mathcal{N}}\|_0^2 + \|u(q^*) - u_{\mathcal{M}}(q^*_{\mathcal{N}})\|_1^2 + \|z(q^*) - z_{\mathcal{M}}(q^*_{\mathcal{N}})\|_1^2 + \epsilon_1^2 + \epsilon_2^2,$$

where ζ, ξ, η are defined by (4.1)-(4.3), and

$$\epsilon_1^2 = \sum_{k=1}^{K} \frac{4}{M_k(M_k+1)} \left\| \bar{u}^{(k)} - \bar{u}^{(k)}_{M_k} \right\|^2_{0,w_k},$$

$$\epsilon_2^2 = \sum_{k=1}^{K} \frac{4}{M_k(M_k+1)} \left\| f^{(k)} - f^{(k)}_{M_k} \right\|^2_{0,w_k}.$$

Remark 4.8. It follows from Theorems 4.3 and 4.7 that the estimators defined in (4.1)-(4.3) are in fact equivalent in the sense that there are two constants $c, C > 0$ such that

$$\begin{aligned} & c(\zeta^2 + \xi^2 + \eta^2) - c(\epsilon_1^2 + \epsilon_2^2) \\ \le \; & \|q^* - q^*_{\mathcal{N}}\|_0^2 + \|u(q^*) - u_{\mathcal{M}}(q^*_{\mathcal{N}})\|_1^2 + \|z(q^*) - z_{\mathcal{M}}(q^*_{\mathcal{N}})\|_1^2 \\ \le \; & C(\zeta^2 + \xi^2 + \eta^2) \end{aligned}$$

where ϵ_1 and ϵ_2 are defined in Theorem 4.7, which are all higher order terms.

5. Numerical results

In this section, we carry out some numerical experiments to demonstrate the error estimators developed in Section 4. In all our calculations, the control, state and adjoint state are all approximated by the piecewise polynomials of degree N. Let $\lambda = 1$, we consider problem (2.7) on $\Lambda = (-1, 1)$ with the exact solutions:

$$u(q^*) = \pi^2 \sin \pi x, \;\; z(q^*) = \sin \pi x, \;\; q^* = -\sin \pi x.$$

The main purpose is to validate the a posteriori error estimators. This is done by comparing the error indicator

$$\mathcal{E} := \zeta + \xi + \eta$$

with ζ, ξ, η defined in (4.1)-(4.3), and the true error of the numerical solution measured by:

$$e := \|q^* - q^*_{\mathcal{N}}\|_0 + \|u(q^*) - u_{\mathcal{M}}(q^*_{\mathcal{N}})\|_1 + \|z(q^*) - z_{\mathcal{M}}(q^*_{\mathcal{N}})\|_1 .$$

On a fixed mesh with 4 elements, we study these two errors in Table 1. It is shown that the error of the spectral element method between the numerical and exact solutions has the same order of accuracy as the a posteriori error indicators, which coincide with the predicted theoretical results.

Table 1. The true error and the posteriori error estimators for varying N.

N	4	6	8	10	12
e	4.3091E-3	1.0661E-5	1.7327E-8	1.9537E-11	2.6331E-13
$\mathcal{E}$	4.5289E-3	1.1109E-5	1.7681E-8	1.9723E-11	2.6444E-13

6. Concluding remarks

In this paper, we discussed a posteriori error estimates of the spectral element method for a distributed convex optimal control problem governed by the two-point boundary value problem. It is shown that a posteriori error estimators derived in this paper provide both upper and lower bounds for the approximation errors, and such a posteriori error estimators is sharp.

There are many important issues that still need to be addressed. First, studies for more complicated control problems and constraint sets are needed. Secondly, a posteriori error analysis for high order methods, such as the spectral method and the spectral element method in two or higher dimensional optimal contro problem is needed. Thirdly, many computational issues have to be addressed. For example, adaptive strategy should be investigated for efficiently implementing adaptive spectral element method for optimal control problems.

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