ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 42 (2016), No. 1, pp. 19–25

Title:

Some connections between powers of conjugacy classes and degrees of irreducible characters in solvable groups

Author(s):

S. M. Robati

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 1, pp. 19–25
Online ISSN: 1735-8515

SOME CONNECTIONS BETWEEN POWERS OF CONJUGACY CLASSES AND DEGREES OF IRREDUCIBLE CHARACTERS IN SOLVABLE GROUPS

S. M. ROBATI

(Communicated by Ali Reza Ashrafi)

Abstract. Let G be a finite group. We say that the derived covering number of G is finite if and only if there exists a positive integer n such that $C^n = G'$ for all non-central conjugacy classes C of G. In this paper we characterize solvable groups G in which the derived covering number is finite.
Keywords: Conjugacy classes, irreducible characters, solvable groups.
MSC(2010): Primary: 20E45, 20C15.

1. Introduction

Let G be a finite group. In this paper $ccl(G)$ denotes the set of the conjugacy class sizes of G, $cd(G)$ denotes the set of degrees of the irreducible characters of G, $Irr(G)$ denotes the set of all the irreducible characters of G, and $Irr_1(G)$ denotes the set of all the non-linear irreducible characters of G.
In [2], the *derived covering number* of G, denoted by $dcn(G)$, is defined as the smallest positive integer n such that $C^n = G'$ for all non-central conjugacy classes C of G. If such integer exists, we say that $dcn(G)$ is finite, otherwise we set $dcn(G) = \infty$. The authors also proved the following result:

Theorem 1.1. If a finite non-abelian group G has a finite derived covering number then one of the following holds:

(1) G' is a minimal normal subgroup of G,
(2) G is a nilpotent group of class two in which $ccl(G) = \{1, |G'|\}$,
(3) G' is perfect and $\frac{G'}{Z(G')}$ is a minimal normal subgroup of $\frac{G}{Z(G')}$, with $Z(G') = G' \cap Z(G)$.

Article electronically published on February 22, 2016.
Received: 5 May 2014, Accepted: 9 October 2014.

In Section 2, we characterize finite nilpotent groups G which have a finite derived covering number and show some properties of such groups.

On the other hand, the following theorem is proved in [1], where characterized non-nilpotent groups G with $cd(G) = \{1, m\}$:

Theorem 1.2. Let G be a non-abelian solvable group with the center Z and suppose that G' is a minimal normal subgroup of G. Then G' is an elementary abelian p-group of order p^r for some $p \in \mathbb{P}$ and $r \in \mathbb{N}$, $cd(G) = \{1, f\}$ for some $f > 1$ and one of the following situations occurs:

(1) $G' \leq Z$, in which case $|G'| = p$, $f = p^u$ for some $u \in \mathbb{N}$, G is nilpotent with an abelian p-complement and G/Z is an elementary abelian p-group of order f^2.

(2) $G' \cap Z = 1$, in which case G/Z is a Frobenius group with the kernel $G' \times Z/Z$ of order p^r and a cyclic complement of order f.

In Section 3, we provide an example of a Frobenius group which satisfies the case (2) of Theorem 1.2 and Corollary 2 of [1], but where G' is not a minimal normal subgroup of G. Moreover, by Theorem 1.2 and Theorem 1.1, we characterize finite non-nilpotent solvable groups G whose derived covering number is finite.

2. Nilpotent groups

Theorem 2.1. Let G be a finite group. Then $ccl(G) = \{1, |G'|\}$ if and only if G is nilpotent with an abelian p-complement and $G/Z(G)$ is an elementary abelian p-group of order p^{2r} such that $cd(G) = \{1, p^r\}$.

Proof. Assume that $ccl(G) = \{1, |G'|\}$ and $a \in G - Z(G)$. Since $a^G = a[a, G]$ and $|a^G| = |[a, G]| = |G'|$ then $[a, G] = G'$ and $a^G = aG'$. Now since $|a^G| = |G/C_G(a)| = |G'|$ then $|G : G'| = |C_G(a)|$ and we can write

$$\begin{aligned} |C_G(a)| &= \sum_{\chi \in Irr(G)} |\chi(a)|^2 \\ &= \sum_{\chi \in Irr_1(G)} |\chi(a)|^2 + \sum_{\chi \in Lin(G)} |\chi(a)|^2 \\ &= \sum_{\chi \in Irr_1(G)} |\chi(a)|^2 + |G : G'| \end{aligned}$$

where $Lin(G)$ denotes the set of all linear irreducible characters of G. Therefore $\sum_{\chi \in Irr_1(G)} |\chi(a)|^2 = 0$ and so $\chi(a) = 0$ for all $\chi \in Irr_1(G)$. By Corollary 2.30 of [4], we have $\chi(1)^2 = |G : Z(\chi)|$ and since $Z(G) \leq Z(\chi)$ and $0 = \chi(b) \neq \chi(1)$ for all $b \in G - Z(G)$, therefore $Z(\chi) = Z(G)$ for all $\chi \in Irr_1(G)$ and $cd(G) =$

$\{1, |G : Z(G)|^{1/2}\}$.
On the other hand, observe that

$$0 = \sum_{\chi \in Irr(G)} \chi(1)\chi(a) = \sum_{\lambda \in Lin(G)} \lambda(1)\lambda(a)$$

and thus $a \in G - G'$. It follows that $G' \leq Z(G)$ and so G is a nilpotent group of class 2. Then $G = P_1 \times ... \times P_k$ in which $P_i \in Syl_{p_i}(G)$ and any irreducible character of G can be written as a product $\chi_1 ... \chi_k$ where $\chi_i \in Irr(P_i)$. Thus since $\chi_1(1)...\chi_k(1) \in cd(G)$ and $|cd(G)| = 2$, then G is the direct product of a p-group of class exactly 2 and an abelian p-complement. By Corollary 2.2 of [5], we also have $G/Z(G)$ is an elementary abelian p-group.

□

Proposition 2.2. Suppose that G satisfies the assumption of Theorem 2.1. Then:

(1) $Z(G) = Z(\chi)$ for all $\chi \in Irr_1(G)$.
(2) $\chi(a) = 0$ for all $\chi \in Irr_1(G)$ and $a \in G - Z(G)$.
(3) $a^G = aG'$ for all $a \in G - Z(G)$.
(4) G' is an elementary abelian p-group.
(5) $G' \leq \Phi(G) \leq Z(G)$.
(6) $ccl(G)$=$\{1, |G'|\}$, with corresponding frequencies $\{|Z(G)|, (|G| - |Z(G)|)/|G'|\}$.
(7) $cd(G) = \{1, |G : Z(G)|^{1/2}\}$, with corresponding frequencies $\{|G : G'|, |Z(G)|(|G'| - 1)/|G'|\}$.

Proof. (1), (2) and (3) follows from proof of Theorem 2.1.

(4) Assume that $x, y \in G$. Since $G' = \langle [x, y] | x, y \in G \rangle$ and $G/Z(G)$ is an elementary abelian p-group, then $[x, y]^p = [x^p, y] = [z, y] = 1$ for some $z \in Z(G)$. Therefore G' is an elementary abelian p-group.
(5) Since $G/Z(G)$ is an elementary abelian p-group and $\Phi(G) = G'G^p$, then $G^p \leq Z(G)$ and it follows that $G' \leq \Phi(G) \leq Z(G)$.
(6) Clearly, all non-central conjugacy classes are of size $|G'|$ and the number of non-central conjugacy classes is $(|G| - |Z(G)|)/|G'|$.
(7) Observe that

$$\begin{aligned} |G| &= \sum_{\chi \in Irr(G)} \chi(1)^2 \\ &= \sum_{\chi \in Irr_1(G)} \chi(1)^2 + \sum_{\chi \in Lin(G)} \chi(1)^2 \\ &= t\frac{|G|}{|Z(G)|} + \frac{|G|}{|G'|} \\ &= |G|\frac{t|G'| + |Z(G)|}{|Z(G)||G'|}, \end{aligned}$$

therefore G has $t = |Z(G)|(|G'|-1)/|G'|$ non-linear irreducible characters.

□

Corollary 2.3. Let G be finite group such that $ccl(G) = \{1, |G'|\}$. Then G is an extra-special group if and only if $|Z(G)| = p$.

Proof. By Proposition 2.2, we have $G' \leq \Phi(G) \leq Z(G)$. Since $G' \neq \{1\}$, if $|Z(G)| = p$ then $G' = \Phi(G) = Z(G)$. □

Example 2.1. Let Q_8 be the quaternion group. We can check that $a^{Q_8} = aZ(Q_8)$, $a^{Q_8}a^{Q_8} = Z(Q_8) = Q_8'$, and so $dcn(Q_8) = 2$.

Theorem 2.4. A finite nilpotent group G has a finite derived covering number if and only if G is nilpotent with an abelian p-complement and $G/Z(G)$ is an elementary abelian p-group of order p^{2r} such that $cd(G) = \{1, p^r\}$.

Proof. Suppose that a finite nilpotent group G has a finite derived covering number. Then the claim follows from Theorem 2.1 and Proposition 2.4 of [2]. Conversely, suppose that $\chi(1)^2 = |G : Z(G)|$ for all $\chi \in Irr_1(G)$, by Theorem 2.1, we can write $a^G = aG'$ for all $a \in G - Z(G)$. Hence $(a^G)^m = (aG')^m = a^mG' = G'$ in which $m = |G : G'|$ and therefore the derived covering number is finite. □

In the following Example, we provide a finite group which satisfies Theorem 2.4 such that $|G'| \geq p$

Example 2.2. The finite Symplectic group $Sp_4(q)$ of dimension 4 over the field F with $q = p^m$ (where p is an odd prime number) elements is the set of all non-singular 4×4 matrices A satisfying $AJA^t = J$, where $J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$. By [6], a Sylow p-subgroup P of $Sp_4(q)$ consists of matrices of the form $\begin{bmatrix} 1 & \lambda & 0 & \lambda\alpha+\beta \\ 0 & 1 & 0 & \alpha \\ -\alpha & \beta & 1 & \mu \\ 0 & 0 & 0 & 1 \end{bmatrix}$ which is denoted by $A(\lambda, \alpha, \mu, \beta)$.

We now define

$$Q = \{A(\alpha, \alpha, \mu, \beta) = \begin{bmatrix} 1 & \alpha & 0 & \alpha^2+\beta \\ 0 & 1 & 0 & \alpha \\ -\alpha & \beta & 1 & \mu \\ 0 & 0 & 0 & 1 \end{bmatrix} |\alpha, \beta, \mu \in F\}$$

We can easily check that Q is a p-group and

$A^{-1}(a, a, c, b)A(\alpha, \alpha, \mu, \beta)A(a, a, c, b) = A(\alpha, \alpha, 2(\beta a - \alpha b) + \alpha a(\alpha - a) + \mu, \beta)$.

Therefore, the element $A(\alpha,\alpha,\mu,\beta)$ is conjugate to $A(\alpha,\alpha,x,\beta)$ for any $x\in F$ if $(\alpha,\beta)\neq(0,0)$, hence Q has q^2-1 non-central conjugacy classes of order q and we can show that

$$Q'=\bigcup_{\alpha,\beta,\mu\in F}A^{-1}(\alpha,\alpha,\mu,\beta)A^{Q}(\alpha,\alpha,\mu,\beta))=\{A(0,0,\mu,0)|\mu\in F\}=Z(Q)$$

and $A^{Q}(\alpha,\alpha,\mu,\beta)=A(\alpha,\alpha,\mu,\beta)Q'$. Furthermore, we can check by induction on k that

$$A^{2k}(\alpha,\alpha,\mu,\beta)=A(2k\alpha,2k\alpha,\delta,2k\beta-k(2k-1)\alpha^2)$$

for any integer $k\geq 1$ and some $\delta\in F$. Therefore

$$\begin{aligned}A^{2k+1}(\alpha,\alpha,\mu,\beta)&=A^{2k}(\alpha,\alpha,\mu,\beta)A(\alpha,\alpha,\mu,\beta)\\&=A((2k+1)\alpha,(2k+1)\alpha,\delta,(2k+1)\beta-k(2k+1)\alpha^2).\end{aligned}$$

Thus for $p\neq 2$, we can write

$$A^{p}(\alpha,\alpha,\mu,\beta)=A(p\alpha,p\alpha,\delta,p\beta-np\alpha^2)=A(0,0,\delta,0)\in Z(Q)$$

for some $n\in\mathbb{N}$ and some $\delta\in F$. Then $Q/Z(Q)$ is an elementary abelian p-group and $Q'=\Phi(Q)=Z(Q)$ because $\Phi(Q)=Q'Q^p$. Now assume that $G=Q\times A$ in which A is an abelian group. Then G satisfies Theorem 2.1 such that $G'\subsetneq Z(G)$.

3. **Solvable groups**

In the next example, we show the existence of a Frobenius group that satisfies case (2) of Theorem 1.2, but whose derived subgroup is not a minimal subgroup.

Example 3.1. Let p be a prime number and $q=p^m$. Let F be the field with q elements. We define

$$E=\{A(a,b,c)=\begin{bmatrix}a&b&c\\0&1&0\\0&0&1\end{bmatrix}|a\neq 0\ \ and\ \ a,b,c\in F\}.$$

We can check that E is a group of order $(q-1)q^2$ and

$$A^{-1}(\alpha,\beta,\gamma)A(a,b,c)A(\alpha,\beta,\gamma)=A(a,((a\beta-\beta)+b)/\alpha,((a\gamma-\gamma)+c)/\alpha).$$

Hence $A(a,b,c)$ belongs to $Z(E)$ if and only if $a=1$ and $b,c=0$ and so $Z(E)=I_{3\times 3}$. Moreover, we obtain the conjugacy classes of E and tabulate them. There are $2q$ conjugacy classes of E.

Table I : Conjugacy classes of E

Type	Class representative	Number	Size	Elements
1	$A(1,0,0)$	1	1	$A(1,0,0)$
2	$A(a,b,c)$, $a\neq 0,1$	$q-2$	q^2	$\{A(a,\beta,\gamma)\|\beta,\gamma\in F\}$
3	$A(1,b,c)$	$q+1$	$q-1$	$\{A(1,b/\alpha,c/\alpha)\|\alpha\in F\}$

We also have

$$E' = \bigcup_{a,b,c\in F} A^{-1}(a,b,c)A^{E}(a,b,c) = \{A(1,\sigma,\zeta)|\sigma,\zeta \in F\}$$

thus $|E'| = q^2$ and E' is an elementary abelian p-group because

$$A^p(1,b,c) = A(1,pb,pc) = A(1,0,0) = I_{3\times 3}.$$

Furthermore, we define

$$T = \{A(a,0,0) = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} |a \neq 0 \ \ and \ \ a \in F\}.$$

and we can easily check that $E' \cap T = I_{3\times 3}$, $E = E'T$, and $T^g \cap T = I$ for all $g \in E - T$. It follows that E is a Frobenius group with kernel E' and the cyclic complement T of order $q-1$. By Theorem 13.8 of [3] E has exactly $\frac{|E'|-1}{|T|} = \frac{q^2-1}{q-1} = q+1$ non-linear irreducible character of degree $[E : E'] = q-1$ and hence $cd(E) = \{1, q-1\}$. Therefore, this example satisfies Theorem 2.1 and Corollary 2 of [1]. On the other hand, we have that

$$(A^G(1,b,0))^n = (\{A(1,b/\alpha,0)|\alpha \in F\})^n = M$$

in which

$$M = \{A(1,b,0) = \begin{bmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} |b \in F\}$$

is a subgroup of E and $M \subsetneq E'$. Thus $dcn(E)$ is not finite and it follows that G is not a minimal normal subgroup of G.

Theorem 3.1. A finite non-nilpotent solvable group G has a finite derived covering number if and only if G' is minimal normal subgroup and $G/Z(G)$ is a Frobenius group with the kernel $(G' \times Z(G))/Z(G)$ of order p^r and a cyclic complement of order f where $cd(G) = \{1, f\}$.

Proof. It follows from the case (1) of Theorem 1.1 and the case (2) of Theorem 1.2. Conversely, by Theorem 2.2 of [2], since G' is a minimal normal subgroup then the derived covering number is finite. □

Example 3.2. Let S_3 be the symmetric group of degree 3. We can check $((123)^{S_3})^2 = ((12)^{S_3})^2 = A_3$ and $S_3 = A_3H$ where $H = \langle(12)\rangle$, $H \cap A_3 = 1$, and $H \cap H^g = \{1\}$ for all $g \in S_3 - H$ because $(12)^{(123)} = (23)$ and $(12)^{(132)} = (13)$. Therefore, S_3 satisfies Theorem 3.1 such that $cd(G) = \{1, |H|\} = \{1, 2\}$.

Proposition 3.2. Let G be a finite group that admits a unique conjugacy class C of size k. Then $Z(G) = \{1\}$.

Proof. Assume that $C = a^G$ is the unique conjugacy class of size k of G and $z \in Z(G)$. Since, $(az)^G = a^G z$ and $|a^G| = |(az)^G|$, then $az \in a^G$. By Problem 3.12 of [4], we can write

$$\begin{aligned}\chi(a)\chi(z) &= \frac{\chi(1)}{|G|}\sum_{h\in G}\chi(az^h)\\ &= \frac{\chi(1)}{|G|}\sum_{h\in G}\chi(a)\\ &= \chi(1)\chi(a).\end{aligned}$$

Therefore, $\chi(1) = \chi(z)$ for all $\chi \in Irr(G)$. Thus $z \in \bigcap_{\chi\in Irr(G)} ker\chi = \{1\}$ and so $Z(G) = \{1\}$. □

Corollary 3.3. A finite non-nilpotent solvable group G has a finite derived covering number and non-central conjugacy classes of distinct sizes if and only if $G \simeq S_3$

Proof. By Corollary 2 of [5], observe that $ccl(G) = \{1, \frac{p^r-1}{s}, p^r\}$, with corresponding frequencies $\{|Z(G)|, s|Z(G)|, (\frac{p^r-1}{s} - 1)|Z(G)|\}$ where s is the number of conjugacy classes of G contained in $G' - 1$. Thus by Proposition 3, we have $|Z(G)| = 1$, $s = 1$, and $\frac{p^r-1}{s} - 1 = 1$. Therefore $r = 1$, $p = 3$, and $ccl(G) = \{1, 2, 3\}$ and it follows that $G \simeq S_3$ □

References

[1] M. Bianchi, A. G. B. Mauri, M. Herzog, G. Qian and W. Shi, Characterization of non-nilpotent groups with two irreducible character degrees, *J. Algebra* **284** (2005), no. 1, 326–332.

[2] M. R. Darafsheh and S. M. Robati, Powers of irreducible characters and conjugacy classes in finite groups, *J. Algebra Appl.* **13** (2014), no. 8, 9 pages.

[3] L. Dornhoff, Group representation theory, Part A: Ordinary representation theory, Marcel Dekker, Inc., New York, 1971.

[4] I. M. Isaacs, Character theory of finite groups, Academic Press, New York-San Francisco-London, 1976.

[5] K. Ishikawa, On finite p-groups which have only two conjugacy lengths, *Israel J. Math.* **129** (2002) 119–123.

[6] B. Srinivasan, The characters of the finite symplectic group $Sp(4, q)$, *Trans. Amer. Math. Soc.* **131** (1968) 488–525.

(S. M. Robati) Department of Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran.

E-mail address: sajjad.robati@gmail.com; mahmoodrobati@sci.ikiu.ac.ir