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CHARACTERIZATION OF SOME PROJECTIVE SPECIAL LINEAR GROUPS IN DIMENSION FOUR BY THEIR ORDERS AND DEGREE PATTERNS

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Abstract. Let G be a finite group. The degree pattern of G denoted by $D(G)$ is defined as follows: If $\pi(G) = \{p_1, p_2, ..., p_k\}$ such that $p_1 < p_2 < ... < p_k$, then $D(G) := (deg(p_1), deg(p_2), ..., deg(p_k))$, where $deg(p_i)$ for $1 \leq i \leq k$, are the degree of vertices p_i in the prime graph of G. In this article, we consider a finite group G under assumptions $|G| = |L_4(2^n)|$ and $D(G) = D(L_4(2^n))$, where $n \in \{5, 6, 7\}$ and we prove that $G \cong L_4(2^n)$.
Keywords: Degree pattern, prime graph, projective special linear group.
MSC(2010): Primary: 20D05; Secondary: 20D06.

1. Introduction

Let G be a finite group and $\omega(G)$ be the set of element orders for G. The set $\omega(G)$ is partially ordered under divisibility and is uniquely determined by a subset $\mu(G)$ of its maximal elements. We put all prime divisors of $|G|$ in $\pi(G)$, and we associate to $\pi(G)$ a simple graph $\Gamma(G)$, called prime graph or *Grunberg-Kegel* graph, whose vertex set is $\pi(G)$ and every two primes p and q are adjacent iff $pq \in \omega(G)$, in this case we write $p \sim q$, and by $p \nsim q$ we mean that any element of order pq does not exist in G.

Definition 1.1. *Let G be a group with $\pi(G) = \{p_1, p_2, ..., p_k\}$. We define degree of p as follows for $p \in \pi(G)$:*

$$deg(p) := |\{q \in \pi(G) | p \sim q\}|.$$

Also $D(G) := (deg(p_1), deg(p_2), ..., deg(p_k))$, where $p_1 < p_2 < ... < p_k$, which is called degree pattern of G.

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A group G is called k-fold OD-characterizable if there exist just k non-isomorphic finite groups H with $|H| = |G|$ and $D(H) = D(G)$. If $k = 1$, a 1-fold OD-characterizable group is simply called OD-characterizable.

Many articles are devoted to characterize some finite simple groups by their orders and degree patterns. As in the present paper we investigate OD-characterizability of some projective special linear groups, we review only some OD-characterizable groups of this kind that have been obtained up to now.

(1) $L_2(2) \cong \mathbb{S}_3$ and $L_2(3) \cong A_4$ [7].
(2) $L_2(2^n)$, where $n \geq 2$ [7].
(3) $L_2(q)$, where $q \geq 4$ is an odd prime power and $\pi(L_2(q)) \leq 4$ [12].
(4) $L_2(q)$, where $q \geq 4$ is an odd prime power and $\pi(L_2(q)) \geq 5$ [13].
(5) $L_4(q)$, where $q = 5, 7, 9, 11, 13, 17$ [1, 2].
(6) $L_4(2^n)$, where $n = 2, 3, 4$ [1].
(7) $L_3(q)$, with $q = p^n$ and $|\frac{\pi(q^2+q+1)}{d}| = 1$ where $d = (3, q-1)$ [8].
(8) $L_3(9)$ and $L_9(2)$ [5, 14].
(9) $L_n(2)$, where $n = p$ or $p+1$, for which $2^p - 1$ is a prime [2].

We add another projective special linear groups to the sixth part of the former list. In fact, we prove that $L_4(2^n)$ for $n = 5, 6$ and 7 are OD-characterizable. All groups in this paper are assumed finite.
Throughout this paper, we use the following notations: We denote the socle of G by $\mathrm{Soc}(G)$, which is the subgroup generated by the set of all minimal normal subgroups of G. For $p \in \pi(G)$, we denote by G_p and $\mathrm{Syl}_p(G)$ a Sylow p-subgroup of G and the set of all Sylow p-subgroups of G respectively. All further unexplained notations are standard and and one may refer to [10].

2. Preliminary lemmas

Given a prime $p \geqslant 5$, we denote by $\mathfrak{S}_p$ the set of all finite non-abelian simple groups G such that $p \in \pi(G) \subseteq \{2, 3, ..., p\}$. It is clear that the set of all finite non-abelian simple groups is the disjoint union of the finite sets $\mathfrak{S}_p$ for all primes $p \geqslant 5$.

Lemma 2.1. *Let P be a non-abelian simple group belongs to $\mathfrak{S}_p$, where $5 \leqslant p \leqslant 997$. Then $\pi(\mathrm{Out}(P)) \subseteq \{2, 3, 5, 7, 11\}$.*

Proof. All finite non-abelian simple groups P in $\mathfrak{S}_p$, for $5 \leqslant p \leqslant 997$, are collected in Table 4 in [11]. So by computing the order of outer automorphism groups of them, we see that $\pi(\mathrm{Out}(P)) \subseteq \{2, 3, 5, 7, 11\}$. In fact, 11 only divides the order of outer automorphism group of $L_2(2^{11})$, where $L_2(2^{11}) \in \mathfrak{S}_{683}$. □

To prove the propositions in the next section, we need degree patterns of the special linear groups under study. Since we obtain these degree patterns by a subset μ of these groups, we give following lemma.

Lemma 2.2. *Let $L = L_4(q)$. Then $\mu(L) = \{(q^2+1)(q+1), (q^3-1), 2(q^2-1), 4(q-1)\}$.*

Proof. The proof follows from the structure of maximal tori in finite simple classical groups, see [3, 6]. □

Lemma 2.3. *[4, Theorem 10.3.1] Let G be a Frobenius group with kernel K and complement H. Then:*

(1) *K is a nilpotent group.*
(2) $|K| \equiv 1(\mathrm{mod}|H|)$.

Definition 2.4. *G is said to be completely reducible group if and only if either $G = 1$ or G is the direct product of a finite number of simple groups. A completely reducible group will be called a CR-group.*

A CR-group has trivial center if and only if it is a direct product of non-abelian simple groups and in this case, it has been named a centerless CR-group. The following lemma determines the structure of the automorphism group of a centerless CR-group.

Lemma 2.5. *[10, Theorem 3.3.20] Let R be a finite centerless CR-group and write $R = R_1 \times R_2 \times ... \times R_k$, where R_i is a direct product of n_i isomorphic copies of a simple group H_i, and H_i and H_j are not isomorphic if $i \neq j$. Then $\mathrm{Aut}(R) = \mathrm{Aut}(R_1) \times \mathrm{Aut}(R_2) \times ... \times \mathrm{Aut}(R_k)$ and $\mathrm{Aut}(R_i) \cong \mathrm{Aut}(H_i) \wr \mathbb{S}_{n_i}$, where in this wreath product $\mathrm{Aut}(H_i)$ appears in its right regular representation and the symmetric group $\mathbb{S}_{n_i}$ in its natural permutation representation. Moreover, these isomorphisms induce isomorphisms $\mathrm{Out}(R) \cong \mathrm{Out}(R_1) \times \mathrm{Out}(R_2) \times ... \times \mathrm{Out}(R_k)$ and $\mathrm{Out}(R_i) \cong \mathrm{Out}(H_i) \wr \mathbb{S}_{n_i}$.*

3. Main results

Proposition 3.1. *If G is a finite group such that $D(G) = D(L_4(2^5))$ and $|G| = |L_4(2^5)|$, then $G \cong L_4(2^5)$.*

Proof. We break the proof of all propositions in this section to three steps. In this case, we know that $|G| = |L_4(2^5)| = 2^{30}.3^2.5^2.7.11^2.31^3.41.151$, now since $\mu(L_4(2^5)) = \{(2^{10}+1)(2^5+1), (2^{15}-1), 2(2^{10}-1), 4(2^5-1)\}$ (by Lemma 2.2), then $D(L_4(2^5)) = (3,5,3,2,5,5,3,2)$. So $D(G) = (3,5,3,2,5,5,3,2)$

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{151, r\}'$-group, where $r \in \{11, 31, 41\}$. In particular, G is non-solvable.
First, we show that K is a $\{151\}'$-group. Assume the contrary and let $151 \in \pi(K)$. Since $deg(151) = 2$, at least one of the primes in $\{11, 31, 41\}$ and 151 aren't adjacent, we put it r. Now we claim that r does not divide the order of K. Otherwise, we may suppose that H is a Hall $\{151, r\}$-subgroup of K of order $151 \cdot r^i$, where $i \in \{1, 2, 3\}$. It is seen that H is a nilpotent subgroup of G, thus $151.r \in \omega(K) \subseteq \omega(G)$, a contradiction. Thus, $\{151\} \subseteq \pi(K) \subseteq \pi(G) - \{r\}$. Let

$K_{151} \in \mathrm{Syl}_{151}(K)$, then by Frattini argument, $G = KN_G(K_{151})$. Therefore, $N_G(K_{151})$ contains an element of order r, say σ. Since G has no element of order $151.r$, $\langle\sigma\rangle$ should act fixed point freely on K_{151}, implying $\langle\sigma\rangle K_{151}$ is a Frobenius group. By Lemma 2.3, $|\langle\sigma\rangle|\,|(|K_{151}| - 1)$. It follows that $r|151 - 1$, which is impossible. So K is a $\{151\}^{'}$-group. Next, we show that K is a $\{r\}'$-group. Assume the contrary, let $r||K|$ and $K_r \in \mathrm{Syl}_r(K)$. Then by Frattini argument $G = KN_G(K_r)$. Since K is a $\{151\}^{'}$-group, 151 must divide $|N_G(K_r)|$, so suppose x is an element of $N_G(K_r)$ of order 151. As $\langle x\rangle \subseteq N_G(K_r)$, then $\langle x\rangle K_r$ is a subgroup of G. Moreover this subgroup is nilpotent and therefore $151 \sim r$, which is a contradiction by assumption. Therefore , r and 151 do not divide $|K|$. In addition, since $G \neq K$, G is non-solvable.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group (recall that a group G is an almost simple group, if $S \trianglelefteq G \lesssim \mathrm{Aut}(S)$, for some non-abelian group S). In fact, $S \leq \frac{G}{K} \lesssim \mathrm{Aut}(S)$, where S is a finite non-abelian simple group isomorphic to $L_4(2^5)$.

Let $\overline{G} = \frac{G}{K}$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times P_2 \times ... \times P_m$, where P_i's are finite non-abelian simple groups and $S \leq \frac{G}{K} \lesssim \mathrm{Aut}(S)$ (see [9, Proposition 3.1, Step 2]). First, we show that $m = 1$. Suppose that $m \geq 2$. We consider these separate parts:

Part A. Let $151 \nsim 2$ or $151 \nsim 3$. We claim 151 does not divide $|S|$. Assume the contrary and let $151 \mid |S|$, on the other hand by Table 1 in [11], $\{2, 3\} \subseteq \pi(P_i)$ for every i, hence $2 \sim 151$ and $3 \sim 151$, which is a contradiction. Now, by Step 1, we observe that $151 \in \pi(\overline{G}) \subseteq \pi(\mathrm{Aut}(S))$. However, $\mathrm{Aut}(S) = \mathrm{Aut}(S_1) \times \mathrm{Aut}(S_2) \times ... \times \mathrm{Aut}(S_r)$, where the groups S_j are direct products of isomorphic P_i's such that $S = S_1 \times S_2 \times ... \times S_r$. Therefore, for some j, 151 divides the order of an automorphism group of a direct product S_j of t isomorphic simple groups P_i. Since $P_i \in \mathfrak{S}_p$ $(5 \leqslant p \leqslant 151)$, Lemma 2.1 follows that $|\mathrm{Out}(P_i)|$ is not divisible by 151, so 151 does not divide the order of $\mathrm{Aut}(P_i)$. Now, by Lemma 2.5, we obtain $|\mathrm{Aut}(S_j)| = |\mathrm{Aut}(P_i)|^{t!}.t!$. Therefore, $t \geq 151$ and so 2^{302} must divide the order of G, which is a contradiction.

Part B. Let $151 \sim 2$ and $151 \sim 3$. Since $deg(151) = 2$, then $151 \nsim \{11, 31, 41\}$. So by Step 1, K is a $\{11, 31, 41, 151\}'$-group. Now since $deg(2) = 3$, 2 and at least one prime in $\{11, 31, 41\}$ are not adjacent, put it u. Now we claim u does not divide $|S|$. Assume the contrary and let $u \mid |S|$. Therefore, $u \sim 2$, which is impossible. Using similar argument as before, we see that $3^u \geqslant 3^{11}$ must divide the order of G, which is a contradiction.

Part A and Part B imply that $m = 1$ and hence $S = P_1$.
By Table 1 and Step 1, it is evident that $|S| = 2^{\alpha_1}.3^{\alpha_2}.5^{\alpha_3}.7^{\alpha_4}.11^{\alpha_5}.31^{\alpha_6}.41^{\alpha_7}.$ 151, where α_i's have the following conditions:

(1) $1 \leq \alpha_1 \leq 30$, $1 \leq \alpha_2 \leq 2$, $0 \leq \alpha_3, \alpha_5 \leq 2$, $0 \leq \alpha_4, \alpha_7 \leq 1$ and $0 \leq \alpha_6 \leq 3$;

(2) $\alpha_5 = 2$, $\alpha_6 = 3$ or $\alpha_7 = 1$.

TABLE 1. Finite simple groups $S \in \mathfrak{S}_{151}$

S	$\|S\|$
$L_3(32)$	$2^{15}.3.7.11.31^2.151$
$L_4(32)$	$2^{30}.3^2.5^2.7.11^2.31^3.41.151$
$L_5(8)$	$2^{30}.3^4.5.7^4.13.31.73.151$
$L_6(8)$	$2^{45}.3^7.5.7^5.13.19.31.73^2.151$
$L_2(151)$	$2^3.3.5^2.19.151$
A_{151}	$3 \times 4 \times 5 \times ... \times 151$
A_{152}	$3 \times 4 \times 5 \times ... \times 152$
A_{153}	$3 \times 4 \times 5 \times ... \times 153$
A_{154}	$3 \times 4 \times 5 \times ... \times 154$
A_{155}	$3 \times 4 \times 5 \times ... \times 155$
A_{156}	$3 \times 4 \times 5 \times ... \times 156$

Now, using Table 1 follows that $S \cong L_4(2^5)$, and this completes the proof of Step 2.

Step 3. G is isomorphic to $L_4(2^5)$.

By Step 2, $L_4(2^5) \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L_4(2^5))$. As $|G| = |L_4(2^5)|$, we deduce $K = 1$, so $G \cong L_4(2^5)$, and the proof is completed. □

Proposition 3.2. *If G is a finite group such that $D(G) = D(L_4(2^6))$ and $|G| = |L_4(2^6)|$, then $G \cong L_4(2^6)$.*

Proof. By Lemma 2.2, $\mu(L_4(2^6)) = \{(2^{12}+1)(2^6+1), (2^{18}-1), 2(2^{12}-1), 4(2^6-1)\}$, then $D(G) = D(L_4(2^6)) = (4, 6, 6, 6, 6, 3, 3, 3, 3)$. Also we have $|G| = |L_4(2^6)| = 2^{36}.3^7.5^2.7^3.13^2.17.19.73.241$.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{73, 241\}'$-group. In particular, G is non-solvable.

First, we show that K is a $\{241\}'$-group. Assume the contrary and let $241 \in \pi(K)$. Since $deg(241) = 3$, at least one of the primes in $\{7, 13, 19, 73\}$ and 241 aren't adjacent, we put it r. Now we claim that r does not divide the order of K. Otherwise, we may suppose that H is a Hall $\{241, r\}$-subgroup of K of order $241 \cdot r^i$, where $i \in \{1, 2, 3\}$. It is seen that H is a nilpotent subgroup of G, thus $241.r \in \omega(K) \subseteq \omega(G)$, a contradiction. Thus, $\{241\} \subseteq \pi(K) \subseteq \pi(G) - \{r\}$. Let $K_{241} \in \mathrm{Syl}_{241}(K)$, then by Frattini argument, $G = KN_G(K_{241})$. Therefore, $N_G(K_{241})$ contains an element of order r, say σ. Since G has no element of order $241.r$, $\langle\sigma\rangle$ should act fixed point freely on K_{241}, implying $\langle\sigma\rangle K_{241}$ is a Frobenius group. By Lemma 2.3, $|\langle\sigma\rangle| | (|K_{241}| - 1)$. It follows that $r | 241 - 1$, which is impossible. So K is a $\{241\}'$-group. Next, we show that K is a $\{73\}'$-group. Assume the contrary, let $73 | |K|$, since $deg(73) = 3$ then at least one of the primes in $\{13, 17, 19, 241\}$ and 73 aren't adjacent, we put it r. By similar way we obtain that $r|73-1$, which is impossible. Therefore K is a $\{73\}^{'}$-group too. In addition since $G \neq K$, G is non-solvable.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \mathrm{Aut}(S)$, where S is a finite non-abelian simple group isomorphic to $L_4(2^6)$ or $O_8^-(8)$.

Let $\overline{G} = \frac{G}{K}$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times P_2 \times ... \times P_m$, where P_i's are finite non-abelian simple groups and $S \leq \frac{G}{K} \lesssim \mathrm{Aut}(S)$. First, we show that $m = 1$. Suppose that $m \geq 2$. We consider these separate parts:

Part A. Let $241 \nsim 2$ or $241 \nsim 3$. By the same way in Proposition 3.1 (Step 2, Part A) we get a contradiction, because 2^{482} must divide the order of G, which is impossible.

Part B. Let $73 \nsim 2$ or $73 \nsim 3$. Similarly to those in Proposition 3.1 (Step 2, Part A), we can prove that 2^{146} must divide the order of G, which is a contradiction.

Part C. Let $241 \sim \{2, 3\}$ and $73 \sim \{2, 3\}$. In this case, we consider the following two subcases:

- Subcase 1. $241 \sim 73$.
  As $deg(241) = 3$, then $241 \nsim \{13, 17, 19\}$. We easily see that K is a $\{13, 17, 19\}'$-group. For example we investigate this fact for 13. Assume the contrary, let $13||K|$ and $K_{13} \in \mathrm{Syl}_{13}(K)$. Then by Frattini argument $G = KN_G(K_{13})$. Since K is a $\{241\}^{'}$-group, 241 must divide $|N_G(K_{13})|$, so suppose x is an element of $N_G(K_{13})$ of order 241. As $\langle x\rangle \subseteq N_G(K_{13})$, then $\langle x\rangle K_{13}$ is a subgroup of G. Moreover, this subgroup is nilpotent and therefore $241 \sim 13$, which is a contradiction by assumption. So K is a $\{13\}^{'}$-group.

  On the other hand $deg(2) = 4$, therefore 2 and at least one of the primes in $\{13, 17, 19\}$ are not adjacent, put it u. similarly to Proposition 3.1 (Step 2, Part B), we conclude that $3^u \geqslant 3^{13}$ must divide the order of G, which is a contradiction.
- Subcase 2. $241 \nsim 73$.
  (i) Suppose there exists one prime in $\{13, 17, 19\}$ which is not adjacent to 241 or 73, and also to 2. we put it u. By a similar way in Subcase 1, it is seen that K is a $\{u\}'$-group. Now, we claim u does not divide $|S|$. Otherwise we must have $u \sim 2$, which is impossible. The same technique in Proposition 3.1 (Step 2, Part B), implies that $3^u \geqslant 3^{13}$ must divide the order of G, which is a contradiction.
  (1) Suppose that we do not have the conditions in [(i)], i.e., two primes in $\{13, 17, 19\} := \{u_1, u_2, u_3\}$ are adjacent to 2, we put them u_2 and u_3, and the other ones, u_1, is adjacent to 73 and 241 simultaneously. By a similar way in Proposition 3.1(Step 1), K is a $\{u_2\}'$-group (because $241 \nsim u_2$), and after that similar as before as K is a $\{u_2\}'$-group (because $u_2 \nsim u_1$). Now similarly to those

Table 2. Finite simple groups $S \in \mathfrak{S}_{241}$

S	$\lvert S\rvert$
$U_3(16)$	$2^{12}.3.5.17^2.241$
${}^3D_4(4)$	$2^{24}.3^4.5^2.7.13^2.241$
$L_2(2^{12})$	$2^{12}.3^2.5.7.13.17.241$
$G_2(16)$	$2^{24}.3^3.5^2.7.13.17^2.241$
$S_4(64)$	$2^{24}.3^3.5^2.7^2.13^2.17.241$
$O_8^-(8)$	$2^{36}.3^7.5.7^3.13.17.19.73.241$
$L_4(64)$	$2^{36}.3^7.5^2.7^3.13^2.17.19.73.241$
$S_8(8)$	$2^{48}.3^9.5^2.7^4.13^3.17.19.241$
$U_4(64)$	$2^{36}.3^4.5^3.7^2.13^3.17.37.109.241$
$O_{10}^+(8)$	$2^{60}.3^9.5^2.7^5.13^2.17^2.19.31.73.151.241$
$L_3(2^{12})$	$2^{36}.3^5.5^2.7^2.13^2.17.19.37.73.109.241$
$S_6(64)$	$2^{54}.3^6.5^3.7^3.13^3.17.19.37.109.241$
$O_8^+(64)$	$2^{72}.3^7.5^3.7^4.13^4.17^2.37.73.109.241^2$
$F_4(8)$	$2^{72}.3^{10}.5^2.7^4.13^2.17.37.73^2.109.241$
$L_2(241)$	$2^4.3.5.11^2.241$
A_{241}	$3 \times 4 \times 5 \times ... \times 241$
.	.
.	.
.	.
A_{250}	$3 \times 4 \times 5 \times ... \times 250$

in Proposition 3.1 (Step 2, Part B), we can prove $3^{u_1} \geqslant 3^{13}$ must divide the order of G, which is a contradiction.

Part A, Part B and Part C imply that $m = 1$ and hence $S = P_1$.
By Table 2 and Step 1, it is evident that $|S| = 2^{\alpha_1}.3^{\alpha_2}.5^{\alpha_3}.7^{\alpha_4}.13^{\alpha_5}.17^{\alpha_6}.19^{\alpha_7}.73.241$, where α_i's have the following conditions:

$$1 \leq \alpha_1 \leq 36, \quad 1 \leq \alpha_2 \leq 7, \quad 0 \leq \alpha_3, \alpha_5 \leq 2, \quad 0 \leq \alpha_4 \leq 3 \quad \text{and} \quad 0 \leq \alpha_6, \alpha_7 \leq 1$$

Now, using Table 2 follows that $S \cong L_4(2^6)$ or $O_8^-(8)$, and this completes the proof of Step 2.

Step 3. G is isomorphic to $\cong L_4(2^6)$.

If $S \cong L_4(2^6)$, as $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$ and $|G| = |L_4(2^6)|$, we deduce $K = 1$, so $G \cong L_4(2^6)$.

If $S \cong O_8^-(8)$, by $S \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(S)$ we have,

$$1 \mid \frac{5.13}{|K|} \mid\mid Out(O_8^-(8)) \mid= 6$$

Therefore $\mid K \mid = 5 \cdot 13$. Then $K \cong \mathbb{Z}_{5.13}$ and therefore $K \leq C_G(K)$. But $\frac{C_G(K)}{K} \trianglelefteq \frac{G}{K} \cong O_8^-(8)$, then simplicity of $O_8^-(8)$ implies that $\frac{C_G(K)}{K} = 1$ or $\frac{C_G(K)}{K} \cong O_8^-(8)$. If $\frac{C_G(K)}{K} = 1$, $K = C_G(K)$ and hence,

$$\mid O_8^-(8) \mid = \mid \frac{G}{K} \mid = \mid \frac{G}{C_G(K)} \mid \mid\mid \mathrm{Aut}(K) \mid = 48$$

which is impossible. Therefore $\frac{C_G(K)}{K} \cong O_8^-(8)$, this implies that $G = C_G(K)$, so $K \leq Z(G)$, that is, G is a central extension of $\mathbb{Z}_{5.13}$ by $O_8^-(8)$. If G splits over K, then $G \cong \mathbb{Z}_{5.13} \times O_8^-(8)$, which is impossible because $deg(5) \neq 8$, by assumption. Otherwise $G \cong \mathbb{Z}_{5.13}.O_8^-(8)$, which is impossible too, because 5.13 must divide the Schur multiplier of $O_8^-(8)$, which is 1. The proof here is completed.

□

Proposition 3.3. *If G is a finite group such that $D(G) = D(L_4(2^7))$ and $|G| = |L_4(2^7)|$, then $G \cong L_4(2^7)$.*

Proof. As $\mu(L_4(2^7)) = \{(2^{14}+1)(2^7+1), (2^{21}-1), 2(2^{14}-1), 4(2^7-1)\}$, $D(G) = D(L_4(2^7)) = (3, 6, 4, 2, 4, 6, 4, 5, 2)$. Also $|G| = |L_4(2^7)| = 2^{42}.3^2.5.7^2$ $.29.43^2.113.127^3.337$.

Step 1. Let K be the maximal normal solvable subgroup of G. Then K is a $\{337, r\}'$-group, where $r \in \{43, 113, 127\}$. In particular, G is non-solvable.
First, we show that K is a $\{337\}'$-group. Assume the contrary and let $337 \in \pi(K)$. Since $deg(337) = 2$, at least one of the primes in $\{43, 113, 127\}$ and 337 are not adjacent, we put it r. Now, we claim that r does not divide the order of K. Otherwise, we may suppose that H is a Hall $\{337, r\}$-subgroup of K of order $337 \cdot r^i$, where $i \in \{1, 2, 3\}$. It is seen that H is a nilpotent subgroup of G, thus $337.r \in \omega(K) \subseteq \omega(G)$, a contradiction. Thus, $\{337\} \subseteq \pi(K) \subseteq \pi(G) - \{r\}$. Let $K_{337} \in \mathrm{Syl}_{337}(K)$, then by Frattini argument, $G = KN_G(K_{337})$. Therefore, $N_G(K_{337})$ contains an element of order r, say σ. Since G has no element of order $337.r$, $\langle\sigma\rangle$ should act fixed point freely on K_{337}, implying $\langle\sigma\rangle K_{337}$ is a Frobenius group. By Lemma 2.3, $|\langle\sigma\rangle| | (|K_{337}| - 1)$. It follows that $r | 337 - 1$, which is impossible. So K is a $\{337\}'$-group. Next, we show that K is a $\{r\}'$-group. Assume the contrary, let $r||K|$ and $K_r \in \mathrm{Syl}_r(K)$. Then by Frattini argument $G = KN_G(K_r)$. Since K is a $\{337\}'$-group, 337 must divide $|N_G(K_r)|$, so suppose x is an element of $N_G(K_r)$ of order 337. As $\langle x\rangle \subseteq N_G(K_r)$, then $\langle x\rangle K_r$ is a subgroup of G. Moreover, this subgroup is nilpotent and therefore $337 \sim r$, which is a contradiction by assumption. Therefore r and 337 do not divide $|K|$. In addition since $G \neq K$, G is non-solvable.

Step 2. The quotient $\frac{G}{K}$ is an almost simple group. In fact, $S \leq \frac{G}{K} \lesssim \mathrm{Aut}(S)$, where S is a finite non-abelian simple group isomorphic to $L_4(2^7)$.

Let $\overline{G} = \frac{G}{K}$. Then $S := \mathrm{Soc}(\overline{G}) = P_1 \times P_2 \times ... \times P_m$, where P_i's are finite non-abelian simple groups and $S \leq \frac{G}{K} \lesssim \mathrm{Aut}(S)$. First, we show that $m = 1$.

Suppose that $m \geq 2$. We consider these separate parts:
Part A. Let $337 \nsim 2$ or $337 \nsim 3$. Similarly to those in Proposition 3.1 (Step 2, Part A), we obtain 2^{674} must divide the order of G, which is a contradiction.
Part B. Let $337 \sim 2$ and $337 \sim 3$. Since $deg(2) = 3$, 2 and at least one prime in $\{43, 113, 127\}$ are not adjacent, put it u. Since $deg(337) = 2$, then $337 \nsim \{43, 113, 127\}$. So by Step 1, K is a $\{43, 113, 127, 337\}'$-group. Now we claim u does not divide $|S|$. Assume the contrary and let $u \mid |S|$. By similar way in Proposition 3.1 (Step 2, Part A), we conclude that $3^u \geqslant 3^{43}$ must divide the order of G, which is a contradiction.

Part A and Part B imply that $m = 1$ and hence $S = P_1$.

Table 3. Finite simple groups $S \in \mathfrak{S}_{337}$

S	$\|S\|$
$L_3(2^7)$	$2^{21}.3.7^2.43.127^2.337$
$L_2(337^2)$	$2^5.3.5.7.13^2.41.277.337$
$S_4(337)$	$2^{10}.3^2.5.7^2.13^4.41.277.337^4$
$L_4(2^7)$	$2^{42}.3^2.5.7^2.29.43^2.113.127^3.337$
$L_7(8)$	$2^{63}.3^7.5.7^6.13.19.31.73.127.151.337$
$L_8(8)$	$2^{84}.3^9.5^2.7^8.13^2.17.19.31.73.127.151.241.337$
$O_{14}^+(8)$	$2^{126}.3^{14}.5^3.7^9.11.13^3.19^2.31.37.73^2.109.127.151.241.331.337$
$L_2(337)$	$2^4.3.7.13^2.337$
A_{337}	$3 \times 4 \times 5 \times ... \times 337$
A_{338}	$3 \times 4 \times 5 \times ... \times 338$

By Table 3 and Step 1, it is evident that $|S| = 2^{\alpha_1}.3^{\alpha_2}.5^{\alpha_3}.7^{\alpha_4}.29^{\alpha_5}.43^{\alpha_6}.113^{\alpha_7}$ $.127^{\alpha_8}.337$, where α_i's have the following conditions:

(1) $1 \leq \alpha_1 \leq 42$, $1 \leq \alpha_2 \leq 2$, $0 \leq \alpha_4, \alpha_6 \leq 2$, $0 \leq \alpha_3, \alpha_5, \alpha_7 \leq 1$ and $0 \leq \alpha_8 \leq 3$;
(2) $\alpha_6 = 2$, $\alpha_7 = 1$ or $\alpha_8 = 3$.

Now, using Table 3 follows that $S \cong L_4(2^7)$, and this completes the proof of Step 2.

Step 3. G is isomorphic to $L_4(2^7)$.

By Step 2, $L_4(2^7) \trianglelefteq \frac{G}{K} \lesssim \mathrm{Aut}(L_4(2^7))$. As $|G| = |L_4(2^7)|$, we deduce $K = 1$, so $G \cong L_4(2^7)$, and the proof is completed. □

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