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Characterization of some projective special linear groups in dimension four by their orders and degree patterns

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CHARACTERIZATION OF SOME PROJECTIVE SPECIAL LINEAR GROUPS IN DIMENSION FOUR BY THEIR ORDERS AND DEGREE PATTERNS

M. SAJJADI, M. BIBAK AND G. R. REZAEEZADEH

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Abstract. Let $G$ be a finite group. The degree pattern of $G$ denoted by $D(G)$ is defined as follows: If $\pi(G) = \{p_1, p_2, \ldots, p_k\}$ such that $p_1 < p_2 < \ldots < p_k$, then $D(G) := (\deg(p_1), \deg(p_2), \ldots, \deg(p_k))$, where $\deg(p_i)$ for $1 \leq i \leq k$, are the degree of vertices $p_i$ in the prime graph of $G$. In this article, we consider a finite group $G$ under assumptions $|G| = |L_4(2^n)|$ and $D(G) = D(L_4(2^n))$, where $n \in \{5, 6, 7\}$ and we prove that $G \cong L_4(2^n)$.

Keywords: Degree pattern, prime graph, projective special linear group.

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1. Introduction

Let $G$ be a finite group and $\omega(G)$ be the set of element orders for $G$. The set $\omega(G)$ is partially ordered under divisibility and is uniquely determined by a subset $\mu(G)$ of its maximal elements. We put all prime divisors of $|G|$ in $\pi(G)$, and we associate to $\pi(G)$ a simple graph $\Gamma(G)$, called prime graph or Grunberg-Kegel graph, whose vertex set is $\pi(G)$ and every two primes $p$ and $q$ are adjacent iff $pq \in \omega(G)$, in this case we write $p \sim q$, and by $p \not\sim q$ we mean that any element of order $pq$ does not exist in $G$.

Definition 1.1. Let $G$ be a group with $\pi(G) = \{p_1, p_2, \ldots, p_k\}$. We define degree of $p$ as follows for $p \in \pi(G)$:

$$\deg(p) := |\{q \in \pi(G) | p \sim q\}|.$$

Also $D(G) := (\deg(p_1), \deg(p_2), \ldots, \deg(p_k))$, where $p_1 < p_2 < \ldots < p_k$, which is called degree pattern of $G$. 

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A group $G$ is called $k$-fold OD-characterizable if there exist just $k$ non-isomorphic finite groups $H$ with $|H| = |G|$ and $D(H) = D(G)$. If $k = 1$, a 1-fold OD-characterizable group is simply called OD-characterizable.

Many articles are devoted to characterize some finite simple groups by their orders and degree patterns. As in the present paper we investigate OD-characterizability of some projective special linear groups, we review only some OD-characterizable groups of this kind that have been obtained up to now.

1. $L_2(2) ≅ S_3$ and $L_2(3) ≅ A_4$ [7].
2. $L_2(2^n)$, where $n \geq 2$ [7].
3. $L_2(q)$, where $q \geq 4$ is an odd prime power and $\pi(L_2(q)) \leq 4$ [12].
4. $L_2(q)$, where $q \geq 4$ is an odd prime power and $\pi(L_2(q)) \geq 5$ [13].
5. $L_2(q)$, where $q = 5, 7, 9, 11, 13, 17$ [1, 2].
6. $L_4(2^n)$, where $n = 2, 3, 4$ [1].
7. $L_3(q)$, with $q = p^n$ and $|\pi((q^{2+n}+q+1)/d)| = 1$ where $d = (3, q-1)$ [8].
8. $L_3(9)$ and $L_9(2)$ [5, 14].
9. $L_n(2)$, where $n = p$ or $p + 1$, for which $2^p - 1$ is a prime [2].

We add another projective special linear groups to the sixth part of the former list. In fact, we prove that $L_4(2^n)$ for $n = 5, 6$ and $7$ are OD-characterizable. All groups in this paper are assumed finite.

Throughout this paper, we use the following notations: We denote the socle of $G$ by $\text{Soc}(G)$, which is the subgroup generated by the set of all minimal normal subgroups of $G$. For $p \in \pi(G)$, we denote by $G_p$ and $\text{Syl}_p(G)$ a Sylow $p$-subgroup of $G$ and the set of all Sylow $p$-subgroups of $G$ respectively. All further unexplained notations are standard and and one may refer to [10].

2. Preliminary lemmas

Given a prime $p \geq 5$, we denote by $\mathcal{S}_p$ the set of all finite non-abelian simple groups $G$ such that $p \in \pi(G) \subseteq \{2, 3, ..., p\}$. It is clear that the set of all finite non-abelian simple groups is the disjoint union of the finite sets $\mathcal{S}_p$ for all primes $p \geq 5$.

Lemma 2.1. Let $P$ be a non-abelian simple group belongs to $\mathcal{S}_p$, where $5 \leq p \leq 997$. Then $\pi(\text{Out}(P)) \subseteq \{2, 3, 5, 7, 11\}$.

Proof. All finite non-abelian simple groups $P$ in $\mathcal{S}_p$, for $5 \leq p \leq 997$, are collected in Table 4 in [11]. So by computing the order of outer automorphism groups of them, we see that $\pi(\text{Out}(P)) \subseteq \{2, 3, 5, 7, 11\}$. In fact, 11 only divides the order of outer automorphism group of $L_2(2^{11})$, where $L_2(2^{11}) \in \mathcal{S}_{683}$. 

To prove the propositions in the next section, we need degree patterns of the special linear groups under study. Since we obtain these degree patterns by a subset $\mu$ of these groups, we give following lemma.
Lemma 2.2. Let \( L = L_4(q) \). Then \( \mu(L) = \{(q^2 + 1)(q + 1), (q^3 - 1), 2(q^2 - 1), 4(q - 1)\} \).

Proof. The proof follows from the structure of maximal tori in finite simple classical groups, see \([3, 6]\). \(\square\)

Lemma 2.3. \([4, \text{Theorem 10.3.1}]\) Let \( G \) be a Frobenius group with kernel \( K \) and complement \( H \). Then:

1. \( K \) is a nilpotent group.
2. \( |K| \equiv 1 (\text{mod} |H|) \).

Definition 2.4. \( G \) is said to be completely reducible group if and only if either \( G = 1 \) or \( G \) is the direct product of a finite number of simple groups. A completely reducible group will be called a CR-group.

A CR-group has trivial center if and only if it is a direct product of non-abelian simple groups and in this case, it has been named a centerless CR-group. The following lemma determines the structure of the automorphism group of a centerless CR-group.

Lemma 2.5. \([10, \text{Theorem 3.3.20}]\) Let \( R \) be a finite centerless CR-group and write \( R = R_1 \times R_2 \times \ldots \times R_k \), where \( R_i \) is a direct product of \( n_i \) isomorphic copies of a simple group \( H_i \), and \( H_i \) and \( H_j \) are not isomorphic if \( i \neq j \). Then \( \text{Aut}(R) = \text{Aut}(R_1) \times \text{Aut}(R_2) \times \ldots \times \text{Aut}(R_k) \) and \( \text{Aut}(R_i) \cong \text{Out}(H_i) \wr S_{n_i} \), where in this wreath product \( \text{Aut}(H_i) \) appears in its right regular representation and the symmetric group \( S_{n_i} \) in its natural permutation representation. Moreover, these isomorphisms induce isomorphisms \( \text{Out}(R) \cong \text{Out}(R_1) \times \text{Out}(R_2) \times \ldots \times \text{Out}(R_k) \cong \text{Out}(H_i) \wr S_{n_i} \).

3. Main results

Proposition 3.1. If \( G \) is a finite group such that \( D(G) = D(L_4(2^5)) \) and \( |G| = |L_4(2^5)| \), then \( G \cong L_4(2^5) \).

Proof. We break the proof of all propositions in this section to three steps. In this case, we know that \( |G| = |L_4(2^5)| = 2^{30} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 2^3 \cdot 31 \cdot 41 \cdot 151 \), now since \( \mu(L_4(2^5)) = \{(2^{10} + 1)(2^5 + 1), (2^{15} - 1), 2(2^{10} - 1), 4(2^5 - 1)\} \) (by Lemma 2.2), then \( D(L_4(2^5)) = (3, 5, 3, 2, 5, 5, 3, 2) \). So \( D(G) = (3, 5, 3, 2, 5, 5, 3, 2) \).

Step 1. Let \( K \) be the maximal normal solvable subgroup of \( G \). Then \( K \) is a \{151, r\}'-group, where \( r \in \{11, 31, 41\} \). In particular, \( G \) is non-solvable. First, we show that \( K \) is a \{151\}'-group. Assume the contrary and let \( 151 \in \pi(K) \). Since \( \text{deg}(151) = 2 \), at least one of the primes in \( \{11, 31, 41\} \) and 151 aren't adjacent, we put it \( r \). Now we claim that \( r \) does not divide the order of \( K \). Otherwise, we may suppose that \( H \) is a Hall \{151, r\}-subgroup of \( K \) of order \( 151 \cdot r^i \), where \( i \in \{1, 2, 3\} \). It is seen that \( H \) is a nilpotent subgroup of \( G \), thus \( 151 \cdot r \in \omega(K) \subseteq \omega(G) \), a contradiction. Thus, \( \{151\} \subseteq \pi(K) \subseteq \pi(G) - \{r\} \). Let
\[ K_{151} \in \text{Syl}_{151}(K), \text{ then by Frattini argument, } G = KN_G(K_{151}). \text{ Therefore, } N_G(K_{151}) \text{ contains an element of order } r, \text{ say } \sigma. \text{ Since } G \text{ has no element of order } 151r, \langle \sigma \rangle \text{ should act fixed point freely on } K_{151}, \text{ implying } \langle \sigma \rangle K_{151} \text{ is a Frobenius group. By Lemma 2.3, } |\langle \sigma \rangle|||K_{151}| - 1|. \text{ It follows that } r|151 - 1, \text{ which is impossible. So } K \text{ is a } \{151\}-\text{group. Next, we show that } K \text{ is a } \{r\}-\text{group. Assume the contrary, let } r|\langle K \rangle \text{ and } K_r \in \text{Syl}_r(K). \text{ Then by Frattini argument } G = KN_G(K_r). \text{ Since } K \text{ is a } \{151\}-\text{group, } 151 \text{ must divide } |N_G(K_r)|, \text{ so suppose } x \in N_G(K_r) \text{ of order } 151. \text{ As } \langle x \rangle \subseteq N_G(K_r), \text{ then } \langle x \rangle K_r \text{ is a subgroup of } G. \text{ Moreover this subgroup is nilpotent and therefore } 151 \sim r, \text{ which is a contradiction by assumption. Therefore, } r \text{ and } 151 \text{ do not divide } |K|. \text{ In addition, since } G \neq K, \text{ } G \text{ is non-solvable.}

\textbf{Step 2.} The quotient } \frac{G}{K} \text{ is an almost simple group (recall that a group } G \text{ is an almost simple group, if } S \subseteq G \subseteq \text{Aut}(S), \text{ for some non-abelian group } S). \text{ In fact, } S \leq \frac{G}{K} \leq \text{Aut}(S), \text{ where } S \text{ is a finite non-abelian simple group isomorphic to } L_4(2^5).

\text{Let } G = \frac{G}{K}. \text{ Then } S := \text{Soc}(G) = P_1 \times P_2 \times \ldots \times P_m, \text{ where } P_i \text{ are finite non-abelian simple groups and } S \leq \frac{G}{K} \leq \text{Aut}(S) \text{ (see [9, Proposition 3.1, Step 2]). First, we show that } m = 1. \text{ Suppose that } m \geq 2. \text{ We consider these separate parts:}

\textbf{Part A.} \text{ Let } 151 \sim 2 \text{ or } 151 \sim 3. \text{ We claim } 151 \text{ does not divide } |S|. \text{ Assume the contrary and let } 151 \mid |S|, \text{ on the other hand by Table 1 in [11], } \{2, 3\} \subseteq \pi(P_i) \text{ for every } i, \text{ hence } 2 \sim 151 \text{ and } 3 \sim 151, \text{ which is a contradiction. Now, by Step 1, we observe that } 151 \in \pi(G) \leq \pi(\text{Aut}(S)). \text{ However, } \text{Aut}(S) = \text{Aut}(S_1) \times \text{Aut}(S_2) \times \ldots \times \text{Aut}(S_t), \text{ where the groups } S_i \text{ are direct products of isomorphic } P_i \text{‘s such that } S = S_1 \times S_2 \times \ldots \times S_t. \text{ Therefore, for some } j, 151 \text{ divides the order of an automorphism group of a direct product } S_j \text{ of } t \text{ isomorphic simple groups } P_i. \text{ Since } P_i \in \mathfrak{S}_p \ (5 \leq p \leq 151), \text{ Lemma 2.1 follows that } |\text{Out}(P_i)| \text{ is not divisible by } 151, \text{ so } 151 \text{ does not divide the order of } \text{Aut}(P_i). \text{ Now, by Lemma 2.5, we obtain } |\text{Aut}(S_j)| = |\text{Aut}(P_i)|^{\alpha_1, \alpha_2, \ldots, \alpha_t}. \text{ Therefore, } t \geq 151 \text{ and so } 2^{302} \text{ must divide the order of } G, \text{ which is a contradiction.}

\textbf{Part B.} \text{ Let } 151 \sim 2 \text{ and } 151 \sim 3. \text{ Since } \text{deg}(151) = 2, \text{ then } 151 \sim \{11, 31, 41\}. \text{ So by Step 1, } K \text{ is a } \{11, 31, 41, 151\}-\text{group. Now since } \text{deg}(2) = 3, 2 \text{ and at least one prime in } \{11, 31, 41\} \text{ are not adjacent, put it } u. \text{ Now we claim } u \text{ does not divide } |S|. \text{ Assume the contrary and let } u \mid |S|. \text{ Therefore, } u \sim 2, \text{ which is impossible. Using similar argument as before, we see that } 3^u \geq 3^{11} \text{ must divide the order of } G, \text{ which is a contradiction.}

\text{Part A and Part B imply that } m = 1 \text{ and hence } S = P_1. \text{ By Table 1 and Step 1, it is evident that } |S| = 2^{\alpha_1, 3^{\alpha_2}, 5^{\alpha_3}, 7^{\alpha_4}, 11^{\alpha_5}, 31^{\alpha_6}, 41^{\alpha_7}}. \text{ 151, where } \alpha_i \text{‘s have the following conditions:}

(1) \ 1 \leq \alpha_1 \leq 30, \ 1 \leq \alpha_2 \leq 2, \ 0 \leq \alpha_3, \alpha_5 \leq 2, \ 0 \leq \alpha_4, \alpha_7 \leq 1 \text{ and } 0 \leq \alpha_6 \leq 3;

(2) \ \alpha_5 = 2, \ \alpha_6 = 3 \text{ or } \alpha_7 = 1.
Finite simple groups

By Lemma 2.3 and 2.2, if $r$ is a prime in $L_2(241)$, then $j(r) = 1$. It follows that $G$ is a finite group such that $D(G) = D(L_4(2^5))$ and $|G| = |L_4(2^5)|$, we deduce $K = 1$, so $G \cong L_4(2^5)$, and the proof is completed.

Proposition 3.2. If $G$ is a finite group such that $D(G) = D(L_4(2^5))$ and $|G| = |L_4(2^5)|$, then $G \cong L_4(2^5)$.

Proof. By Lemma 2.2, $\mu(L_4(2^5)) = \{(2^{12}+1)(2^6+1), (2^{18}-1), 2(2^{12}-1), 4(2^6-1)\}$, then $D(G) = D(L_4(2^5)) = \{4, 6, 6, 6, 3, 3, 3, 3\}$. Also we have $|G| = |L_4(2^5)| = 2^{36} \cdot 3^7 \cdot 5^2 \cdot 7^3 \cdot 13^2 \cdot 17 \cdot 19 \cdot 73 \cdot 241$.

Step 1. Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{73, 241\}$-group. In particular, $G$ is non-solvable.

First, we show that $K$ is a $\{241\}$-group. Assume the contrary and let $241 \in \pi(K)$. Since $\deg(241) = 3$, at least one of the primes in $\{7, 13, 19, 73\}$ and 241 aren’t adjacent, we put it $r$. Now we claim that $r$ does not divide the order of $K$. Otherwise, we may suppose that $H$ is a Hall $\{241, r\}$-subgroup of $K$ of order $241 \cdot r^i$, where $i \in \{1, 2, 3\}$. It is seen that $H$ is a nilpotent subgroup of $G$, thus $241 \cdot r^i \in \omega(K) \subseteq \omega(G)$, a contradiction. Thus, $\{241\} \subseteq \pi(K) \subseteq \pi(G) - \{r\}$. Let $K_{241} \in \text{Syl}_{241}(K)$, then by Frattini argument, $G = KN_G(K_{241})$. Therefore, $N_G(K_{241})$ contains an element of order $r$, say $\sigma$. Since $G$ has no element of order $241 \cdot r$, $\langle \sigma \rangle$ should act fixed point freely on $K_{241}$, implying $\langle \sigma \rangle K_{241}$ is a Frobenius group. By Lemma 2.3, $|\langle \sigma \rangle||(|K_{241}| - 1)$, which is impossible. So $K$ is a $\{241\}$-group. Next, we show that $K$ is a $\{73\}$-group. Assume the contrary, let $73 || K$, since $\deg(73) = 3$ then at least one of the primes in $\{13, 17, 19, 241\}$ and 73 aren’t adjacent, we put it $r$. By similar
way we obtain that \( r|73 - 1 \), which is impossible. Therefore \( K \) is a \( \{73\}'\)-group too. In addition since \( G \neq K \), \( G \) is non-solvable.

**Step 2.** The quotient \( \frac{G}{K} \) is an almost simple group. In fact, \( S \leq \frac{G}{K} \leq \text{Aut}(S) \), where \( S \) is a finite non-abelian simple group isomorphic to \( L_4(2^p) \) or \( O_8^+(2) \).

Let \( G = \frac{G}{K} \). Then \( S := \text{Soc}(G) = P_1 \times P_2 \times \ldots \times P_m \), where \( P_i \)'s are finite non-abelian simple groups and \( S \leq \frac{G}{K} \leq \text{Aut}(S) \). First, we show that \( m = 1 \).

Suppose that \( m \geq 2 \). We consider these separate parts:

**Part A.** Let \( 241 \sim 2 \) or \( 241 \sim 3 \). By the same way in Proposition 3.1 (Step 2, Part A) we get a contradiction, because \( 2^{462} \) must divide the order of \( G \), which is impossible.

**Part B.** Let \( 73 \sim 2 \) or \( 73 \sim 3 \). Similarly to those in Proposition 3.1 (Step 2, Part A), we can prove that \( 2^{466} \) must divide the order of \( G \), which is a contradiction.

**Part C.** Let \( 241 \sim \{2, 3\} \) and \( 73 \sim \{2, 3\} \). In this case, we consider the following two subcases:

- **Subcase 1.** \( 241 \sim 73 \).
  
  As \( \text{deg}(241) = 3 \), then \( 241 \sim \{13, 17, 19\} \). We easily see that \( K \) is a \( \{13, 17, 19\}'\)-group. For example we investigate this fact for \( 13 \). Assume the contrary, let \( 13 || K \) and \( K_{13} \in \text{Syl}_{13}(K) \). Then by Frattini argument \( G = KN_G(K_{13}) \). Since \( K \) is a \( \{241\}'\)-group, \( 241 \) must divide \( |N_G(K_{13})| \), so suppose \( x \) is an element of \( N_G(K_{13}) \) of order \( 241 \). As \( \langle x \rangle \leq N_G(K_{13}) \), then \( \langle x \rangle K_{13} \) is a subgroup of \( G \). Moreover, this subgroup is nilpotent and therefore \( 241 \sim 13 \), which is a contradiction by assumption. So \( K \) is a \( \{13\}'\)-group.

  On the other hand \( \text{deg}(2) = 4 \), therefore \( 2 \) and at least one of the primes in \( \{13, 17, 19\} \) are not adjacent, put it \( u \). Similarly to Proposition 3.1 (Step 2, Part B), we conclude that \( 3^u \geq 3^{13} \) must divide the order of \( G \), which is a contradiction.

- **Subcase 2.** \( 241 \sim 73 \).
  
  (i) Suppose there exists one prime in \( \{13, 17, 19\} \) which is not adjacent to \( 241 \) or \( 73 \), and also to \( 2 \). we put it \( u \). By a similar way in Subcase 1, it is seen that \( K \) is a \( \{u\}'\)-group. Now, we claim \( u \) does not divide \( |S| \). Otherwise we must have \( u \sim 2 \), which is impossible. The same technique in Proposition 3.1 (Step 2, Part B), implies that \( 3^u \geq 3^{13} \) must divide the order of \( G \), which is a contradiction.

(1) Suppose that we do not have the conditions in [(i)], i.e., two primes in \( \{13, 17, 19\} := \{u_1, u_2, u_3\} \) are adjacent to \( 2 \), we put them \( u_2 \) and \( u_3 \), and the other ones, \( u_1 \), is adjacent to \( 73 \) and \( 241 \) simultaneously. By a similar way in Proposition 3.1(Step 1), \( K \) is a \( \{u_2\}'\)-group (because \( 241 \sim u_2 \)), and after that similar as before as \( K \) is a \( \{u_2\}'\)-group (because \( u_2 \sim u_1 \)). Now similarly to those
Table 2. Finite simple groups $S \in \mathcal{S}_{241}$

| $S$          | $|S|$                     |
|--------------|---------------------------|
| $U_3(16)$    | $2^{12}.3.5.17^2.241$    |
| $^{3}D_4(4)$ | $2^{24}.3^4.5^2.7.13^2.241$ |
| $L_2(2^{12})$ | $2^{12}.3^2.5.7.13.17.241$ |
| $G_2(16)$    | $2^{24}.3^3.5^2.7.13.17^2.241$ |
| $S_4(64)$    | $2^{24}.3^3.5^2.7^2.13^2.17.241$ |
| $O_8^+(8)$   | $2^{36}.3^3.5^7.13.17.19.73.241$ |
| $L_4(64)$    | $2^{36}.3^7.5^2.7^2.13^2.17.19.73.241$ |
| $S_8(8)$     | $2^{48}.3^9.5^2.7^2.13^2.17.19.241$ |
| $U_4(64)$    | $2^{36}.3^3.5^3.7^2.13^2.17.37.109.241$ |
| $O_{10}^{-}(8)$ | $2^{60}.3^9.5^2.7^3.13^2.17^2.19.31.73.151.241$ |
| $L_3(2^{12})$ | $2^{36}.3^3.5^2.7^2.13^2.17.19.37.73.109.241$ |
| $S_6(64)$    | $2^{54}.3^6.5^3.7^2.13^2.17.19.37.109.241$ |
| $O_8^{-}(64)$ | $2^{72}.3^7.5^2.7^4.13^4.17^2.37.73.109.241^2$ |
| $F_4(8)$     | $2^{72}.3^{10}.5^2.7^4.13^2.17.37.73^2.109.241$ |
| $L_2(241)$   | $2^{12}.3.5.11^2.241$     |
| $A_{241}$    | $3 \times 4 \times 5 \times ... \times 241$ |
| $A_{245}$    | $3 \times 4 \times 5 \times ... \times 250$ |

in Proposition 3.1 (Step 2, Part B), we can prove $3^{m_1} \geq 3^{13}$ must divide the order of $G$, which is a contradiction.

Part A, Part B and Part C imply that $m = 1$ and hence $S = P_1$.

By Table 2 and Step 1, it is evident that $|S| = 2^{241}.3^{242}.5^{103}.7^{109}.13^{152}.17^{109}.19^{17}$. 73.241, where $\alpha_i$’s have the following conditions:

$$1 \leq \alpha_1 \leq 36, \quad 1 \leq \alpha_2 \leq 7, \quad 0 \leq \alpha_3, \alpha_5 \leq 2, \quad 0 \leq \alpha_4 \leq 3 \quad \text{and} \quad 0 \leq \alpha_6, \alpha_7 \leq 1$$

Now, using Table 2 follows that $S \cong L_4(2^6)$ or $O_{8}^{-}(8)$, and this completes the proof of Step 2.

**Step 3.** $G$ is isomorphic to $\cong L_4(2^6)$.

If $S \cong L_4(2^6)$, as $S \cong G \cong \text{Aut}(S)$ and $|G| = |L_4(2^6)|$, we deduce $K = 1$, so $G \cong L_4(2^6)$.

If $S \cong O_{8}^{-}(8)$, by $S \cong G \cong \text{Aut}(S)$ we have,

$$1 \mid \frac{513}{14} \mid \| \text{Out}(O_{8}^{-}(8)) \| = 6$$
Therefore $|K| = 5 \cdot 13$. Then $K \cong \mathbb{Z}_{5,13}$ and therefore $K \leq C_G(K)$. But $C_G(K) / K \cong O_8^-(8)$, thus simplicity of $O_8^-(8)$ implies that $C_G(K) / K = 1$ or $O_8^-(8)$ is impossible too, because $\langle G \rangle$.

Thus, $O_8^-(8)$ is impossible, because $G = C_G(K)$, so $K \leq Z(G)$, that is, $G$ is a central extension of $\mathbb{Z}_{5,13}$ by $O_8^-(8)$. If $G$ splits over $K$, then $G \cong \mathbb{Z}_{5,13} \times O_8^-(8)$, which is impossible because $\deg(5) \neq 8$, by assumption. Otherwise $G \cong \mathbb{Z}_{5,13} \times O_8^-(8)$, which is impossible too, because 5.13 must divide the Schur multiplier of $O_8^-(8)$, which is 1. The proof here is

\[ \text{Proposition 3.3.} \quad \text{If } G \text{ is a finite group such that } D(G) = D(L_4(2^7)) \text{ and } |G| = |L_4(2^7)|, \text{ then } G \cong L_4(2^7). \]

Proof. As $\mu(L_4(2^7)) = \{(2^{14} + 1)(2^7 + 1), (2^{21} - 1), 2(2^{14} - 1), 4(2^7 - 1)\}$, $D(G) = D(L_4(2^7)) = (3, 6, 4, 2, 4, 6, 4, 5, 2)$. Also $|G| = |L_4(2^7)| = 2^{42}.3^2.5.7^2.29.43.113.127^3.337$.

\textbf{Step 1.} Let $K$ be the maximal normal solvable subgroup of $G$. Then $K$ is a $\{337, r\}$-group, where $r \in \{43, 113, 127\}$. In particular, $G$ is non-solvable. First, we show that $K$ is a $\{337\}$-group. Assume the contrary and let $337 \in \pi(K)$. Since $\deg(337) = 2$, at least one of the primes in $\{43, 113, 127\}$ and 337 are not adjacent, we put it $r$. Now, we claim that $r$ does not divide the order of $K$. Otherwise, we may suppose that $H$ is a Hall $\{337, r\}$-subgroup of $K$ of order $337 \cdot r^i$, where $i \in \{1, 2, 3\}$. It is seen that $H$ is a nilpotent subgroup of $G$, thus $337 \cdot r \in \omega(K) \subseteq \omega(G)$, a contradiction. Thus, $\{337\} \subseteq \pi(K) \subseteq \pi(G) - \{r\}$. Let $K_{337} \in \text{Syl}_{337}(K)$, then by Frattini argument, $G = KN_G(K_{337})$. Therefore, $N_G(K_{337})$ contains an element of order $r$, say $\sigma$. Since $G$ has no element of order $337 \cdot r$, $\langle \sigma \rangle$ should act fixed point freely on $K_{337}$, implying $\langle \sigma \rangle K_{337}$ is a Frobenius group. By Lemma 2.3, $|\langle \sigma \rangle||K_{337} - 1|$. It follows that $r|337 - 1$, which is impossible. So $K$ is a $\{337\}$-group. Next, we show that $K$ is a $\{r\}$-group. Assume the contrary, let $r|\bar{K}$ and $K_r \in \text{Syl}_r(K)$. Then by Frattini argument $G = KN_G(K_r)$. Since $K$ is a $\{337\}$-group, 337 must divide $|N_G(K_r)|$, so suppose $x$ is an element of $N_G(K_r)$ of order 337. As $\langle x \rangle \subseteq N_G(K_r)$, then $\langle x \rangle K_r$ is a subgroup of $G$. Moreover, this subgroup is nilpotent and therefore $337 \sim r$, which is a contradiction by assumption. Therefore $r$ and 337 do not divide $|K|$. In addition since $G \neq K$, $G$ is non-solvable.

\textbf{Step 2.} The quotient $G / K$ is an almost simple group. In fact, $S \leq G / K \leq \text{Aut}(S)$, where $S$ is a finite non-abelian simple group isomorphic to $L_4(2^7)$.

Let $\overline{G} = G / K$. Then $S := \text{Soc}(\overline{G}) = P_1 \times P_2 \times \ldots \times P_m$, where $P_i$'s are finite non-abelian simple groups and $S \leq G / K \leq \text{Aut}(S)$. First, we show that $m = 1$. 


Suppose that \( m \geq 2 \). We consider these separate parts:

**Part A.** Let \( 337 \sim 2 \) or \( 337 \sim 3 \). Similarly to those in Proposition 3.1 (Step 2, Part A), we obtain \( 2^74 \) must divide the order of \( G \), which is a contradiction.

**Part B.** Let \( 337 \sim 2 \) and \( 337 \sim 3 \). Since \( deg(2) = 3, 2 \) and at least one prime in \( \{43, 113, 127\} \) are not adjacent, put it \( u \). Since \( deg(337) = 2 \), then \( 337 \sim \{43, 113, 127\} \). So by Step 1, \( K \) is a \( \{43, 113, 127, 337\} \)-group. Now we claim \( u \) does not divide \( |S| \). Assume the contrary and let \( u \mid |S| \). By similar way in Proposition 3.1 (Step 2, Part A), we conclude that \( 3^a \geq 3^{47} \) must divide the order of \( G \), which is a contradiction.

Part A and Part B imply that \( m = 1 \) and hence \( S = P_1 \).

**Table 3.** Finite simple groups \( S \in \mathcal{S}_{337} \)

| \( S \)      | \( |S| \)                                    |
|------------|--------------------------------------------|
| \( L_3(2^7) \) | \( 2^{41}.3.7^2.43.127^2.337 \)            |
| \( L_2(337^2) \) | \( 2^5.3.5.7.13^2.41.277.337 \)            |
| \( S_4(337) \) | \( 2^{10}.3^2.5.7^2.13^2.41.277.337^4 \)  |
| \( L_4(2^7) \) | \( 2^{42}.3^2.5.7^2.29.43^2.113.127^3.337 \) |
| \( L_7(8) \) | \( 2^{63}.3.7.5.7^2.13.19.31.73.127.151.337 \) |
| \( L_8(8) \) | \( 2^{84}.3^3.5^2.7^2.13^2.17.19.31.73.127.151.241.337 \) |
| \( O_4^+(8) \) | \( 2^{126}.3^3.5^2.7^2.11.13.19.31.37.73^2.109.127.151.242.331.337 \) |
| \( L_4(337) \) | \( 2^{43}.3.7^2.13^2.337 \)                |
| \( A_{437} \) | \( 3 \times 4 \times 5 \times \ldots \times 337 \) |
| \( A_{438} \) | \( 3 \times 4 \times 5 \times \ldots \times 338 \) |

By Table 3 and Step 1, it is evident that \( |S| = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \cdot 7^{\alpha_4} \cdot 29^{\alpha_5} \cdot 43^{\alpha_6} \cdot 113^{\alpha_7} \cdot 127^{\alpha_8} \cdot 337 \), where \( \alpha_i \)'s have the following conditions:

1. \( 1 \leq \alpha_1 \leq 42, \ 1 \leq \alpha_2 \leq 2, \ 0 \leq \alpha_4, \alpha_6 \leq 2, \ 0 \leq \alpha_3, \alpha_5, \alpha_7 \leq 1 \) and \( 0 \leq \alpha_8 \leq 3 \).
2. \( \alpha_6 = 2, \ \alpha_7 = 1 \) or \( \alpha_8 = 3 \).

Now, using Table 3 follows that \( S \cong L_4(2^7) \), and this completes the proof of Step 2.

**Step 3.** \( G \) is isomorphic to \( L_4(2^7) \).

By Step 2, \( L_4(2^7) \leq \overline{G} \leq \text{Aut}(L_4(2^7)) \). As \( |G| = |L_4(2^7)| \), we deduce \( K = 1 \), so \( G \cong L_4(2^7) \), and the proof is completed. \( \Box \)

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REFERENCES


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