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ON THE TYCHONOFF'S TYPE THEOREM VIA GRILLS

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Abstract. Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of topological spaces, and $\mathcal{G}_\alpha$ be a grill on X_α for each $\alpha \in \Lambda$. We consider Tychonoff's type Theorem for $X = \prod_{\alpha\in\Lambda} X_\alpha$ via the above grills and a natural grill on X related to these grills, and present a simple proof to this theorem. This immediately yields the classical theorem of Tychonoff. We shall also observe that the above result is also equivalent to the Axiom of Choice.
Keywords: Grill, compactness via grills, Axiom of Choice.
MSC(2010): Primary: 54D35.

1. Introduction and preliminaries

For every grill $\mathcal{G}$ on a topological space (X, τ), B. Roy and M.N. Mukherjee [5], have recently introduced a certain compactness on (X, τ), called $\mathcal{G}$-compact and in [5, Theorem 2.18], have shown that a Tychonoff's type theorem for the Cartesian product of topological spaces holds, with respect to this compactness.

The concept of grill was first introduced by Choquet in [1]. Later this concept, which has some advantages over similar concepts in topology, has received attention by some authors, see [5] for some pertinent references. Let us recall that a grill on a topological space (X, τ) is a non-null collection $\mathcal{G}$ of nonempty subsets of X satisfying two conditions: (i) If $A \in \mathcal{G}$ and $A \subseteq B \subseteq X$ then $B \in \mathcal{G}$, and (ii) for each $A, B \subseteq X$ if $A \cup B \in \mathcal{G}$ then $A \in \mathcal{G}$ or $B \in \mathcal{G}$, or equivalently for each $A, B \subseteq X (A \cup B \in \mathcal{G} \iff A \in \mathcal{G}$ or $B \in \mathcal{G})$, see also [5].

We cite the next two definitions from [5].

Definition 1.1. Let $\mathcal{G}$ be a grill on a topological space (X, τ). A cover $\{U_\alpha : \alpha \in \Lambda\}$ of X is said to be a $\mathcal{G}$-cover of X, if there exists a finite subset Λ_0 of Λ such that $X \setminus \cup_{\alpha\in\Lambda_0} U_\alpha \notin \mathcal{G}$.

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Definition 1.2. Let $\mathcal{G}$ be a grill on a topological space (X,τ). Then (X,τ) is said to be compact with respect to the grill $\mathcal{G}$ or simply $\mathcal{G}$-compact if every open cover of X is a $\mathcal{G}$-cover of X.

Clearly, every compact space (X,τ) is $\mathcal{G}$-compact, where $\mathcal{G}$ is any grill on X. It is also manifest that if $\mathcal{G} = P(X) \setminus \{\varnothing\}$, then (X,τ) is compact if and only it is $\mathcal{G}$-compact.

The following definition is the counterpart of the concept of the finite intersection property.

Definition 1.3. Let $\mathcal{G}$ be a grill on a topological space (X,τ). A familly $\mathcal{F}$ of subsets of X is said to have $\mathcal{G}$-finite intersection property (briefly, $\mathcal{G}$-F.I.P), if $\cap_{A\in\mathcal{S}} A \in \mathcal{G}$, where $\mathcal{S}$ is any finite nonempty subset of $\mathcal{F}$.

The next proposition is the counterpart of the well-known fact that a space (X,τ) is compact if and only if any nonempty family of closed subsets of X with the finite intersection property, has nonempty intersection.

Proposition 1.4. *Let $\mathcal{G}$ be a grill on a topological space (X,τ), then (X,τ) is $\mathcal{G}$ -compact if whenever $\mathcal{F}$ is a collection of closed subsets of X with $\cap\mathcal{F} = \varnothing$ then there exists a finite subfamily $\mathcal{S}$ of $\mathcal{F}$ such that $\cap_{A\in\mathcal{S}} A \notin \mathcal{G}$ (or equivalently, if every finite intersection of elements of $\mathcal{F}$ is in $\mathcal{G}$, then $\cap\mathcal{F} \neq \varnothing$).*

Proof. Suppose that X is a $\mathcal{G}$-compact space and let $\mathcal{F}$ be a collection of closed subsets of X with $\cap\mathcal{F} = \varnothing$. Clearly $\{A^c : A \in \mathcal{F}\}$ is an open cover of X and therefore by the $\mathcal{G}$-compactness of X there exists a finite subcollection $\mathcal{S}$ of $\mathcal{F}$ such that $X \setminus \cup_{A\in\mathcal{S}} A^c \notin \mathcal{G}$, that is to say, $\cap_{A\in\mathcal{S}} A \notin \mathcal{G}$ and we are done.
Conversly, we are to show that X is $\mathcal{G}$-compact. Let $\{U_\alpha : \alpha \in \Lambda\}$ be an open cover of X, then $\{U_\alpha^c : \alpha \in \Lambda\}$ is a family of closed subsets of X with the empty intersection. Thus by our assumption there exists a finite subset Λ_0 of Λ such that $\cap_{\alpha\in\Lambda_0} U_\alpha^c \notin \mathcal{G}$, i.e. there exists a finite subset Λ_0 of Λ such that $X \setminus \cup_{\alpha\in\Lambda_0} U_\alpha \notin \mathcal{G}$ hence X is $\mathcal{G}$-compact.

□

Let us also emphasize that by using Zorn's Lemma, one can trivially observe that if $\mathcal{F}$ is a family of subsets of a topological space (X,τ) with the $\mathcal{G}$-F.I.P, where $\mathcal{G}$ is a grill on X, then $\mathcal{F}$ is contained in a maximal family with the $\mathcal{G}$-F.I.P.

2. **Main results**

Now we are ready to give our simple proof to the result that we promised in the abstract. As we know, Tychonoff's theorem on the product of compact spaces is a fundamental cornerstone in the subject of compact spaces and as it is, rightly believed by many authors, it is the most important single theorem in general topology, see for example [2, 3]. There are several proofs of this theorem in the literature. Roy and Mukherjee in [5], similarly to the classical proof of Tychonoff's theorem, first prove the Alexander's subbase theorem for $\mathcal{G}$-compactness and then apply it to give an interesting proof to the second part of the next theorem which generalizes the theorem of Tychonoff. In what follows we give a direct proof to this generalized result without using the Alexander's subbase theorem.

Theorem 2.1. *Let $\{X_\alpha : \alpha \in \Lambda\}$ be a family of topological spaces, and $\mathcal{G}_\alpha$ be a grill on X_α for each $\alpha \in \Lambda$. Let $\mathcal{G}$ be any grill on the Cartesian product space $X = \prod_{\alpha\in\Lambda} X_\alpha$ such that $p_\alpha^{-1}(\mathcal{G}_\alpha) \subseteq \mathcal{G}$ (resp., $p_\alpha^{-1}(\mathcal{G}_\alpha) \supseteq \mathcal{G}$) for each $\alpha \in \Lambda$, where $p_\alpha : X \to X_\alpha$ is the natural αth projection map. If X is $\mathcal{G}$-compact, then X_α is $\mathcal{G}_\alpha$-compact for each $\alpha \in \Lambda$ (resp., if X_α is $\mathcal{G}_\alpha$-compact for each $\alpha \in \Lambda$, then X is $\mathcal{G}$-compact, too).*

Proof. First, let us assume that X is $\mathcal{G}$-compact and $\{U_i : i \in I\}$ be an open cover for X_α, where $\alpha \in \Lambda$. Hence $X = \bigcup_{i\in I} p_\alpha^{-1}(U_i)$. Since X is $\mathcal{G}$-compact, we infer that there exists a finite subset J of I such that $X \setminus \bigcup_{i\in J} p_\alpha^{-1}(U_i) \notin \mathcal{G}$. This immediately implies that $X_\alpha \setminus \bigcup_{i\in J} U_i \notin \mathcal{G}_\alpha$ and we are done, otherwise $A = X_\alpha \setminus \bigcup_{i\in J} U_i \in \mathcal{G}_\alpha$, which in turn, implies that $p_\alpha^{-1}(A) \in p_\alpha^{-1}(\mathcal{G}_\alpha) \subseteq \mathcal{G}$, by our hypothesis and therefore $p_\alpha^{-1}(A) = X \setminus \bigcup_{i\in J} p_\alpha^{-1}(U_i) \in \mathcal{G}$, which is absurd. Now we suppose that X_α is $\mathcal{G}_\alpha$-compact for each $\alpha \in \Lambda$, and we are to show that X is $\mathcal{G}$-compact. In view of the above proposition it suffices to show that if $\mathcal{F}$ is a collection of closed subsets of X with the $\mathcal{G}$-F.I.P, then $\bigcap \mathcal{F} \neq \varnothing$. By our comment preceding the theorem, $\mathcal{F}$ is contained in a maximal family of subsets of X, say $\mathcal{H}$, with the $\mathcal{G}$-F.I.P. If we prove that $\bigcap_{A\in\mathcal{H}} \bar{A} \neq \varnothing$, where $\bar{A}$ is the closure of A in X, we are done. Hence without loss of generality we may suppose that $\mathcal{F}$ is a maximal collection of subsets of X (note, not necessarily consisting of closed subsets of X) with respect to the property $\mathcal{G}$-F.I.P., and try to show that $\bigcap_{A\in\mathcal{F}} \bar{A} \neq \varnothing$, which completes the proof. For each $\alpha \in \Lambda$, we claim that $p_\alpha(\mathcal{F}) = \{p_\alpha(A) : A \in \mathcal{F}\}$ has the $\mathcal{G}_\alpha$-F.I.P. To see this, let $\mathcal{S}$ be a nonempty finite subset of $\mathcal{F}$ and we must show that $\bigcap_{A\in\mathcal{S}} p_\alpha(A) \in \mathcal{G}_\alpha$. Since $\mathcal{F}$ is maximal with respect to the $\mathcal{G}$-F.I.P., we infer that it is closed under the finite intersection. Hence $B = \bigcap_{A\in\mathcal{S}} A \in \mathcal{F}$, i.e., $p_\alpha(B) \in p_\alpha(\mathcal{F})$. Since $\mathcal{F}$ has the $\mathcal{G}$-F.I.P., we infer that $B \in \mathcal{G}$. Consequently, by our assumption $B \in p_\alpha^{-1}(\mathcal{G}_\alpha)$, which implies that $p_\alpha(B) \in \mathcal{G}_\alpha$. The latter containment and the trivial fact that $\bigcap_{A\in\mathcal{S}} p_\alpha(A) \supseteq p_\alpha(B)$ imply that $\bigcap_{A\in\mathcal{S}} p_\alpha(A) \in \mathcal{G}_\alpha$ (note, $\mathcal{G}_\alpha$ is a grill). Thus we have already shown that $p_\alpha(\mathcal{F})$ has the $\mathcal{G}_\alpha$-F.I.P., as we claimed above. This immediately implies that $\{\overline{p_\alpha(A)} : A \in \mathcal{F}\}$ has the $\mathcal{G}_\alpha$-F.I.P., too, where $\overline{p_\alpha(A)}$ is the closure of $p_\alpha(A)$, in X_α for each $A \in \mathcal{F}$. Since X_α is $\mathcal{G}_\alpha$-compact, we infer that $\bigcap_{A\in\mathcal{F}} \overline{p_\alpha(A)} \neq \varnothing$ (by the above proposition). For each $\alpha \in \Lambda$, take $x_\alpha \in X_\alpha$ to be an element in the latter nonempty intersection and let $x \in X$ to be the unique element with $p_\alpha(x) = x_\alpha$ for each $\alpha \in \Lambda$. Now we claim that $x \in \bar{A}$ for each element $A \in \mathcal{F}$, that is to say, $\bigcap_{A\in\mathcal{F}} \bar{A} \neq \varnothing$, which completes the proof by our comment at the beginning of the second part of the proof. Before proving our latter claim, let us emphasize that $\mathcal{F}$ is in fact an ultrafilter and therefore whenever a subset B of X intersects each member of $\mathcal{F}$, $B \in \mathcal{F}$. Let $B = \bigcap_{\alpha\in\mathcal{S}} p_\alpha^{-1}(G_\alpha)$, where $\mathcal{S}$ is a finite subset of Λ and G_α is an open set in X_α for each α, be an arbitrary member of the canonical base for X and let $x \in B$. Since $x_\alpha \in \overline{p_\alpha(A)}$ for each element $A \in \mathcal{F}$, we infer that for each $\alpha \in \mathcal{S}$, $G_\alpha \bigcap p_\alpha(A) \neq \varnothing$, where A is any member of $\mathcal{F}$. Consequently, $A \bigcap p_\alpha^{-1}(G_\alpha) \neq \varnothing$ for each $\alpha \in \mathcal{S}$. It follows that $p_\alpha^{-1}(G_\alpha) \in \mathcal{F}$ for each $\alpha \in \mathcal{S}$, hence $B = \bigcap_{\alpha\in\mathcal{S}} p_\alpha^{-1}(G_\alpha) \in \mathcal{F}$. This implies that $B \bigcap A \neq \varnothing$, for all $A \in \mathcal{F}$ which means that $x \in \bar{A}$ for all $A \in \mathcal{F}$ and we are done. □

Remark 2.2. If in the above theorem for each α we take $\mathcal{G}_\alpha = P(X_\alpha) \setminus \{\varnothing\}$ and let $\mathcal{G} = P(X) \setminus \{\varnothing\}$, then we immediately obtain the celebrated Tychonoff's Theorem. We should emphasize that the converse of each part in the previous theorem may not hold in general. To see this, let (X, τ) be a non-compact space and let $x \in X$ be a fixed point. Now put $X_\alpha = X$ for each $\alpha \in \Lambda$ and consider $Y = \prod_{\alpha\in\Lambda} X_\alpha$ and put $\mathcal{G}_\alpha = \{A \subseteq X : x \in A\}$. Clearly, $\mathcal{G}_\alpha$ is a grill on X_α for each α and evidently each X_α is $\mathcal{G}_\alpha$-compact. But if we take the grill $\mathcal{G} = P(Y) \setminus \{\varnothing\}$ on Y, then clearly $p_\alpha^{-1}(\mathcal{G}_\alpha) \subseteq \mathcal{G}$ and Y is not $\mathcal{G}$-compact. For the second part, consider $\mathcal{G}_\alpha = P(X) \setminus \{\varnothing\}$ for each $\alpha \in \Lambda$ and for a fixed element $y \in Y$, let $\mathcal{G} = \{A \subseteq Y : y \in A\}$. Now it is manifest that $p_\alpha^{-1}(\mathcal{G}_\alpha) \supseteq \mathcal{G}$ for each $\alpha \in \Lambda$ and Y is $\mathcal{G}$-compact, but no X_α is $\mathcal{G}_\alpha$-compact.

Kelly in [4], shows that Tychonoff's Theorem is equivalent to the Axiom of Choice. Next, we aim to show that the Axiom of Choice is also equivalent to the above theorem. But, before presenting our proof let us make it clear that we are not using the previous result of Kelley.

Theorem 2.3. *The above theorem is equivalent to the Axiom of Choice.*

Proof. Clearly we have used Zorn's lemma in the proof of the above theorem. Hence it remains to be shown that the theorem implies the Axiom of Choice. Let $\{X_i : i \in I\}$ be a collection of nonempty sets. We must show that $\prod_{i\in I} X_i \neq \varnothing$. Let t be an element such that $t \notin X_i$ for all $i \in I$. Define $Y_i = X_i \bigcup \{t\}$ and for each $i \in I$ define the topology $\tau_i = \{\varnothing, Y_i, \{t\}\}$ on Y_i. Now for each $i \in I$ we define $\mathcal{G}_i = P(Y_i) \setminus \{\varnothing\}$. We also put $Y = \prod_{i\in I} Y_i$ which is nonempty, for each component Y_i contains t. Let $\mathcal{G}$ be a grill on Y such that it contains all the subsets of Y which are of the form $\prod_{i\in J} X_i \times \prod_{i\in I\setminus J} Y_i$, where J is a finite subset of I (e.g., $\mathcal{G} = P(Y) \setminus \{\varnothing\}$). The latter subsets are clearly nonempty (note, no Axiom of Choice is needed). Now $\mathcal{F} = \{p_i^{-1}(X_i) : i \in I\}$

is a collection of closed subsets of Y. Clearly $p_i^{-1}(\mathcal{G}_i) \supset \mathcal{G}$ for each $i \in I$. It is manifest that Y_i is $\mathcal{G}_i$-compact for each $i \in I$, hence by the above theorem Y is $\mathcal{G}$-compact, too. Now any finite intersection of elements of $\mathcal{F}$ is of the form $\bigcap_{i\in J} p_i^{-1}(X_i)$, where J is a finite subset of I. The latter finite intersection can be written as $\prod_{i\in J} X_i \times \prod_{i\in I\setminus J} Y_i$ and is contained in $\mathcal{G}$, by our hypothesis. This means $\mathcal{F}$ has the $\mathcal{G}$-F.I.P, hence by Proposition 1, $\bigcap \mathcal{F} \neq \varnothing$. But clearly $\prod_{i\in I} X_i = \bigcap_{i\in I} p_i^{-1}(X_i) = \bigcap \mathcal{F} \neq \varnothing$ and we are done.

□

Remark 2.4. Clearly the spaces Y_i for all $i \in I$ and Y are compact. Hence the above proof can also be considered as a simple proof of the the well-known fact that Tychonoff's Theorem implies the Axiom of Choice, see [4].

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