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ON THE DECOMPOSITION OF NONCOSINGULAR $\sum$-LIFTING MODULES

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Abstract. Let R be a right artinian ring or a perfect commutative ring. Let M be a noncosingular self-generator $\sum$-lifting module. Then M has a direct decomposition $M = \oplus_{i\in I} M_i$, where each M_i is noetherian quasi-projective and each endomorphism ring $End(M_i)$ is local.
Keywords: Noncosingular module, lifting module, $\sum$-lifting module.
MSC(2010): Primary: 16D10; Secondary: 16D80.

1. **Introduction**

Throughout this paper R will denote an associative ring with identity. Modules over R will be right R-modules. We will use the notation $N \ll M$ to indicate that N is small in M (i.e. $\forall L \lneq M, L + N \neq M$). $Rad(M)$ will denote the Jacobson radical of M. A non-zero module M is called *hollow* if every proper submodule of M is small in M. M is called *local* if the sum of all proper submodules of M is also a proper submodule of M. It is clear that every local module is hollow. A module M is called *lifting* if for every submodule $A \leq M$, there exists a direct summand B of M such that $B \leq A$ and $A/B \ll M/B$. M is said to be $\sum$*-lifting* if every direct sum of copies of M is lifting. Lifting modules are dual notions of extending modules and [4] deals with different aspects of lifting modules. A module M is amply supplemented and every coclosed submodule of M is a direct summand of M if and only if M is lifting by [4, 22.3(d)]. In [8] Talebi and Vanaja defined $\overline{Z}(M)$ as follows:

$$\overline{Z}(M) = \mathrm{Re}\,(\mathrm{M}, \mathcal{S}) = \bigcap \{\mathrm{Ker(g)} \mid \mathrm{g} \in \mathrm{Hom}\,(\mathrm{M}, \mathrm{L}), \mathrm{L} \in \mathcal{S}\},$$

where $\mathcal{S}$ denotes the class of all small modules. They called M a *cosingular* (*noncosingular*) module if $\overline{Z}(M) = 0$ ($\overline{Z}(M) = M$).

In this note, we study the decomposition of noncosingular ($\sum$-) lifting modules. Following [5], Er asked the following question:

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(∗) When does the lifting condition on $M^{(\mathbb{N})}$ imply the same on $M^{(I)}$ for arbitrary set I?

Er provides some answers to this question [5, Corollary 6 and Corollary 7]. In this paper we give another answer to this question (Proposition 3.3).

In Section 2, we prove the following proposition:

Let R be a right artinian ring or a perfect commutative ring. Let M be a noncosingular lifting module which has no relatively projection component. Then $M = \oplus_{i=1}^{n} M_i$, where each endomorphism ring $End(M_i)$ is local and the following statements satisfy:

(1) The decomposition complements direct summands.

(2) Every local summand of M is a summand.

(3) M has the exchange property.

(4) The radical factor ring $S/J(S)$ of the endomorphism ring S of M is von Neumann regular, and idempotents lift modulo $J(S)$.

In Section 3, as we stated in the abstract, we prove the following main theorem:

Let R be a right artinian ring or a perfect commutative ring. Let M be a noncosingular self-generator $\sum$-lifting module. Then M has a direct decomposition $M = \oplus_{i\in I} M_i$, where each M_i is noetherian quasi-projective and each endomorphism ring $End(M_i)$ is local.

A family $\{X_\lambda : \lambda \in \Lambda\}$ of submodules of a module M is called a *local summand* of M, if $\sum_{\lambda\in\Lambda} X_\lambda$ is direct and $\sum_{\lambda\in F} X_\lambda$ is a summand of M for every finite subset $F \subseteq \Lambda$. If even $\sum_{\lambda\in\Lambda} X_\lambda$ is a summand of M, we say that *the local summand is a summand*. A module M is said to have the *(finite) exchange property* if for any (finite) index set I, whenever $M \oplus N = \oplus_{i\in I} A_i$ for modules N and A_i, then $M \oplus N = M \oplus (\oplus_{i\in I} B_i)$ for submodules $B_i \leq A_i$. Let $M = \oplus_I M_i$ be a decomposition of the module M into nonzero summands M_i. This decomposition is said to *complement direct summands* if, whenever A is a direct summand of M, there is a subset J of I for which $M = (\oplus_J M_j) \oplus A$. The module M is called *quasi-discrete* if M is lifting and satisfies the following condition:

For every direct summands K and L of M with $M = K + L$, $K \cap L$ is a direct summand of M.

2. **Noncosingular lifting modules**

Lemma 2.1. [1, Lemma 2.2] *Let $M = \oplus_{i=1}^{\infty} M_i$, where each M_i is local noncosingular. If, for each i, there is an epimorphism $f_i : M_i \longrightarrow M_{i+1}$ which is non-isomorphism, then M is not lifting.*

Proposition 2.2. *Let R be an arbitrary ring and M a noncosingular local module. If M is not noetherian, then there exists a countable family $\{N_i \mid i \in \mathbb{N}\}$ of non-noetherian images of M such that $\oplus_{i\in\mathbb{N}} N_i$ is not lifting.*

Proof. Assume that $\oplus_{i\in\mathbb{N}}N_i$ is a lifting module for any countably family $\{N_i \mid i \in \mathbb{N}\}$ of non-noetherian images of M. We will prove that M is noetherian. Let $A_1 \subset A_2 \subset \cdots$ be a strictly ascending chain of submodules of M. Let $N_i = M/A_i$ and $f_i : N_i \to N_{i+1}$ be the obvious projections. Hence there is an infinite sequence of non-isomorphism epimorphisms

$$N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} \cdots \longrightarrow N_n \xrightarrow{f_n} \cdots .$$

By Lemma 2.1, $\oplus_{i\in\mathbb{N}}N_i$ is not a lifting module which is a contradiction. Thus M is noetherian. □

Recall that a family of modules $\{M_i \mid i \in I\}$ is called *(locally) semi-T-nilpotent* if, for any countable set of non-isomorphisms $\{f_n : M_{i_n} \to M_{i_{n+1}}\}_{\mathbb{N}}$ with all i_n distinct in I, (and for any $x \in M_{i_1}$), there exists $k \in \mathbb{N}$ (depending on x) such that $f_k...f_1 = 0$ ($f_k...f_1(x) = 0$). It is obvious that if each M_i is a local module, then the family $\{M_i \mid i \in I\}$ of modules is locally semi-T-nilpotent if and only if it is semi-T-nilpotent.

Proposition 2.3. *Let $M = \oplus_{i=1}^{\infty}M_i$ with M_i local noncosingular and M_j-projective whenever $j \neq i$. If M is a lifting module, then:*
(1) *$\{M_i\}$ is locally semi-T-nilpotent.*
(2) *M is quasi-discrete.*
(3) *$Rad(M) \ll M$.*
(4) *The decomposition $M = \oplus_{i=1}^{\infty}M_i$ complements summands.*

Proof. By [1, Corollary 2.1], $\{M_i\}$ is locally semi-T-nilpotent. By [7, Theorem 4.53], (2), (3) and (4) hold. □

Recall that a module M is said to be *Hopfian* if any epimorphism is an isomorphism.

Lemma 2.4. *Let R be a right artinian ring or a perfect commutative ring. Then every noncosingular hollow R-module M has a local endomorphism ring.*

Proof. Let M be a noncosingular hollow R-module. Assume that $\phi : M \to M$ is a nonzero endomorphism. Since M is noncosingular and hollow, ϕ is an epimorphism. Let R be right artinian. From the fact that hollow modules over artinian rings are noetherian and so Hopfian, $End(M)$ is local.

Now let R be a perfect commutative ring. Note that every hollow module over a perfect ring is local. Thus M is local and so is cyclic. As finitely generated modules over commutative rings are Hopfian, M is Hopfian. Thus $End(M)$ is local. □

A module M is said to have *finite hollow dimension* if there exists an epimorphism from M to a finite direct sum of n hollow factor modules with small kernel.

Theorem 2.5. [1, Theorem 2.1] *Let R be a right perfect ring. Let M be a noncosingular lifting module that does not have relatively projective component. Then M has finite hollow dimension.*

Proposition 2.6. *Let R be a right artinian ring or a perfect commutative ring. Let M be a noncosingular lifting module that does not have relatively projection component. Then $M = \oplus_{i=1}^{n} M_i$, where each endomorphism ring $End(M_i)$ is local and the following statements satisfy:*

(1) *The decomposition complements direct summands.*
(2) *Every local summand of M is a summand.*
(3) *M has the exchange property.*
(4) *The radical factor ring $S/J(S)$ of the endomorphism ring S of M is von Neumann regular, and idempotents lift modulo $J(S)$.*

Proof. By Theorem 2.5, there exist hollow submodules M_i $(i \in \{1, 2, \cdots, n\})$ such that $M = \oplus_{i=1}^{n} M_i$. By Lemma 2.4, $End(M_i)$ is local for all $i \in \{1, 2, \cdots, n\}$. Using [2, Corollary 12.7], this decomposition complements direct summands. By [7, Theorem 2.25], (2), (3) and (4) hold. □

Proposition 2.7. *Let M be a noetherian noncosingular lifting module. Then there exists a decomposition $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ where for each i, M_i is a noetherian hollow module with $End(M_i)$ a local ring.*

Proof. Since M is noetherian, it has a finite decomposition with indecomposable noetherian direct summands. Since every direct summand is hollow noncosingular, every non-zero homomorphism is an epimorphism. As every noetherian module is Hopfian, each noncosingular hollow direct summand has a local endomorphism ring. □

3. Noncosingular $\sum$-lifting modules

Lemma 3.1. [1, Lemma 2.3] *Let U and V be noncosingular hollow modules such that the module $U \oplus V$ is lifting. Then there exists an epimorphism from U to V or V is U-projective.*

Proposition 3.2. *Let M be a nonzero noncosingular $\sum$-lifting module. If M is a local module, then $End(M)$ is a division ring and M is quasi-projective.*

Proof. Let $\phi \in End(M)$. Since M is hollow and noncosingular, ϕ is an epimorphism. Suppose ϕ is not monomorphism. By [1, Corollary 2.1], the family $\{M_n\}_{n\in\mathbb{N}}$, where $M_n = M$ for all $n \in \mathbb{N}$, is semi-T-nilpotent. Consider $\phi_n = \phi : M_n \to M_{n+1}$, for all $n \in \mathbb{N}$. Since $\{M_n\}_{n\in\mathbb{N}}$ is semi-T-nilpotent, there exists a positive number k such that $\phi^k : M_1 = M \to M_k = M$ is a zero epimorphism, which is a contradiction. As $M \oplus M$ is lifting and M is not isomorphic to any nonzero image of M, M is quasi-projective (Lemma 3.1). □

The next proposition addresses Question (*).

Proposition 3.3. *Let R be a right artinian ring or a perfect commutative ring. If M is a noncosingular hollow R-module such that $M^{(\mathbb{N})}$ is a lifting module, then for any set I, $M^{(I)}$ is a lifting module with the exchange property.*

Proof. First note that M has a local endomorphism ring (Lemma 2.4). By [3, Theorem 2 and Lemma 3], any family of copies of M is locally semi-T-nilpotent and M is almost M-projective. Now, by [3, Theorem 2 and Lemma 3] again, we obtain that $M^{(I)}$ is a lifting module for any set I. Also, by locally semi-T-nilpotent property and [7, Theorem 2.25], $M^{(I)}$ has the exchange property for any set I. □

Theorem 3.4. *Let R be a right artinian ring or a perfect commutative ring. Let M be a noncosingular self-generator $\sum$-lifting module. Then M has a direct decomposition $M = \oplus_{i\in I} M_i$, where each M_i is noetherian quasi-projective and each endomorphism ring $End(M_i)$ is local.*

Proof. By [6, Theorem 2.14 and Corollary 2.6(ii)], there exist an index set I and hollow submodules M_i $(i \in I)$ such that $M = \oplus_{i\in I} M_i$. By Lemma 2.4, $End(M_i)$ is local for all $i \in I$.

Since $M_i \oplus M_i$ is lifting and M_i is not isomorphic to any nonzero image of M_i, so it follows that M_i is M_i-projective by Lemma 3.1. Now we show that M_i is noetherian for each $i \in I$. Let A be any submodule of M_i. Then, since M is self-generator, there exists an exact sequence

$$M^{(J)} \xrightarrow{f} A \longrightarrow 0$$

for some index set J. But $M^{(J)}/\mathrm{Ker} f \cong A$, hence A is noncosingular and so A is a coclosed submodule of M_i by [8, Lemma 2.3(2)]. Since M_i is lifting, A is a direct summand of M_i and so a direct summand of M. As $End(M_i)$ is local for all $i \in I$, we get by the Krull-Schmidt-Azumaya theorem ([2, Theorem 12.6]) that $A \cong M_j$ for some $j \in I$.

Now suppose that there exists a strictly ascending chain of submodules of M_i

$$A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots \subseteq M_i.$$

Then, by the above argument, each A_n is isomorphic to M_{i_n} for some $i_n \in I$. Hence the external direct sum $D = \oplus_{n=1}^{\infty} A_n$ is isomorphic to $\oplus_{n=1}^{\infty} M_n$ which is a direct summand of $M^{(\mathbb{N})}$. So D is a lifting module. Clearly there exists an infinite sequence of non-isomorphic epimorphisms

$$M_i/A_1 \xrightarrow{f_1} M_i/A_2 \xrightarrow{f_2} \cdots \longrightarrow M_i/A_n \xrightarrow{f_n} \cdots,$$

where f_n is the projection map on A_n. By Lemma 2.1, we get a contradiction which proves that M_i is noetherian for each $i \in I$. □

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