ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 42 (2016), No. 1, pp. 53–60

Title:

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Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 1, pp. 53–60
Online ISSN: 1735-8515

THE REVERSE ORDER LAW FOR MOORE-PENROSE INVERSES OF OPERATORS ON HILBERT C*-MODULES

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(Communicated by Behzad Djafari-Rouhani)

Abstract. Suppose T and S are bounded adjointable operators between Hilbert C*-modules admitting bounded Moore-Penrose inverse operators. Some necessary and sufficient conditions are given for the reverse order law $(TS)^\dagger = S^\dagger T^\dagger$ to hold. In particular, we show that the equality holds if and only if $Ran(T^*TS) \subseteq Ran(S)$ and $Ran(SS^*T^*) \subseteq Ran(T^*)$, which was studied first by Greville [*SIAM Rev. 8 (1966) 518–521*] for matrices.

Keywords: Bounded adjointable operator, Hilbert C*-module, Moore-Penrose inverse, reverse order law.
MSC(2010): Primary 47A05; Secondary 46L08, 15A09.

1. Introduction and preliminaries.

It is well-known that for invertible operators (or nonsingular matrices) T, S and TS, $(TS)^{-1} = S^{-1}T^{-1}$. However, this so-called reverse order law is not necessarily true for other kind of generalized inverses. An interesting problem is, for given operators (or matrices) TS with TS meaningful, under what conditions, $(TS)^\dagger = S^\dagger T^\dagger$? The problem first studied by Greville [7] and then reconsidered by Bouldin and Izumino [2, 9]. Many authors discussed the problem like this, see e.g. [3–5, 11, 13] and references therein. An special case, when $S = T^*$, was given by Moslehian et al. [14] for a Moore-Penrose invertible operator T on Hilbert C*-modules. The later paper and the work of [5, 7] motivated us to study the problem in the framework of Hilbert C*-modules.

The notion of a Hilbert C*-module is a generalization of the notion of a Hilbert space. However, some well known properties of Hilbert spaces like Pythagoras' equality, self-duality, and even decomposition into orthogonal complements do not hold in the framework of Hilbert modules. The first use of such

Article electronically published on February 22, 2016.
Received: 16 August 2012, Accepted: 26 October 2014.
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objects was made by I. Kaplansky [10] and then studied more in the work of W. L. Paschke [15]. Let us quickly recall the definition of a Hilbert C*-module.

Suppose that $\mathcal{A}$ is an arbitrary C*-algebra and E is a linear space which is a right $\mathcal{A}$-module and the scalar multiplication satisfies $\lambda(xa) = x(\lambda a) = (\lambda x)a$ for all $x \in E,\ a \in \mathcal{A}, \lambda \in \mathbb{C}$. The $\mathcal{A}$-module E is called a *pre-Hilbert $\mathcal{A}$-module* if there exists an $\mathcal{A}$-valued map $\langle .,. \rangle : E \times E \to \mathcal{A}$ with the following properties:

(i) $\langle x, y + \lambda z\rangle = \langle x, y\rangle + \lambda\langle x, z\rangle$; for all $x, y, z \in E, \lambda \in \mathbb{C}$,
(ii) $\langle x, ya\rangle = \langle x, y\rangle a$; for all $x, y \in E$ and $a \in A$,
(iii) $\langle x, y\rangle^* = \langle y, x\rangle$; for all $x, y \in E$,
(iv) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

The $\mathcal{A}$-module E is called a *Hilbert C*-module* if E is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$. For any pair of Hilbert C*-modules E_1 and E_2, we define $E_1 \oplus E_2 = \{(e_1, e_2)|\ e_1 \in E_1 \text{ and } e_2 \in E_2\}$ which is also a Hilbert C*-module whose $\mathcal{A}$-valued inner product is given by

$$\langle (x_1, y_1), (x_2, y_2)\rangle = \langle x_1, x_2\rangle + \langle y_1, y_2\rangle, \text{ for } x_1, x_2 \in E_1 \text{ and } y_1, y_2 \in E_2.$$

If V is a (possibly non-closed) $\mathcal{A}$-submodule of E, then $V^\perp := \{y \in E : \langle x, y\rangle = 0, \text{ for all } x \in V\}$ is a closed $\mathcal{A}$-submodule of E and $\overline{V} \subseteq V^{\perp\perp}$. A Hilbert $\mathcal{A}$-submodule V of a Hilbert $\mathcal{A}$-module E is orthogonally complemented if V and its orthogonal complement $V^\perp$ yield $E = V \oplus V^\perp$, in this case, V and its biorthogonal complement $V^{\perp\perp}$ coincide. For the basic theory of Hilbert C*-modules we refer to the book by E. C. Lance [12]. Note that every Hilbert space is a Hilbert $\mathbb{C}$-module and every C*-algebra $\mathcal{A}$, can be regarded as a Hilbert $\mathcal{A}$-module via $\langle a, b\rangle = a^*b$ when $a, b \in \mathcal{A}$.

Throughout this paper we assume that $\mathcal{A}$ is an arbitrary C*-algebra. We use $[\cdot,\cdot]$ for commutator of two elements. The notations $Ker(\cdot)$ and $Ran(\cdot)$ stand for kernel and range of operators, respectively. Suppose E and F are Hilbert $\mathcal{A}$-modules, $\mathcal{L}(E, F)$ denotes the set of all bounded adjointable operators from E to F, that is, all operator $T : E \to F$ for which there exists $T^* : F \to E$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$, for all $x \in E$ and $y \in F$.

Closed submodules of Hilbert modules need not to be orthogonally complemented at all, however we have the following well known results. Suppose T in $\mathcal{L}(E, F)$, the operator T has closed range if and only if T^* has. In this case, $E = Ker(T) \oplus Ran(T^*)$ and $F = Ker(T^*) \oplus Ran(T)$, cf. [12, Theorem 3.2]. In view of [16, Lemma 2.1], $Ran(T)$ is closed if and only if $Ran(TT^*)$ is, and in this case, $Ran(T) = Ran(TT^*)$.

Let $T \in \mathcal{L}(E, F)$. The Moore–Penrose inverse $T^\dagger$ of T (if it exists) is an element $X \in \mathcal{L}(F, E)$ which satisfies

(1) $TXT = T$,
(2) $XTX = X$,
(3) $(TX)^* = TX$,
(4) $(XT)^* = XT$.

If $\theta \subseteq \{1,2,3,4\}$, and X satisfies the equation (i) for all $i \in \theta$, then X is an θ-inverse of T. The set of all θ-inverses of T is denoted by $T\{\theta\}$. In particular, $T\{1,2,3,4\} = \{T^\dagger\}$. The properties (1) to (4) imply that $T^\dagger$ is unique and $T^\dagger T$ and $TT^\dagger$ are orthogonal projections. Moreover, $Ran(T^\dagger) = Ran(T^\dagger T)$, $Ran(T) = Ran(TT^\dagger)$, $Ker(T) = Ker(T^\dagger T)$ and $Ker(T^\dagger) = Ker(TT^\dagger)$ which lead us to $E = Ker(T^\dagger T) \oplus Ran(T^\dagger T) = Ker(T) \oplus Ran(T^\dagger)$ and $F = Ker(T^\dagger) \oplus Ran(T)$. We also have $Ran(T^\dagger) = Ran(T^*)$ and $Ker(T^\dagger) = Ker(T^*)$.

Xu and Sheng in [19] have shown that a bounded adjointable operator between two Hilbert C*-modules admits a bounded Moore-Penrose inverse if and only if the operator has closed range. The reader should be aware of the fact that a bounded adjointable operator may admit an unbounded operator as its Moore-Penrose, see [6,8,16,18] for more detailed information.

It is a classical result of Greville [7], that $(TS)^\dagger = S^\dagger T^\dagger$ if and only if $T^\dagger TSS^*T^* = SS^*T^*$ and $SS^\dagger T^*TS = T^*TS$ (or equivalently, $Ran(SS^*T^*) \subseteq Ran(T^*)$ and $Ran(T^*TS) \subseteq Ran(S)$) for Moore-Penrose invertible matrices T and S. The present paper is an extension of some results of [5,7,14] to Hilbert C*-modules settings. Indeed, we give some necessary and sufficient conditions for reverse order law for the Moore–Penrose inverse by using the matrix form of bounded adjointable module maps. These enable us to derive Greville's result for bounded adjointable module maps.

The matrix form of a bounded adjointable operator $T \in \mathcal{L}(E,F)$ is induced by some natural decompositions of Hilbert C*-modules. If $F = M \oplus M^\perp, E = K \oplus K^\perp$ then T can be written as the following 2×2 matrix

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \tag{1.1}$$

with operator entries, $T_1 \in \mathcal{L}(K,M), T_2 \in \mathcal{L}(K^\perp, M), T_3 \in \mathcal{L}(K, M^\perp)$ and $T_4 \in \mathcal{L}(K^\perp, M^\perp)$.

Lemma 1.1. *Let $T \in \mathcal{L}(E,F)$ have a closed range. Then T has the following matrix decomposition with respect to the orthogonal decompositions of submodules $E = Ran(T^*) \oplus Ker(T)$ and $F = Ran(T) \oplus Ker(T^*)$:*

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(T^*) \\ Ker(T) \end{bmatrix} \to \begin{bmatrix} Ran(T) \\ Ker(T^*) \end{bmatrix},$$

where T_1 is invertible. Moreover,

$$T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(T) \\ Ker(T^*) \end{bmatrix} \to \begin{bmatrix} Ran(T^*) \\ Ker(T) \end{bmatrix}.$$

Proof. The operator T and its adjoint T^* have the following representations:

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} : \begin{bmatrix} Ran(T^*) \\ Ker(T) \end{bmatrix} \to \begin{bmatrix} Ran(T) \\ Ker(T^*) \end{bmatrix},$$

$$T^* = \begin{bmatrix} T_1^* & T_3^* \\ T_2^* & T_4^* \end{bmatrix} : \begin{bmatrix} Ran(T) \\ Ker(T^*) \end{bmatrix} \to \begin{bmatrix} Ran(T^*) \\ Ker(T) \end{bmatrix}.$$

From $T^*(Ker(T^*)) = \{0\}$ we obtain $T_3^* = 0$ and $T_4^* = 0$, thus $T_3 = 0$ and $T_4 = 0$. Since $T(Ker(T)) = \{0\}$, $T_2 = 0$ therefore $T = \left[\begin{smallmatrix} T_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

Moreover since $Ran(T)$ is closed, T_1 possesses a bounded adjointable inverse from $Ran(T)$ onto $Ran(T^*)$. Now, it is easy to check that the matrix $\left[\begin{smallmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is the Moore–Penrose inverse of $T = \left[\begin{smallmatrix} T_1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. □

Lemma 1.2. *Let $T \in \mathcal{L}(E,F)$ have a closed range. Let E_1, E_2 be closed submodules of E and F_1, F_2 be closed submodules of F such that $E = E_1 \oplus E_2$ and $F = F_1 \oplus F_2$. Then the operator T has the following matrix representations with respect to the orthogonal sums of submodules $E = Ran(T^*) \oplus Ker(T)$ and $F = Ran(T) \oplus Ker(T^*)$:*

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \to \begin{bmatrix} Ran(T) \\ Ker(T^*) \end{bmatrix}, \tag{1.2}$$

where $D = T_1T_1^* + T_2T_2^* \in \mathcal{L}(Ran(T))$ is positive and invertible. Moreover,

$$T^\dagger = \begin{bmatrix} T_1^*D^{-1} & 0 \\ T_2^*D^{-1} & 0 \end{bmatrix}. \tag{1.3}$$

$$T = \begin{bmatrix} T_1 & 0 \\ T_2 & 0 \end{bmatrix} : \begin{bmatrix} Ran(T^*) \\ Ker(T) \end{bmatrix} \to \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \tag{1.4}$$

where $\mathfrak{D} = T_1^*T_1 + T_2^*T_2 \in \mathcal{L}(Ran(T^*))$ is positive and invertible. Moreover,

$$T^\dagger = \begin{bmatrix} \mathfrak{D}^{-1}T_1^* & \mathfrak{D}^{-1}T_2^* \\ 0 & 0 \end{bmatrix}. \tag{1.5}$$

Proof. We prove only the matrix representations (1.2) and (1.3), the proof of (1.4) and (1.5) are analogous. The operator T has the following representation:

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} : \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \to \begin{bmatrix} Ran(T) \\ Ker(T^*) \end{bmatrix},$$

which yields

$$T^* = \begin{bmatrix} T_1^* & T_3^* \\ T_2^* & T_4^* \end{bmatrix} : \begin{bmatrix} Ran(T) \\ Ker(T^*) \end{bmatrix} \to \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}.$$

From $T^*(Ker(T^*)) = \{0\}$ we obtain $T_3^* = 0$ and $T_4^* = 0$. Then $T_3 = 0$ and $T_4 = 0$ which yield the matrix form (1.2) of T. Consequently, the adjoint operator T^* has the matrix representation

$$T^* = \begin{bmatrix} T_1^* & 0 \\ T_2^* & 0 \end{bmatrix} : \begin{bmatrix} Ran(T) \\ Ker(T^*) \end{bmatrix} \to \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}.$$

We therefore have

$$T\,T^* = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(T) \\ Ker(T^*) \end{bmatrix} \to \begin{bmatrix} Ran(T) \\ Ker(T^*) \end{bmatrix}, \tag{1.6}$$

where $D = T_1T_1^* + T_2T_2^* : Ran(T) \to Ran(T)$. From $Ker(TT^*) = Ker(T^*)$, it follows that D is injective. From $Ran(TT^*) = Ran(T)$ it follows that D is surjective. Hence, D is invertible. Now using [14, Corollary 2.4] and (1.6) we obtain

$$T^\dagger = T^*(TT^*)^\dagger = \begin{bmatrix} T_1^* & 0 \\ T_2^* & 0 \end{bmatrix} \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1^*D^{-1} & 0 \\ T_2^*D^{-1} & 0 \end{bmatrix}.$$

□

2. The reverse order law

We begin this section with the following useful facts about the product of module maps with closed range. Suppose E, F and G are Hilbert C*-modules and $S \in \mathcal{L}(E,F)$ and $T \in \mathcal{L}(F,G)$ are bounded adjointable operators with closed ranges. Then TS has closed range if and only if $T^\dagger TSS^\dagger$ has, if and only if $Ker(T) + Ran(S)$ is an orthogonal summand in F, if an only if $Ker(S^*) + Ran(T^*)$ is an orthogonal summand in F. For the proofs of the results and historical notes about the problem we refer to [17] and references therein.

Theorem 2.1. *Suppose E, F and G are Hilbert C^*-modules and $S \in \mathcal{L}(E,F)$, $T \in \mathcal{L}(F,G)$ and $TS \in \mathcal{L}(E,G)$ have closed ranges. Then following statements are equivalent:*

(i) $TS(TS)^\dagger = TSS^\dagger T^\dagger$,
(ii) $T^*TS = SS^\dagger T^*TS$,
(iii) $S^\dagger T^\dagger \in (TS)\{1,2,3\}$.

Proof. Using Lemma 1.1, the operator S and its Moore-Penrose inverse $S^\dagger$ have the following matrix forms:

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} \quad : \quad \begin{bmatrix} Ran(S^*) \\ Ker(S) \end{bmatrix} \to \begin{bmatrix} Ran(S) \\ Ker(S^*) \end{bmatrix},$$

$$S^\dagger = \begin{bmatrix} S_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \quad : \quad \begin{bmatrix} Ran(S) \\ Ker(S^*) \end{bmatrix} \to \begin{bmatrix} Ran(S^*) \\ Ker(S) \end{bmatrix}.$$

From Lemma 1.2 it follows that the operator T and $T^\dagger$ have the following matrix forms:

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(S) \\ Ker(S^*) \end{bmatrix} \to \begin{bmatrix} Ran(T) \\ Ker(T^*) \end{bmatrix},$$

$$T^\dagger = \begin{bmatrix} T_1^*D^{-1} & 0 \\ T_2^*D^{-1} & 0 \end{bmatrix},$$

where $D = T_1T_1^* + T_2T_2^*$ is invertible and positive in $\mathcal{L}(Ran(T))$. Then we have the following products

$$TS = \begin{bmatrix} T_1S_1 & 0 \\ 0 & 0 \end{bmatrix},\ (TS)^\dagger = \begin{bmatrix} (T_1S_1)^\dagger & 0 \\ 0 & 0 \end{bmatrix},\ S^\dagger T^\dagger = \begin{bmatrix} S_1^{-1}T_1^*D^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

It is easy to check that the following three expressions in terms of T_1, T_2 and S_1 are equivalent to our statements.

(1) $T_1S_1(T_1S_1)^\dagger = T_1T_1^*D^{-1}$, which is equivalent to (i).
(2) $T_2^*T_1 = 0$, which is equivalent to (ii).
(3) $T_1T_1^*D^{-1}T_1 = T_1$ and $[T_1T_1^*, D^{-1}] = 0$, which are equivalent to (iii).

Note that $[T_1T_1^*, D^{-1}] = 0$, since $T_1S_1(T_1S_1)^\dagger$ is self-adjoint. We show that (3) $\Rightarrow$ (2) $\Leftrightarrow$ (1) $\Rightarrow$ (3).

To prove (1) $\Leftrightarrow$ (2), we observe that $T_1S_1(T_1S_1)^\dagger = T_1T_1^*D^{-1}$ if and only if $(T_1S_1)^\dagger = (T_1S_1)^\dagger T_1T_1^*D^{-1}$. The last statement is obtained by multiplying the first expression by $(T_1S_1)^\dagger$ from the left side, or multiplying the second expression by T_1S_1 from the left side, and using $T_1T_1^* = T_1S_1S_1^{-1}T_1^*$. We therefore have

$$\begin{aligned}
(T_1S_1)^\dagger = (T_1S_1)^\dagger T_1T_1^*D^{-1} \quad &\Leftrightarrow \quad (T_1S_1)^\dagger(T_1T_1^* + T_2T_2^*) = (T_1S_1)^\dagger T_1T_1^* \\
&\Leftrightarrow \quad (T_1S_1)^\dagger T_2T_2^* = 0 \\
&\Leftrightarrow \quad Ran(T_2T_2^*) \subseteq Ker((T_1S_1)^\dagger) = Ker((T_1S_1)^*) \\
&\Leftrightarrow \quad S_1^*T_1^*T_2T_2^* = 0 \;\Leftrightarrow\; T_2T_2^*T_1 = 0 \\
&\Leftrightarrow \quad Ran(T_1) \subseteq Ker(T_2T_2^*) = Ker(T_2^*) \\
&\Leftrightarrow \quad T_2^*T_1 = 0.
\end{aligned}$$

To demonstrate (1) $\Rightarrow$ (3), we multiply $T_1S_1(T_1S_1)^\dagger = T_1T_1^*D^{-1}$ by T_1S_1 from the right side, we find $T_1T_1^*D^{-1}T_1 = T_1$, i.e. (3) holds.

Finally, we prove (3) $\Rightarrow$ (2). If $T_1T_1^*D^{-1}T_1 = T_1$ and $[T_1T_1^*, D^{-1}] = 0$, then $T_1T_1^*T_1 = DT_1 = T_1T_1^*T_1 + T_2T_2^*T_1$. Consequently, $T_2T_2^*T_1 = 0$ which implies $T_2T_1^* = 0$, since $Ran(T_1) \subseteq Ker(T_2T_2^*) = Ker(T_2^*)$. $\square$

Theorem 2.2. *Suppose E, F and G are Hilbert C^*-modules and $S \in \mathcal{L}(E,F)$, $T \in \mathcal{L}(F,G)$ and $TS \in \mathcal{L}(E,G)$ have closed ranges. Then following statements are equivalent:*

(i) $(TS)^\dagger TS = S^\dagger T^\dagger TS$,
(ii) $TSS^* = TSS^*T^\dagger T$,
(iii) $S^\dagger T^\dagger \in (TS)\{1,2,4\}$.

Proof. The operators T, S and TS and their Moore-Penrose inverses have the same matrix representations as in the previous theorem. To prove the assertions, we first find the equivalent expressions for our statements in terms of T_1, T_2 and S_1.

(1) $(T_1S_1)^\dagger T_1S_1 = S_1^{-1}T_1^*D^{-1}T_1S_1$, which is equivalent to (i).
(2) $T_1S_1S_1^*T_1^*D^{-1}T_1 = T_1S_1S_1^*$ and $T_1S_1S_1^*T_1^*D^{-1}T_2 = 0$, which are equivalent to (ii).
(3) $T_1T_1^*D^{-1}T_1 = T_1$ and $[S_1S_1^*, T_1^*D^{-1}T_1] = 0$, which are equivalent to (iii).

Note that $[S_1S_1^*, T_1^*D^{-1}T_1] = 0$, since $(T_1S_1)^\dagger T_1S_1$ is self-adjoint. We show that $(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$.

Suppose (1) holds. If we multiply $(T_1S_1)^\dagger T_1S_1 = S_1^{-1}T_1^*D^{-1}T_1S_1$ by T_1S_1 from the left side, we obtain $T_1 = T_1T_1^*D^{-1}T_1$. Furthermore, $[S_1S_1^*, T_1^*D^{-1}T_1] = 0$, i.e. (3) holds.

Suppose (3) holds. Obviously, $T_1S_1S_1^*T_1^*D^{-1}T_1 = T_1T_1^*D^{-1}T_1S_1S_1^* = T_1S_1S_1^*$, that is, the first equality of (2) holds. According to the fact that $(T_1T_1^* + T_2T_2^*)D^{-1}T_1 = T_1$ and the assumption $T_1T_1^*D^{-1}T_1 = T_1$, we have $T_2^*D^{-1}T_1 = 0$. Consequently,

$$Ran(D^{-1}T_1) \subseteq Ker(T_2T_2^*) = Ker(T_2^*),$$

which yields $T_2^*D^{-1}T_1 = 0$. Therefore, $T_1^*D^{-1}T_2 = 0$ which establishes the second equality of (2).

In order to prove $(2) \Rightarrow (1)$, we multiply $T_1S_1S_1^*T_1^*D^{-1}T_1 = T_1S_1S_1^*$ by $(T_1S_1)^\dagger$ from the left side. In view of $[S_1S_1^*, T_1^*D^{-1}T_1] = 0$, we find

$$\begin{aligned} S_1^*T_1^*D^{-1}T_1 = (T_1S_1)^\dagger T_1S_1S_1^* \quad &\Rightarrow \quad (T_1S_1)^\dagger T_1S_1 = S_1^*T_1^*D^{-1}T_1(S_1^*)^{-1} \\ &\Leftrightarrow \quad (T_1S_1)^\dagger T_1S_1 = S_1^{-1}T_1^*D^{-1}T_1S_1. \end{aligned}$$

□

Now we are ready to derive Greville's result, which also gives an answer to a problem of [17] about the reverse order law for Moore-Penrose inverses of modular operators. The operators $SS^\dagger$ and $T^\dagger T$ are orthogonal projections onto $Ran(S)$ and $Ran(T^\dagger) = Ran(T^*)$, respectively. These facts together with Theorems 2.1 and 2.2 lead us to the following result.

Corollary 2.3. *Suppose E, F and G are Hilbert C^*-modules and $S \in \mathcal{L}(E,F)$, $T \in \mathcal{L}(F,G)$ and $TS \in \mathcal{L}(E,G)$ have closed ranges. Then following statements are equivalent:*

(i) $(TS)^\dagger = S^\dagger T^\dagger$,
(ii) $TS(TS)^\dagger = TSS^\dagger T^\dagger$ *and* $(TS)^\dagger TS = S^\dagger T^\dagger TS$,
(iii) $SS^\dagger T^*TS = T^*TS$ *and* $TSS^*T^\dagger T = TSS^*$,
(iv) $Ran(T^*TS) \subseteq Ran(S)$ *and* $Ran(SS^*T^*) \subseteq Ran(T^*)$.

Acknowledgments

This research was done while the first author stayed at Mathematisches Institut of the Westfälische Wilhelms-Universität in Münster, Germany. He would like to express his thanks to Professor Wend Werner and his colleagues in functional analysis and operator algebras group for warm hospitality and great scientific atmosphere. The first author gratefully acknowledges financial support from the Alexander von Humboldt Foundation. This research was partially supported by a grant from Shahrood University of Technology. The authors are grateful to the referee for his/her careful reading and useful comments.

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