Title:
Frames in right ideals of $C^*$-algebras

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Abstract. We investigate the problem of the existence of a frame for right ideals of a $C^*$-algebra, without using the Kasparov stabilization theorem.

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1. Introduction

Frank and Larsen generalized the notion of a frame in Hilbert spaces to Hilbert $C^*$-modules [7]. They showed, using the Kasparov stabilization theorem [8], that every finitely or countably generated Hilbert $C^*$-module has a standard frame.

The characterization problem of those $C^*$-algebras $A$ for which all Hilbert $A$-modules have a standard frame is open until now [7]. In 2011, Li solved the problem for commutative unital $C^*$-algebras [10, Theorem 1.1]. In fact, Li shows that for a commutative unital $C^*$-algebra $A$, every Hilbert $A$-module has a frame if and only if $A$ is finite dimensional.

On the other hand, a $C^*$-algebra $A$ is a $C^*$-algebra of compact operators if and only if every Hilbert $A$-module has a basis [2,3]. Also, it is well known that each unital $C^*$-algebra of compact operators is finite dimensional and a commutative $C^*$-algebra $A = C_0(Z)$ is a $C^*$-algebra of compact operators exactly when $Z$ is discrete.

Hence, as mentioned in [1], a non-unital version of Li’s theorem can be obtained as follows.

Theorem 1.1. Let $A$ be a commutative $C^*$-algebra. Then $A$ is a $C^*$-algebra of compact operators if and only if every Hilbert $A$-module has a frame.

Therefore, for general case, the following conjecture arises [1].
Conjecture 1.2. If every Hilbert $C^*$-module over a $C^*$-algebra $A$ has a frame, then $A$ is a $C^*$-algebra of compact operators.

In [1], it is shown that the above conjecture has an affirmative for certain classes of $C^*$-algebras.

In this note, we investigate the problem of the existence of a frame for right ideals of a $C^*$-algebra $A$, without using the Kasparov stabilization theorem. We show that this property cannot characterize $A$ as a $C^*$-algebra of compact operators.

2. Frames and Ideals

Let $A$ be a $C^*$-algebra and $E$ be a Hilbert $A$-module. A family $\{x_i\}_{i \in I}$ of elements in $E$ is called a frame if there are real constants $C, D > 0$ such that $\sum_{i \in I} \langle x, x_i \rangle_A \langle x_i, x \rangle_A$ converges, in the ultraweak topology of the universal enveloping von Neumann algebra, to some element in $A^*$ and

$$C \langle x, x \rangle_A \leq \sum_{i \in I} \langle x, x_i \rangle_A \langle x_i, x \rangle_A \leq D \langle x, x \rangle_A$$

for every $x \in E$. A frame is said to be standard if the sum in the middle of the above inequality converges in norm for every $x \in E$, and is said to be normalized if $C = D = 1$.

There are some results in the literature on the characterization of a $C^*$-algebra of compact operators by certain properties of its (right) ideals. For instance, Magajna in [11] showed that if $A$ is a $C^*$-algebra and there exists a full Hilbert $A$-module $E$ such that each closed submodule of $E$ is orthogonally complemented, then $A$ is a $C^*$-algebra of compact operators. Schweizer in [14] remarked that this problem on Hilbert $A$-submodules of $E$ can be reformulated as a problem on right ideals of $A$ and consequently the result can be obtained easily.

Therefore, one may expect that the problem of the existence of a frame for each Hilbert $A$-module can be reformulated as the problem of the existence of a frame for each right ideal of $A$. Hereinafter, by ideal we mean closed ideals.

Definition 2.1. We say that a right Hilbert $C^*$-module $E$ over a $C^*$-algebra $A$ is countably generated if there is a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $E$ such that the $A$-linear hull of $\{x_n : n \in \mathbb{N}\}$ is norm-dense in $E$.

Note that our definition of being countably generated really means “topologically countably generated” and this differs from being algebraically countably generated. Surprisingly, it is shown in [5] that if every closed right ideal of a Banach algebra $A$ is algebraically countably generated, then $A$ is finite dimensional. Recently, Blecher and Kania gave a characterization of Hilbert $C^*$-modules which are algebraically (countably) finitely generated [4].
Lemma 2.2. Let $H$ be a Hilbert space. Then $K(H)$ is countably generated, as a $K(H)$-module, if and only if $H$ is separable.

Proof. Let $H$ be a separable Hilbert space with a fixed orthonormal basis $\{e_n\}_{n=1}^{\infty}$. We have $T = \sum_{n=1}^{\infty} P_n T$, for all $T \in K(H)$, where $P_n$ is the orthogonal projection to the one-dimensional subspace spanned by $e_n$. Therefore $\{P_n\}_{n=1}^{\infty}$ is a countable set of generators for $K(H)$.

Conversely, let $\{T_n\}_{n=1}^{\infty}$ be a countable set of generators for $K(H)$. Then $H$ is equal to the closed linear span of $\cup_{n \in \mathbb{N}} R(T_n)$, where $R(T_n)$ is the range of $T_n$. Also, it is well known that the range of each compact operator is separable. Therefore, $H$ is separable. \qed

Note that, in the above lemma, $H$ is separable if and only if the $C^*$-algebra $K(H)$ is separable. For a general $C^*$-algebra $A$, if it is topologically countably generated as an $A$-module, one cannot conclude that $A$ is separable. Instead, we have the following characterization.

Proposition 2.3. For a $C^*$-algebra $A$, the following statements are equivalent:

(i) $A$ is $\sigma$-unital;
(ii) $A$ has a strictly positive element;
(iii) $A$ has a countable standard normalized frame;
(iv) $A$ is countably generated as an $A$-module.

Proof. (i) $\Leftrightarrow$ (ii): This is a well-known fact in the $C^*$-algebra literature [13].

(ii) $\Rightarrow$ (iii): Let $h \in A$ be a strictly positive element. We set $v_0 = 0$, $v_n = h(h + \frac{1}{n})^{-1}$ and $u_n = (v_n - v_{n-1})^{\frac{1}{2}}$ for each $n \in \mathbb{N}$. As mentioned in [13], the sequence $\{v_n\}_{n=1}^{\infty}$ is a countable approximate unit for $A$. Then, for every $a \in A$, we have $a = \lim_n v_n a = \lim_n \sum_{j=1}^{n} (u_j)^2 a = \sum_{n=1}^{\infty} u_n (u_n, a)$. Hence, $\{u_n\}_{n=1}^{\infty}$ is a countable standard normalized frame for $A$.

(iii) $\Rightarrow$ (iv): Obviously, if $\{u_n\}_{n=1}^{\infty}$ is a standard normalized frame for $A$, then $\{u_n\}_{n=1}^{\infty}$ is a countable set of generators for $A$, by the reconstruction formula.

(iv) $\Rightarrow$ (ii): Let $\{u_n\}_{n=1}^{\infty}$ be a bounded set of generators for $A$, then $p = \sum_{n=1}^{\infty} \frac{1}{n} u_n u_n^*$ is a strictly positive element. In fact, if $\varphi$ is a positive functional on $A$ such that $\varphi(p) = 0$, then $\varphi(u_n u_n^*) = 0$, for all $n$. It follows that $\varphi(u_n b_n) = 0$, for all $n \in \mathbb{N}$ and $b_n \in A$. Therefore, $\varphi(a) = 0$ for each $a \in A$, i.e., $\varphi \equiv 0$. \qed

We recall that if $B$ is a hereditary $C^*$-subalgebra of $A$, then there is a unique right ideal $L$ such that $B = L \cap L^*$ [13, Theorem 3.2.1]. Similar to the proof of (ii) $\Rightarrow$ (iii) in the above proposition, one can show that if $B$ has a strictly positive element, then $L$, as a Hilbert $A$-module, has a countable standard normalized frame.

Corollary 2.4. For a $C^*$-algebra $A$, the following statements are equivalent:
(i) \( A \) is completely \( \sigma \)-unital, i.e., every hereditary \( C^* \)-subalgebra of \( A \) is \( \sigma \)-unital;
(ii) every hereditary \( C^* \)-subalgebra of \( A \) has a strictly positive element;
(iii) every right ideal \( I \) of \( A \) is countably generated as an \( A \)-module;
(iv) every right ideal \( I \) of \( A \) has a countable standard normalized frame.

If \( Z \) is a locally compact Hausdorff space, then the \( C^* \)-algebra \( C_0(Z) \) is separable if and only if \( Z \) is \( \sigma \)-compact and metrizable, if and only if \( Z \) is second countable. Also, the \( C^* \)-algebra \( C_0(Z) \) is \( \sigma \)-unital if and only if \( Z \) is \( \sigma \)-compact.

We recall that if a locally compact Hausdorff space \( Z \) is \( \sigma \)-compact, then \( Z \) is paracompact. Also, whenever a locally compact Hausdorff space \( Z \) is paracompact (or \( \sigma \)-compact), for any open cover \( U \) of \( Z \), there exists a continuous partition of unity subordinated to \( U \). In fact, there exists a partition of unity \( \{f_j\}_{j \in J} \) (or \( \{f_n\}_{n \in \mathbb{N}} \)) in \( C_c(Z) \).

It is well known that ideals of \( A = C_0(Z) \) correspond bijectively to closed sets of \( Z \). More precisely, \( I \) is an ideal of \( A \) if and only if there is a closed set \( F \subseteq Z \) such that

\[
I = \{ f \in C_0(Z) : f(z) = 0 \text{ for all } z \in F \}.
\]

The following proposition can be derived easily from Proposition 2.3, however we supply a direct proof of it.

**Proposition 2.5.** Let \( Z \) be a locally compact Hausdorff space and let \( A = C_0(Z) \). Then the following statements are equivalent:

(i) \( A \) is completely \( \sigma \)-unital;
(ii) \( Z \) is hereditary \( \sigma \)-compact, i.e., every open subset of \( Z \) is \( \sigma \)-compact;
(iii) every ideal \( I \) of \( A \) has a countable standard normalized frame.

**Proof.** (i) \( \iff \) (ii): Since \( A \) is commutative, hereditary \( C^* \)-subalgebras are exactly ideals of \( A \). Also, if \( F \) is a closed subset of \( Z \) and \( I_F = \{ f \in C_0(Z) : f(z) = 0 \text{ for all } z \in F \} \), then it is easy to see that \( I_F \) is \( \sigma \)-unital if and only if \( F^c \) is \( \sigma \)-compact.

(ii) \( \implies \) (iii): Let \( I \) be an ideal of \( A = C_0(Z) \) and \( F \) be a closed subset of \( Z \) such that \( I = I_F \). By assumption, \( F^c \) is \( \sigma \)-compact (and so paracompact), thus there exists a partition of unity of \( F^c \) as \( \{f_n\}_{n \in \mathbb{N}} \) in \( C_c(F^c) \). Since for each \( f_n \), \( \text{Supp}(f_n) = \{ z \in F^c : f_n(z) \neq 0 \} \) is compact and \( \text{Supp}(f_n) \cap F = \emptyset \), if one extends each \( f_n \) on \( Z \) by setting zero on \( F \), then \( f_n \in I_F \), for all \( n \). It is easy to see that, \( \{ f_n^2 \}_{n \in \mathbb{N}} \) is a standard normalized frame for \( I_F = I \).

(iii) \( \implies \) (ii): Let \( F \) be a closed subset of \( Z \) and the sequence \( \{f_n\}_{n \in \mathbb{N}} \) be a standard normalized frame for \( I_F \). Then we have

\[
|f(z)|^2 = \sum_{n=1}^{\infty} |f(z)|^2 |f_n(z)|^2,
\]
for all \( f \in I_F \) and \( z \in Z \). On the other hand, for each \( z \in F^c \) there is some \( f \in I_F \) such that \( f(z) = 1 \). Then \( 1 = \sum_{n=1}^{\infty} |f_n(z)|^2 \), for all \( z \in F^c \). Now, we have \( F^c = \bigcup_{m,n=1}^{\infty} K_m(f_n) \), where \( K_m(f_n) = \{ z \in Z : |f_n(z)|^2 \geq \frac{1}{m} \} \), for all \( m,n \in \mathbb{N} \). Therefore \( F^c \) is \( \sigma \)-compact, because \( K_m(f_n) \) is compact for all \( m,n \).

\begin{proposition}
Let \( Z \) be a locally compact metrizable space and \( A = C_0(Z) \). Then the following statements are equivalent:
\begin{enumerate}
  \item \( A \) is separable;
  \item \( A \) is completely \( \sigma \)-unital;
  \item every ideal \( I \) of \( A \) has a countable standard normalized frame.
\end{enumerate}
\end{proposition}

\begin{proof}
(i) \( \Rightarrow \) (ii): Let \( I \) be an ideal of \( A = C_0(Z) \) and \( F \) be a closed subset of \( Z \) such that \( I = I_F \). Also, let \( \{f_j\}_{j \in I} \) be a (standard normalized) frame for \( I_F \). Then we have \( |f(z)|^2 = \sum_{j \in J} |f(z)|^2 |f_j(z)|^2 \), for all \( f \in I_F \) and \( z \in Z \).

By assumption, there is a countable subset \( W \) of \( F^c \), such that \( W \supseteq F^c \). By Urysohn’s Lemma for locally compact Hausdorff spaces [6], for every \( z \in F^c \) there is an \( f \in I_F \) such that \( f(z) = 1 \) which implies \( \sum_{j \in J} |f_j(z)|^2 = 1 \) for all \( z \in F^c \). In particular, for each \( z \in W \) the set \( J_z = \{ j \in J : f_j(z) \neq 0 \} \) is countable. If \( J_W = \bigcup_{z \in W} J_z \), then \( J_W \) is countable and we have \( f_j(z) = 0 \), for all \( j \in J \setminus J_W \) and \( z \in F^c \), because every \( f_j \) is continuous and \( W \supseteq F^c \).

Therefore, we have
\[ |f(z)|^2 = \sum_{j \in J_W} |f(z)|^2 |f_j(z)|^2, \]
for all \( f \in I_F \) and \( z \in Z \). This means that \( \{f_j\}_{j \in J_W} \) is a countable standard normalized frame for \( I \).

(ii) \( \Rightarrow \) (i): This is evident.
\end{proof}

Proposition 2.7 can be used to derive the following standard fact from Topology:

\begin{proposition}
Let \( Z \) be a separable locally compact Hausdorff space. Then \( Z \) is paracompact if and only if it is \( \sigma \)-compact.
\end{proposition}

Similarly, we can obtain the following result.

\begin{proposition}
Let \( Z \) be a locally compact Hausdorff space and \( A = C_0(Z) \). Then every ideal \( I \) of \( A \) has a standard normalized frame exactly when every open subset of \( Z \) is paracompact.
\end{proposition}
Since every metric space is hereditary paracompact, we also have the following result.

**Corollary 2.10.** If a locally compact Hausdorff space $Z$ is metrizable, then every ideal of the $C^*$-algebra $A = C_0(Z)$ has a standard normalized frame.

As seen in the above results, for a $C^*$-algebra $A$, the fact that “every right ideal of $A$ has a (countable) standard normalized frame” cannot characterize $A$ as a $C^*$-algebra of compact operators. In fact, if every Hilbert $C^*$-module over $A$ has a (countable) standard frame, then every right ideal of $A$ has a (countable) standard frame, but the converse might not hold.

Finally, we remark that in the category of $C^*$-algebras, being separable is strictly stronger than being completely $\sigma$-unital. For instance, according to a classical example, due to Alexandroff and Urysohn, the double arrow space is a compact Hausdorff and perfectly normal space [15]. The latter implies that all open subsets of the double arrow space are $\sigma$-compact, while this space is not second countable and thus it is not metrizable. Therefore, if $Z$ is the double arrow space, then $C(Z)$ is completely $\sigma$-unital, while it is not separable.

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**References**


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