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FRAMES IN RIGHT IDEALS OF C^*-ALGEBRAS

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Abstract. We investigate the problem of the existence of a frame for right ideals of a C^*-algebra, without using the Kasparov stabilization theorem.

Keywords: Hilbert C^*-modules, frames, C^*-algebras.
MSC(2010): Primary: 46L08; Secondary: 42C15, 46L05.

1. Introduction

Frank and Larsen generalized the notion of a frame in Hilbert spaces to Hilbert C^*-modules [7]. They showed, using the Kasparov stabilization theorem [8], that every finitely or countably generated Hilbert C^*-module has a standard frame.

The characterization problem of those C^*-algebras A for which all Hilbert A-modules have a standard frame is open until now [7]. In 2011, Li solved the problem for commutative unital C^*-algebras [10, Theorem 1.1]. In fact, Li shows that for a commutative unital C^*-algebra A, every Hilbert A-module has a frame if and only if A is finite dimensional.

On the other hand, a C^*-algebra A is a C^*-algebra of compact operators if and only if every Hilbert A-module has a basis [2,3].

Also, it is well known that each unital C^*-algebra of compact operators is finite dimensional and a commutative C^*-algebra $A = C_0(Z)$ is a C^*-algebra of compact operators exactly when Z is discrete.

Hence, as mentioned in [1], a non-unital version of Li's theorem can be obtained as follows.

Theorem 1.1. *Let A be a commutative C^*-algebra. Then A is a C^*-algebra of compact operators if and only if every Hilbert A-module has a frame.*

Therefore, for general case, the following conjecture arises [1].

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Conjecture 1.2. If every Hilbert C^*-module over a C^*-algebra A has a frame, then A is a C^*-algebra of compact operators.

In [1], it is shown that the above conjecture has an affirmative for certain classes of C^*-algebras.

In this note, we investigate the problem of the existence of a frame for right ideals of a C^*-algebra A, without using the Kasparov stabilization theorem. We show that this property cannot characterize A as a C^*-algebra of compact operators.

2. **Frames and Ideals**

Let A be a C*-algebra and E be a Hilbert A-module. A family $\{x_i\}_{i\in I}$ of elements in E is called a frame if there are real constants $C, D > 0$ such that $\sum_{i\in I}\langle x, x_i\rangle_A\langle x_i, x\rangle_A$ converges, in the ultraweak topology of the universal enveloping von Neumann algebra, to some element in A^{**} and

$$C\langle x, x\rangle_A \leq \sum_{i\in I}\langle x, x_i\rangle_A\langle x_i, x\rangle_A \leq D\langle x, x\rangle_A$$

for every $x \in E$. A frame is said to be standard if the sum in the middle of the above inequality converges in norm for every $x \in E$, and is said to be normalized if $C = D = 1$.

There are some results in the literature on the characterization of a C^*-algebra of compact operators by certain properties of its (right) ideals. For instance, Magajna in [11] showed that if A is a C*-algebra and there exists a full Hilbert A-module E such that each closed submodule of E is orthogonally complemented, then A is a C*-algebra of compact operators. Schweizer in [14] remarked that this problem on Hilbert A-submodules of E can be reformulated as a problem on right ideals of A and consequently the result can be obtained easily.

Therefore, one may expect that the problem of the existence of a frame for each Hilbert A-module can be reformulated as the problem of the existence of a frame for each right ideal of A. Hereinafter, by *ideal* we mean closed ideals.

Definition 2.1. We say that a right Hilbert C^*-module E over a C^*-algebra A is countably generated if there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in E such that the A-linear hull of $\{x_n : n \in \mathbb{N}\}$ is norm-dense in E.

Note that our definition of being countably generated really means "topologically countably generated" and this differs from being algebraically countably generated. Surprisingly, it is shown in [5] that if every closed right ideal of a Banach algebra A is algebraically countably generated, then A is finite dimensional. Recently, Blecher and Kania gave a characterization of Hilbert C^*-modules which are algebraically (countably) finitely generated [4].

Lemma 2.2. *Let H be a Hilbert space. Then $\mathcal{K}(H)$ is countably generated, as a $\mathcal{K}(H)$-module, if and only if H is separable.*

Proof. Let H be a separable Hilbert space with a fixed orthonormal basis $\{e_n\}_{n=1}^{\infty}$. We have $T = \sum_{n=1}^{\infty} P_n T$, for all $T \in \mathcal{K}(H)$, where P_n is the orthogonal projection to the one-dimensional subspace spanned by e_n. Therefore $\{P_n\}_{n=1}^{\infty}$ is a countable set of generators for $\mathcal{K}(H)$.

Conversely, let $\{T_n\}_{n=1}^{\infty}$ be a countable set of generators for $\mathcal{K}(H)$. Then H is equal to the closed linear span of $\cup_{n\in\mathbb{N}} R(T_n)$, where $R(T_n)$ is the range of T_n. Also, it is well known that the range of each compact operator is separable. Therefore, H is separable. □

Note that, in the above lemma, H is separable if and only if the C^*-algebra $\mathcal{K}(H)$ is separable. For a general C^*-algebra A, if it is topologically countably generated as an A-module, one cannot conclude that A is separable. Instead, we have the following characterization.

Proposition 2.3. *For a C^*-algebra A, the following statements are equivalent:*

(i) *A is σ-unital;*
(ii) *A has a strictly positive element;*
(iii) *A has a countable standard normalized frame;*
(iv) *A is countably generated as an A-module.*

Proof. $(i) \Leftrightarrow (ii)$: This is a well-known fact in the C^*-algebra literature [13].

$(ii) \Rightarrow (iii)$: Let $h \in A$ be a strictly positive element. We set $v_0 = 0$, $v_n = h(h + \frac{1}{n})^{-1}$ and $u_n = (v_n - v_{n-1})^{\frac{1}{2}}$ for each $n \in \mathbb{N}$. As mentioned in [13], the sequence $\{v_n\}_{n=1}^{\infty}$ is a countable approximate unit for A. Then, for every $a \in A$, we have $a = \lim_n v_n a = \lim_n \sum_{j=1}^{n} (u_j)^2 a = \sum_{n=1}^{\infty} u_n \langle u_n, a\rangle$. Hence, $\{u_n\}_{n=1}^{\infty}$ is a countable standard normalized frame for A.

$(iii) \Rightarrow (iv)$: Obviously, if $\{u_n\}_{n=1}^{\infty}$ is a standard normalized frame for A, then $\{u_n\}_{n=1}^{\infty}$ is a countable set of generators for A, by the reconstruction formula.

$(iv) \Rightarrow (ii)$: Let $\{u_n\}_{n=1}^{\infty}$ be a bounded set of generators for A, then $p = \sum_{n=1}^{\infty} \frac{1}{2^n} u_n u_n^*$ is a strictly positive element. In fact, if φ is a positive functional on A such that $\varphi(p) = 0$, then $\varphi(u_n u_n^*) = 0$, for all n. It follows that $\varphi(u_n b_n) = 0$, for all $n \in \mathbb{N}$ and $b_n \in A$. Therefore, $\varphi(a) = 0$ for each $a \in A$, i.e., $\varphi \equiv 0$. □

We recall that if B is a hereditary C^*-subalgebra of A, then there is a unique right ideal L such that $B = L \cap L^*$ [13, Theorem 3.2.1]. Similar to the proof of $(ii) \Rightarrow (iii)$ in the above proposition, one can show that if B has a strictly positive element, then L, as a Hilbert A-module, has a countable standard normalized frame.

Corollary 2.4. *For a C^*-algebra A, the following statements are equivalent:*

(i) *A is completely σ-unital, i.e., every hereditary C^*-subalgebra of A is σ-unital;*
(ii) *every hereditary C^*-subalgebra of A has a strictly positive element;*
(iii) *every right ideal I of A is countably generated as an A-module;*
(iv) *every right ideal I of A has a countable standard normalized frame.*

If Z is a locally compact Hausdorff space, then the C^*-algebra $C_0(Z)$ is separable if and only if Z is σ-compact and metrizable, if and only if Z is second countable. Also, the C^*-algebra $C_0(Z)$ is σ-unital if and only if Z is σ-compact.

We recall that if a locally compact Hausdorff space Z is σ-compact, then Z is paracompact. Also, whenever a locally compact Hausdorff space Z is paracompact (or σ-compact), for any open cover $\mathcal{U}$ of Z, there exists a continuous partition of unity subordinated to $\mathcal{U}$. In fact, there exists a partition of unity $\{f_j\}_{j\in J}$ (or $\{f_n\}_{n\in\mathbb{N}}$) in $C_c(Z)$.

It is well known that ideals of $A = C_0(Z)$ correspond bijectively to closed sets of Z. More precisely, I is an ideal of A if and only if there is a closed set $F \subseteq Z$ such that

$$I = \{f \in C_0(Z) \colon f(z) = 0 \text{ for all } z \in F\}.$$

The following proposition can be derived easily from Proposition 2.3, however we supply a direct proof of it.

Proposition 2.5. *Let Z be a locally compact Hausdorff space and let $A = C_0(Z)$. Then the following statements are equivalent:*

(i) *A is completely σ-unital;*
(ii) *Z is hereditary σ-compact, i.e., every open subset of Z is σ-compact;*
(iii) *every ideal I of A has a countable standard normalized frame.*

Proof. $(i) \Leftrightarrow (ii)$: Since A is commutative, hereditary C^*-subalgebras are exactly ideals of A. Also, if F is a closed subset of Z and $I_F = \{f \in C_0(Z) \colon f(z) = 0 \text{ for all } z \in F\}$, then it is easy to see that I_F is σ-unital if and only if F^c is σ-compact.

$(ii) \Rightarrow (iii)$: Let I be an ideal of $A = C_0(Z)$ and F be a closed subset of Z such that $I = I_F$. By assumption, F^c is σ-compact (and so paracompact), thus there exists a partition of unity of F^c as $\{f_n\}_{n\in\mathbb{N}}$ in $C_c(F^c)$. Since for each f_n, $Supp(f_n) = \overline{\{z \in F^c : f_n(z) \neq 0\}}$ is compact and $Supp(f_n) \cap F = \emptyset$, if one extends each f_n on Z by setting zero on F, then $f_n \in I_F$, for all n. It is easy to see that, $\{f_n^{\frac{1}{2}}\}_{n\in\mathbb{N}}$ is a standard normalized frame for $I_F = I$.

$(iii) \Rightarrow (ii)$: Let F be a closed subset of Z and the sequence $\{f_n\}_{n\in\mathbb{N}}$ be a standard normalized frame for I_F. Then we have

$$|f(z)|^2 = \sum_{n=1}^{\infty} |f(z)|^2 |f_n(z)|^2,$$

for all $f \in I_F$ and $z \in Z$. On the other hand, for each $z \in F^c$ there is some $f \in I_F$ such that $f(z) = 1$. Then $1 = \sum_{n=1}^{\infty} |f_n(z)|^2$, for all $z \in F^c$. Now, we have $F^c = \cup_{n,m=1}^{\infty} K_m(f_n)$, where $K_m(f_n) = \{z \in Z : |f_n(z)|^2 \geq \frac{1}{m}\}$, for all $m, n \in \mathbb{N}$. Therefore F^c is σ-compact, because $K_m(f_n)$ is compact for all m, n. □

Corollary 2.6. *Let Z be a locally compact metrizable space and $A = C_0(Z)$. Then the following statements are equivalent:*

(i) *A is separable;*
(ii) *A is completely σ-unital;*
(iii) *every ideal I of A has a countable standard normalized frame.*

Proposition 2.7. *Let $A = C_0(Z)$, where Z is a locally compact Hausdorff space such that every open subset of Z is separable. Then the following statements are equivalent:*

(i) *every ideal of A has a standard normalized frame;*
(ii) *every ideal of A has a countable standard normalized frame.*

Proof. $(i) \Rightarrow (ii)$: Let I be an ideal of $A = C_0(Z)$ and F be a closed subset of Z such that $I = I_F$. Also, let $\{f_j\}_{j \in J}$ be a (standard normalized) frame for I_F. Then we have $|f(z)|^2 = \sum_{j \in J} |f(z)|^2 |f_j(z)|^2$, for all $f \in I_F$ and $z \in Z$. By assumption, there is a countable subset W of F^c, such that $\overline{W} \supseteq F^c$. By Urysohn's Lemma for locally compact Hausdorff spaces [6], for every $z \in F^c$ there is an $f \in I_F$ such that $f(z) = 1$ which implies $\sum_{j \in J} |f_j(z)|^2 = 1$ for all $z \in F^c$. In particular, for each $z \in W$ the set $J_z = \{j \in J : f_j(z) \neq 0\}$ is countable. If $J_W = \cup_{z \in W} J_z$, then J_W is countable and we have $f_j(z) = 0$, for all $j \in J \setminus J_W$ and $z \in F^c$, because every f_j is continuous and $\overline{W} \supseteq F^c$. Therefore, we have

$$|f(z)|^2 = \sum_{j \in J_W} |f(z)|^2 |f_j(z)|^2,$$

for all $f \in I_F$ and $z \in Z$. This means that $\{f_j\}_{j \in J_W}$ is a countable standard normalized frame for I.

$(ii) \Rightarrow (i)$: This is evident. □

Proposition 2.7 can be used to derive the following standard fact from Topology:

Proposition 2.8. *Let Z be a separable locally compact Hausdorff space. Then Z is paracompact if and only if it is σ-compact.*

Similarly, we can obtain the following result.

Proposition 2.9. *Let Z be a locally compact Hausdorff space and $A = C_0(Z)$. Then every ideal I of A has a standard normalized frame exactly when every open subset of Z is paracompact.*

Since every metric space is hereditary paracompact, we also have the following result.

Corollary 2.10. *If a locally compact Hausdorff space Z is metrizable, then every ideal of the C^*-algebra $A = C_0(Z)$ has a standard normalized frame.*

As seen in the above results, for a C^*-algebra A, the fact that "every right ideal of A has a (countable) standard normalized frame" cannot characterize A as a C^*-algebra of compact operators. In fact, if every Hilbert C^*-module over A has a (countable) standard frame, then every right ideal of A has a (countable) standard frame, but the converse might not hold.

Finally, we remark that in the category of C^*-algebras, being separable is strictly stronger than being completely σ-unital. For instance, according to a classical example, due to Alexandroff and Urysohn, the double arrow space is a compact Hausdorff and perfectly normal space [15]. The latter implies that all open subsets of the double arrow space are σ-compact, while this space is not second countable and thus it is not metrizable. Therefore, if Z is the double arrow space, then $C(Z)$ is completely σ-unital, while it is not separable.

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