ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 42 (2016), No. 1, pp. 69–78

Title:

Linear operators of Banach spaces with range in Lipschitz algebras

Author(s):

A. Golbaharan and H. Mahyar

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 1, pp. 69–78
Online ISSN: 1735-8515

LINEAR OPERATORS OF BANACH SPACES WITH RANGE IN LIPSCHITZ ALGEBRAS

A. GOLBAHARAN AND H. MAHYAR *

(Communicated by Hamid Reza Ebrahimi Vishki)

Abstract. In this paper, a complete description concerning linear operators of Banach spaces with range in Lipschitz algebras $\mathrm{lip}_\alpha(X)$ is provided. Necessary and sufficient conditions are established to ensure boundedness and (weak) compactness of these operators. Finally, a lower bound for the essential norm of such operators is obtained.
Keywords: Lipschitz algebra, compact linear operator, weakly compact linear operator, essential norm.
MSC(2010): Primary: 47B38; Secondary: 46E15.

1. Introduction and preliminaries

Let (X, d) be a compact metric space, $(E, \|\cdot\|)$ be a Banach space, and $\alpha \in (0, 1]$. The space of E-valued functions f on X for which

$$p_\alpha(f) = \sup\left\{\frac{\|f(x) - f(y)\|}{d^\alpha(x, y)} : x, y \in X, x \neq y\right\} < \infty,$$

is denoted by $\mathrm{Lip}_\alpha(X, E)$. The subspace of those functions f with

$$\lim_{d(x,y)\to 0} \frac{\|f(x) - f(y)\|}{d^\alpha(x, y)} = 0,$$

is denoted by $\mathrm{lip}_\alpha(X, E)$. The spaces $\mathrm{Lip}_\alpha(X, E)$ and $\mathrm{lip}_\alpha(X, E)$ are Banach spaces with the norm $\|f\|_\alpha = \|f\|_X + p_\alpha(f)$, where $\|f\|_X = \sup_{x\in X} \|f(x)\|$ is the uniform norm. If $\mathbb{K}$ is the scalar field of the real or complex numbers, to simplify the notation, we put $\mathrm{Lip}_\alpha(X) = \mathrm{Lip}_\alpha(X, \mathbb{K})$ and $\mathrm{lip}_\alpha(X) = \mathrm{lip}_\alpha(X, \mathbb{K})$. In this case, $\mathrm{Lip}_\alpha(X)$ for $0 < \alpha \leq 1$ and $\mathrm{lip}_\alpha(X)$ for $0 < \alpha < 1$ are Banach function algebras which are called Lipschitz algebras and their character spaces (maximal ideal spaces) coincide with X. We are concerned with the spaces $\mathrm{Lip}_\alpha(X, E)$ for $0 < \alpha \leq 1$ and $\mathrm{lip}_\alpha(X, E)$ for $0 < \alpha < 1$. The study of these

Article electronically published on February 22, 2016.
Received: 10 April 2014, Accepted: 1 November 2014.
*Corresponding author.

algebras started by de Leeuw [6] and Sherbert [7, 8]. The interested reader is referred to [1, 5, 9] for further details on these algebras.

It is interesting to characterize compact and weakly compact operators between various function spaces or algebras. Recall that $C(X)$ is the Banach space of scalar-valued continuous functions on a compact Hausdorff space X with the uniform norm. In [2, Theorem VI, 7.1] a complete description of bounded, compact and weakly compact linear operators from a Banach space to $C(X)$ was obtained as follows.

Theorem 1.1. *[2, Theorem VI, 7.1] Let X be a compact Hausdorff space and let T be a bounded linear operator from a Banach space E into $C(X)$. Then there exists a mapping $\tau : X \to E^*$ which is continuous with the w^*-topology on E^* such that*

(i) $Te(x) = \tau(x)e, \quad e \in E, x \in X;$
(ii) $\|T\| = \sup_{x\in X} \|\tau(x)\|$.

Conversely, if such a map τ is given, then the operator T defined by (i) is a bounded linear operator from E into $C(X)$ with the norm given by (ii).

The operator T is compact if and only if τ is continuous with the norm topology on E^*.

The operator T is weakly compact if and only if τ is continuous with the weak topology on E^*.

In this paper, we consider these properties for linear operators from a Banach space E into Lipschitz algebras $\mathrm{lip}_\alpha(X)$ and give a complete description of these operators. We also obtain a lower bound for the essential norm of these operators. As a consequence of our results, if T is a map from $\mathrm{lip}_\alpha(X)$ into $\mathrm{lip}_\alpha(Y)$, then Theorem 5.1 in [7], Theorem 1 in [5] and Theorem 3.1 in [4] will follow immediately. Also using the results of this paper, we give a short proof to the aforementioned theorems in [7] and [5] provided T maps $\mathrm{Lip}_\alpha(X)$ into $\mathrm{Lip}_\alpha(Y)$ and $0 < \alpha < 1$. In addition, one can conclude Theorem 2.10 (ii) in [3] in the scalar case, provided $0 < \alpha < 1$.

2. **Results**

We begin by introducing some notation. Let E be a Banach space. Then $E^\times$, the algebraic dual space of E, is the space of all linear functionals on E. The topological dual space of E is the Banach space E^* whose elements are the bounded linear functionals on E.

Let T be a linear operator (not necessarily bounded) from a Banach space E into $\mathrm{lip}_\alpha(X)$. The restriction of the algebraic adjoint $T^\times : \mathrm{lip}_\alpha(X)^* \to E^\times$ of T to the space X is denoted by ψ. Then, by the definition of adjoint, the function $\psi = T^\times|_X : X \to E^\times$ is defined by $\psi(x) = T^\times(\delta_x) = \delta_x \circ T$, where $\delta_x \in \mathrm{lip}_\alpha(X)^*$ is the evaluation functional at point $x \in X$ defined by $\delta_x(f) = f(x)$ for every $f \in \mathrm{lip}_\alpha(X)$. One can say that the linear operator T is induced by the function ψ or that ψ induces T by means of $\psi(x) = \delta_x \circ T$ or equivalently, $(Te)(x) = \psi(x)(e)$ for each $e \in E$ and $x \in X$. In the case where T is bounded, the function ψ maps X into E^*. In fact, ψ is the restriction of the topological adjoint $T^* : \mathrm{lip}_\alpha(X)^* \to E^*$ of T to the space X, and it is continuous with the weak*-topology on E^*.

In the following theorem, we shall explore the possibility of inducing a linear operator $T : E \to \mathrm{lip}_\alpha(X)$ by a function $\psi : X \to E^\times$.

Theorem 2.1. *Let E be a Banach space. If T is a linear operator from E into $\mathrm{lip}_\alpha(X)$, then the function $\psi = T^\times|_X$ satisfies*

$$\lim_{d(x,y)\to 0} \frac{\psi(x) - \psi(y)}{d^\alpha(x,y)} = 0, \tag{2.1}$$

in the pointwise convergence topology of $E^\times$. Conversely, if a function $\psi : X \to E^\times$ satisfies (2.1) in the pointwise convergence topology of $E^\times$, then the linear operator T defined by $Te(x) = \psi(x)e$ maps E into $\mathrm{lip}_\alpha(X)$.

Proof. If E is a Banach space and T is a linear operator that maps E into $\mathrm{lip}_\alpha(X)$, then $Te \in \mathrm{lip}_\alpha(X)$ for each $e \in E$. Thus,

$$\lim_{d(x,y)\to 0} \frac{\psi(x)e - \psi(y)e}{d^\alpha(x,y)} = \lim_{d(x,y)\to 0} \frac{Te(x) - Te(y)}{d^\alpha(x,y)} = 0 \quad (e \in E).$$

Conversely, suppose that a function $\psi : X \to E^\times$ satisfies (2.1) in the pointwise convergence topology of $E^\times$. It follows that

$$\lim_{d(x,y)\to 0} \frac{Te(x) - Te(y)}{d^\alpha(x,y)} = \lim_{d(x,y)\to 0} \frac{\psi(x)e - \psi(y)e}{d^\alpha(x,y)} = 0,$$

from which we obtain $Te \in \mathrm{lip}_\alpha(X)$ for each $e \in E$. □

We are interested in the problem of using function theoretic properties of ψ to determine when the linear operator T is bounded, compact or weakly compact. If the linear operator T maps a Banach space E into $\mathrm{lip}_\alpha(X)$, then we have the following result on the boundedness of T.

Theorem 2.2. *Suppose that E is a Banach space and $T : E \to \mathrm{lip}_\alpha(X)$ is a linear operator induced by $\psi : X \to E^\times$. Then T is bounded if and only if $\psi \in \mathrm{Lip}_\alpha(X, E^*)$. Moreover, $\|T\| \le \|\psi\|_\alpha \le 2\|T\|$.*

Proof. If T is bounded, then for every $x, y \in X$ with $x \neq y$,

$$\begin{aligned} \frac{\|\psi(x) - \psi(y)\|}{d^\alpha(x,y)} &= \sup_{\|e\|\le 1} \frac{|\psi(x)(e) - \psi(y)(e)|}{d^\alpha(x,y)} = \sup_{\|e\|\le 1} \frac{|Te(x) - Te(y)|}{d^\alpha(x,y)} \\ &\le \sup_{\|e\|\le 1} p_\alpha(Te) \le \sup_{\|e\|\le 1} \|Te\|_\alpha = \|T\|. \end{aligned}$$

Thus $\psi \in \mathrm{Lip}_\alpha(X, E^*)$ and $p_\alpha(\psi) \le \|T\|$. Moreover,

$$\|\psi(x)\| = \|T^*(\delta_x)\| \le \|T^*\|\|\delta_x\| = \|T\|,$$

for every $x \in X$. Therefore, $\|\psi\|_X \leq \|T\|$ and $\|\psi\|_\alpha = \|\psi\|_X + p_\alpha(\psi) \leq 2\|T\|$.

Conversely, let $\psi \in \mathrm{Lip}_\alpha(X, E^*)$. Then for every $x, y \in X$ with $x \neq y$,

$$\begin{aligned} \frac{|(Te)(x) - (Te)(y)|}{d^\alpha(x,y)} &= \frac{|(\psi(x) - \psi(y))(e)|}{d^\alpha(x,y)} \\ &\leq \frac{\|\psi(x) - \psi(y)\|}{d^\alpha(x,y)}\|e\| \leq p_\alpha(\psi)\|e\|. \end{aligned} \tag{2.2}$$

Hence, $p_\alpha(Te) \leq p_\alpha(\psi)\|e\|$. Also, we have

$$|(Te)(x)| = |\psi(x)(e)| \leq \|\psi(x)\|\|e\| \leq \|\psi\|_X\|e\|,$$

for every $x \in X$. Thus $\|Te\|_X \leq \|\psi\|_X\|e\|$. The addition of these inequalities yields

$$\|Te\|_\alpha = p_\alpha(Te) + \|Te\|_X \leq (p_\alpha(\psi) + \|\psi\|_X)\|e\| = \|\psi\|_\alpha\|e\|,$$

which implies that T is bounded and $\|T\| \leq \|\psi\|_\alpha$. □

Note that every $\psi \in \mathrm{Lip}_\alpha(X, E^*)$ induces a bounded linear operator from E into $\mathrm{Lip}_\alpha(X)$. Using the same argument as in the proof of Theorem 2.2, we can obtain the following result for $\mathrm{Lip}_\alpha(X)$.

Theorem 2.3. *Let E be a Banach space. Then the operator $T : E \to \mathrm{Lip}_\alpha(X)$ is a bounded linear operator induced by $\psi : X \to E^*$ if and only if $\psi \in \mathrm{Lip}_\alpha(X, E^*)$. Moreover, $\|T\| \leq \|\psi\|_\alpha \leq 2\|T\|$.*

Considering Theorem 2.1, a function $\psi : X \to E^*$ may not, in general, induce a linear operator $T : E \to \mathrm{lip}_\alpha(X)$. Even if $\psi \in \mathrm{Lip}_\alpha(X, E^*)$, the operator T defined by $(Te)(x) = \psi(x)e$ does not, in general, map E into $\mathrm{lip}_\alpha(X)$. For example, set $E = \mathrm{lip}_\alpha(X)$ and let $\lambda_0 \in \mathrm{lip}_\alpha(X)^*$, $f_0 \in \mathrm{Lip}_\alpha(X) \setminus \mathrm{lip}_\alpha(X)$ and define $\psi : X \to \mathrm{lip}_\alpha(X)^*$ by $\psi(x) = f_0(x)\lambda_0$. Note that $\psi \in \mathrm{Lip}_\alpha(X, \mathrm{lip}_\alpha(X)^*)$. Let T be the induced operator by ψ. Then $(Tf)(x) = \psi(x)f = f_0(x)\lambda_0(f)$ for each $f \in \mathrm{lip}_\alpha(X)$ and $x \in X$. Thus, $Tf = \lambda_0(f)f_0$ is not in $\mathrm{lip}_\alpha(X)$ for any $f \in \mathrm{lip}_\alpha(X)$ with $\lambda_0(f) \neq 0$. Therefore, T does not map $E = \mathrm{lip}_\alpha(X)$ into $\mathrm{lip}_\alpha(X)$.

We now address the compactness of these operators.

Theorem 2.4. *Let E be a Banach space. Then a linear operator $T : E \to \mathrm{lip}_\alpha(X)$ is a compact operator induced by ψ if and only if $\psi \in \mathrm{lip}_\alpha(X, E^*)$.*

Proof. Let $T : E \to \mathrm{lip}_\alpha(X)$ be a compact linear operator induced by ψ. Then $T(B_E) = \{Te : e \in B_E\}$ is totally bounded in $\mathrm{lip}_\alpha(X)$ where B_E is the open unit ball in E. Given $\varepsilon > 0$, there exist $f_1, \ldots, f_n \in \mathrm{lip}_\alpha(X)$ such that $T(B_E) \subseteq \cup_{i=1}^n \mathbb{B}(f_i, \varepsilon/2)$ where $\mathbb{B}(f, r) = \{g \in \mathrm{lip}_\alpha(X) : \|g - f\|_\alpha < r\}$ for $f \in \mathrm{lip}_\alpha(X)$ and $r > 0$. One can choose $\delta > 0$ such that $|f_i(x) - f_i(y)|/d^\alpha(x,y) < \varepsilon/2$ for every $x, y \in X$ with $0 < d(x,y) < \delta$ and for each $i \in \{1, \ldots, n\}$. Let

$e \in B_E$. Hence $\|Te - f_i\|_\alpha < \varepsilon/2$ for some $i \in \{1, \ldots, n\}$. Let $x, y \in X$ with $0 < d(x,y) < \delta$. Then,

$$\begin{aligned}\frac{|(\psi(x)-\psi(y))(e)|}{d^\alpha(x,y)} &= \frac{|(Te)(x)-(Te)(y)|}{d^\alpha(x,y)} \\ &\leq \frac{|(Te-f_i)(x)-(Te-f_i)(y)|}{d^\alpha(x,y)} + \frac{|f_i(x)-f_i(y)|}{d^\alpha(x,y)} \\ &\leq p_\alpha(Te-f_i) + \frac{\varepsilon}{2} \leq \|Te-f_i\|_\alpha + \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

Therefore, $\|\psi(x) - \psi(y)\|/d^\alpha(x,y) < \varepsilon$ whenever $0 < d(x,y) < \delta$, that is $\psi \in \mathrm{lip}_\alpha(X, E^*)$.

Conversely, suppose that $\psi \in \mathrm{lip}_\alpha(X, E^*)$. It follows from (2.2) that $Te \in \mathrm{lip}_\alpha(X)$ for every $e \in E$. Hence, T maps the space E into $\mathrm{lip}_\alpha(X)$. Therefore, by Theorem 2.2, the operator $T : E \to \mathrm{lip}_\alpha(X)$ is bounded. Furthermore, given $\varepsilon > 0$, there exists $0 < \delta < 1$ such that $\|\psi(x) - \psi(y)\|/d^\alpha(x,y) < \varepsilon/4$ for every $x, y \in X$ with $0 < d(x,y) < \delta$. For the compactness of T, we assume that $\{e_n\}$ is a sequence in B_E. Then the boundedness of T and the fact that $Te_n \in \mathrm{Lip}_\alpha(X)$ imply that $\{Te_n\}$ is a bounded and equicontinuous sequence of functions on the compact metric space X. Hence by Arzela-Ascoli theorem, there exists a subsequence $\{e_{n_i}\}$ of $\{e_n\}$ such that $\{Te_{n_i}\}$ is uniformly Cauchy on X. Hence, $\|Te_{n_i} - Te_{n_j}\|_X < \varepsilon\delta^\alpha/4 < \varepsilon/2$, for large enough i, j.

Let $x, y \in X$ with $x \neq y$. If $d(x,y) < \delta$, then

$$\begin{aligned}\frac{|(Te_{n_i}-Te_{n_j})(x)-(Te_{n_i}-Te_{n_j})(y)|}{d^\alpha(x,y)} &= \frac{|(\psi(x)-\psi(y))(e_{n_i}-e_{n_j})|}{d^\alpha(x,y)} \\ &\leq \frac{\|\psi(x)-\psi(y)\|}{d^\alpha(x,y)}\|e_{n_i}-e_{n_j}\| \\ &< 2\frac{\varepsilon}{4} = \frac{\varepsilon}{2}.\end{aligned}$$

If $d(x,y) \geq \delta$, then

$$\frac{|(Te_{n_i}-Te_{n_j})(x)-(Te_{n_i}-Te_{n_j})(y))|}{d^\alpha(x,y)} \leq 2\frac{\|Te_{n_i}-Te_{n_j}\|_X}{\delta^\alpha} < \frac{2}{\delta^\alpha}\frac{\varepsilon\delta^\alpha}{4} = \frac{\varepsilon}{2}.$$

Hence $p_\alpha(Te_{n_i} - Te_{n_j}) < \varepsilon/2$, for large enough i, j. Therefore,

$$\|Te_{n_i}-Te_{n_j}\|_\alpha = \|Te_{n_i}-Te_{n_j}\|_X + p_\alpha(Te_{n_i}-Te_{n_j}) < \varepsilon,$$

for large enough i, j, that is $\{Te_{n_i}\}$ is a Cauchy sequence in $\mathrm{lip}_\alpha(X)$ and hence it is convergent in $\mathrm{lip}_\alpha(X)$. Therefore, T is compact and this completes the proof of the theorem. □

Using Theorem 2.4, we get a lower bound for the essential norm of a bounded linear operator $T : E \to \mathrm{lip}_\alpha(X)$. We recall that the essential norm $\|T\|_e$ of a bounded linear operator T is defined by

$$\|T\|_e = \inf_K \|T - K\|,$$

where the infimum is taken over all compact operators $K : E \to \mathrm{lip}_\alpha(X)$. Note that $\|T\|_e = 0$ if and only if T is compact.

Theorem 2.5. *If E is a Banach space and $T : E \to \mathrm{lip}_\alpha(X)$ is a bounded linear operator induced by a function $\psi : X \to E^*$, then*

$$\lim_{\delta\to 0} \sup_{0<d(x,y)<\delta} \frac{\|\psi(x) - \psi(y)\|}{d^\alpha(x, y)} \le \|T\|_e.$$

Proof. Suppose that $K : E \to \mathrm{lip}_\alpha(X)$ is a compact linear operator. Then by Theorem 2.4, $\varphi = K^*|_X \in \mathrm{lip}_\alpha(X, E^*)$ and $Ke(x) = \varphi(x)e \quad (e \in E, x \in X)$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\frac{\|\varphi(x) - \varphi(y)\|}{d^\alpha(x, y)} < \varepsilon, \tag{2.3}$$

whenever $0 < d(x, y) < \delta$ for $x, y \in X$.

Let $x, y \in X$ with $0 < d(x, y) < \delta$, and $e \in E$ with $\|e\| \le 1$. Then using (2.3), we have

$$\begin{aligned}
\|T - K\| \ge \|Te - Ke\|_\alpha \ge p_\alpha(Te - Ke) &\ge \frac{|(Te - Ke)(x) - (Te - Ke)(y)|}{d^\alpha(x, y)} \\
&\ge \frac{|Te(x) - Te(y)|}{d^\alpha(x, y)} - \frac{|Ke(x) - Ke(y)|}{d^\alpha(x, y)} \\
&= \frac{|\psi(x)(e) - \psi(y)(e)|}{d^\alpha(x, y)} - \frac{|\varphi(x)(e) - \varphi(y)(e)|}{d^\alpha(x, y)} \\
&\ge \frac{|\psi(x)(e) - \psi(y)(e)|}{d^\alpha(x, y)} - \frac{\|\varphi(x) - \varphi(y)\|}{d^\alpha(x, y)} \\
&\ge \frac{|\psi(x)(e) - \psi(y)(e)|}{d^\alpha(x, y)} - \varepsilon.
\end{aligned}$$

By taking supremum over all e in the closed unit ball of E we obtain that

$$\|T - K\| \ge \frac{\|\psi(x) - \psi(y)\|}{d(x, y)^\alpha} - \varepsilon,$$

and then

$$\lim_{\delta\to 0} \sup_{0<d(x,y)<\delta} \frac{\|\psi(x) - \psi(y)\|}{d^\alpha(x, y)} \le \|T - K\|.$$

Now by taking infimum over all compact operators $K : E \to \mathrm{lip}_\alpha(X)$ we conclude the desired result. □

We now give a necessary and sufficient condition for a bounded linear operator $T : E \to \mathrm{lip}_\alpha(X)$ to be weakly compact.

Theorem 2.6. *Let E be a Banach space. Then a linear operator $T : E \to \mathrm{lip}_\alpha(X)$ induced by $\psi : X \to E^*$ is a weakly compact operator if and only if*

$$\lim_{d(x,y)\to 0} \frac{\psi(x)-\psi(y)}{d^\alpha(x,y)} = 0,$$

in the weak topology of E^*.

Proof. Note first that $T^{**} : E^{**} \to \mathrm{lip}_\alpha(X)^{**}$. By [2, Theorem VI.4.2], T is weakly compact if and only if T^{**} maps E^{**} into $\mathrm{lip}_\alpha(X)$, that is for each $\Lambda \in E^{**}$, $T^{**}\Lambda = \delta_f$ for some $f \in \mathrm{lip}_\alpha(X)$. By the definition of the adjoint of an operator and the mentioned fact, $T^{**}E^{**} \subseteq \mathrm{lip}_\alpha(X) \subseteq \mathrm{lip}_\alpha(X)^{**}$ if and only if

$$\begin{aligned} \lim_{d(x,y)\to 0} \frac{\Lambda(\psi(x)-\psi(y))}{d^\alpha(x,y)} &= \lim_{d(x,y)\to 0} \frac{\Lambda(T^*(\delta_x)-T^*(\delta_y))}{d^\alpha(x,y)} \\ &= \lim_{d(x,y)\to 0} \frac{T^{**}\Lambda(\delta_x)-T^{**}\Lambda(\delta_y)}{d^\alpha(x,y)} \\ &= \lim_{d(x,y)\to 0} \frac{f(x)-f(y)}{d^\alpha(x,y)} = 0, \end{aligned}$$

for each $\Lambda \in E^{**}$, which is equivalent to

$$\lim_{d(x,y)\to 0} \frac{\psi(x)-\psi(y)}{d^\alpha(x,y)} = 0,$$

in the weak topology of E^*. This completes the proof. □

Using Theorems 2.3 and 2.4, one may immediately conclude Theorem 5.1 in [7], Theorem 1 in [5] and Theorem 3.1 in [4] when T is a map from $\mathrm{lip}_\alpha(X)$ into $\mathrm{lip}_\alpha(Y)$. Finally, using Theorems 2.3, 2.4 and the following lemma, we give a short proof for the aforementioned theorems in [7], [5] and Theorem 2.10 (ii) in [3] if T maps $\mathrm{Lip}_\alpha(X)$ into $\mathrm{Lip}_\alpha(Y)$ provided $0 < \alpha < 1$.

Lemma 2.7. *Let (X, d) be a compact metric space. Then,*

$$\|\delta_x - \delta_y\| = \frac{2d^\alpha(x,y)}{2+d^\alpha(x,y)}$$

for each $x, y \in X$ where δ_x and δ_y are regarded as elements of the dual space $\mathrm{lip}_\alpha(X)^*$.

Proof. Let $x, y \in X$ with $x \neq y$ and $\alpha < \beta < 1$, and define the function f on X by

$$f(t) = \frac{d^\beta(t,y)-d^\beta(t,x)}{2d^{\beta-\alpha}(x,y)+d^\beta(x,y)} \qquad (t \in X).$$

It is clear that

$$|f(x)| = |f(y)| = \frac{d^\alpha(x,y)}{2+d^\alpha(x,y)},$$

and

$$|f(t)| \le \frac{d^{\beta}(x,y)}{2d^{\beta-\alpha}(x,y)+d^{\beta}(x,y)} = \frac{d^{\alpha}(x,y)}{2+d^{\alpha}(x,y)},$$

for every $t \in X$. Hence,

$$\|f\|_X = \frac{d^{\alpha}(x,y)}{2+d^{\alpha}(x,y)}. \tag{2.4}$$

On the other hand,

$$\frac{|f(x)-f(y)|}{d^{\alpha}(x,y)} = \frac{2d^{\beta-\alpha}(x,y)}{2d^{\beta-\alpha}(x,y)+d^{\beta}(x,y)} = \frac{2}{2+d^{\alpha}(x,y)},$$

and for each $t, s \in X$ with $t \neq s$, considering two cases either $d(x,y) \le d(t,s)$ or $d(t,s) \le d(x,y)$, we get

$$\frac{|f(t)-f(s)|}{d^{\alpha}(t,s)} \le \frac{2}{2+d^{\alpha}(x,y)}.$$

Therefore,

$$p_{\alpha}(f) = \frac{2}{2+d^{\alpha}(x,y)}. \tag{2.5}$$

The addition of (2.4) and (2.5) yields $\|f\|_{\alpha} = \|f\|_X + p_{\alpha}(f) = 1$.

Moreover, for each $t, s \in X$ with $t \neq s$, we have

$$\frac{|f(t)-f(s)|}{d^{\alpha}(t,s)} \le \frac{2d^{\beta-\alpha}(t,s)}{2d^{\beta-\alpha}(x,y)+d^{\beta}(x,y)},$$

which implies that $f \in \mathrm{lip}_{\alpha}(X)$. Using the equality,

$$|(\delta_x - \delta_y)(f)| = \frac{2d^{\beta}(x,y)}{2d^{\beta-\alpha}(x,y)+d^{\beta}(x,y)} = \frac{2d^{\alpha}(x,y)}{2+d^{\alpha}(x,y)},$$

we conclude that $\|\delta_x - \delta_y\| \ge 2d^{\alpha}(x,y)/(2+d^{\alpha}(x,y))$.

Next, take any $g \in \mathrm{lip}_{\alpha}(X)$ with $\|g\|_{\alpha} = 1$. Without loss of generality suppose that $|g(x)| \ge |g(y)|$. Set $c = |g(x)-g(y)|/d^{\alpha}(x,y) + |g(x)|$. Then $0 \le c \le 1$ and we have

$$c = \frac{|g(x)-g(y)|}{d^{\alpha}(x,y)} + |g(x)| \le 2\frac{|g(x)|}{d^{\alpha}(x,y)} + |g(x)| = |g(x)|\frac{2+d^{\alpha}(x,y)}{d^{\alpha}(x,y)}.$$

Thus, $|g(x)| \ge cd^{\alpha}(x,y)/(2+d^{\alpha}(x,y))$. It follows that

$$\begin{aligned}|(\delta_x - \delta_y)(g)| &= |g(x)-g(y)| = (c - |g(x)|)d^{\alpha}(x,y)\\ &\le (c - c\frac{d^{\alpha}(x,y)}{2+d^{\alpha}(x,y)})d^{\alpha}(x,y) = \frac{2cd^{\alpha}(x,y)}{2+d^{\alpha}(x,y)} \le \frac{2d^{\alpha}(x,y)}{2+d^{\alpha}(x,y)}.\end{aligned}$$

Therefore, $\|\delta_x - \delta_y\| = 2d^{\alpha}(x,y)/(2+d^{\alpha}(x,y))$. □

Corollary 2.8. *Let* $0 < \alpha < 1$.

(*i*) [7, *Theorem 5.1*]. *The operator* $T : \mathrm{Lip}_\alpha(X) \to \mathrm{Lip}_\alpha(Y)$ *is a unital homomorphism if and only if there exists a map* $\varphi : Y \to X$ *such that* $Tf = f \circ \varphi$ *for all* $f \in \mathrm{Lip}_\alpha(X)$ *and* $d(\varphi(x), \varphi(y)) \le Cd(x, y)$ *for all* $x, y \in Y$ *where* $C > 0$ *is a constant.*

(*ii*) [5, *Theorem 1*]. *The unital homomorphism* $T : \mathrm{Lip}_\alpha(X) \to \mathrm{Lip}_\alpha(Y)$ *is compact if and only if* φ *is a supercontraction, i.e.*

$$\lim_{d(x,y)\to 0} \frac{d(\varphi(x), \varphi(y))}{d(x, y)} = 0.$$

(*iii*) [3, *Theorem 2.10 (ii)*]. *Suppose that* $T = uC_\varphi : \mathrm{Lip}_\alpha(X) \to \mathrm{Lip}_\alpha(Y)$ *is a weighted composition operator. If* $u \in \mathrm{lip}_\alpha(Y)$ *and* φ *is a supercontraction on* $coz(u) = \{x : u(x) \neq 0\}$, *then* T *is compact.*

Proof. i). It is known that if $T : \mathrm{Lip}_\alpha(X) \to \mathrm{Lip}_\alpha(Y)$ is a unital homomorphism, then for each character m on $\mathrm{Lip}_\alpha(Y)$, $m \circ T$ is a character on $\mathrm{Lip}_\alpha(X)$. We recall that the character space of $\mathrm{Lip}_\alpha(X)$ coincide with X, that is every character on $\mathrm{Lip}_\alpha(X)$ is an evaluation homomorphism at some unique point of X. Thus $\psi = T^*|_Y$ maps Y into X. In fact, there exists a map $\varphi : Y \to X$ such that $\psi(x) = \delta_{\varphi(x)}$ for each $x \in Y$ and

$$Tf(x) = \psi(x)f = \delta_{\varphi(x)}f = f(\varphi(x)),$$

for every $f \in \mathrm{Lip}_\alpha(X)$ and $x \in Y$. Furthermore, by Theorem 2.3, $\psi \in \mathrm{Lip}_\alpha(Y, \mathrm{Lip}_\alpha(X)^*)$ and by using Lemma 2.7, we have

$$\begin{aligned} d^\alpha(\varphi(x), \varphi(y)) &= \frac{1}{2}\|\delta_{\varphi(x)} - \delta_{\varphi(y)}\|(2 + d^\alpha(\varphi(x), \varphi(y))) \\ &\le \frac{1}{2}\|\psi(x) - \psi(y)\|(2 + \mathrm{diam}(X)^\alpha) \\ &\le \frac{1}{2}(2 + \mathrm{diam}(X)^\alpha)p_\alpha(\psi)d^\alpha(x, y) \le Cd^\alpha(x, y), \end{aligned}$$

for each $x, y \in Y$, where $C = \frac{1}{2}(2 + \mathrm{diam}(X)^\alpha)p_\alpha(\psi)$. The converse is clear.

ii). By [1, Theorem 3.5], we have $\mathrm{lip}_\alpha(X)^{**} = \mathrm{Lip}_\alpha(X)$. Let S be the restriction of T to $\mathrm{lip}_\alpha(X)$. Then, by (i), S maps $\mathrm{lip}_\alpha(X)$ into $\mathrm{lip}_\alpha(Y)$ and $T = S^{**}$. By [2, Theorem VI.5.2], T is compact if and only if S is compact. Using Lemma 2.7, we have

$$\frac{d^\alpha(\varphi(x), \varphi(y))}{d^\alpha(x, y)} = \frac{2 + d^\alpha(\varphi(x), \varphi(y))}{2} \frac{\|\psi(x) - \psi(y)\|}{d^\alpha(x, y)}.$$

Hence, the result follows from Theorem 2.4.

iii). In this case $\psi(x) = u(x)\delta_{\varphi(x)}$ for each $x \in Y$. Thus for every $x, y \in Y$,

$$\begin{aligned}\frac{\|\psi(x)-\psi(y)\|}{d^\alpha(x,y)} &= \frac{\|u(x)\delta_{\varphi(x)} - u(y)\delta_{\varphi(y)}\|}{d^\alpha(x,y)} \\ &\leq \frac{|u(x)-u(y)|}{d^\alpha(x,y)}\|\delta_{\varphi(x)}\| + |u(y)|\frac{\|\delta_{\varphi(x)} - \delta_{\varphi(y)}\|}{d^\alpha(x,y)} \\ &= \frac{|u(x)-u(y)|}{d^\alpha(x,y)} + |u(y)|\frac{d^\alpha(\varphi(x),\varphi(y))}{d^\alpha(x,y)}\frac{2}{2+d^\alpha(\varphi(x),\varphi(y))}.\end{aligned}$$

Considering two cases either both x and y belong to $coz(u)$ or at least one of them, say y, does not belong to $coz(u)$, one can conclude that $\lim_{d(x,y)\to 0} \frac{\|\psi(x)-\psi(y)\|}{d^\alpha(x,y)} = 0$. As in the proof of part (ii), let S be the restriction of T to $\mathrm{lip}_\alpha(X)$. Then, by Theorem 2.4, S and consequently T is compact. □

Acknowledgement

The authors would like to thank the referees for useful comments and suggestions.

References

[1] W. G. Bade, P. C. Curtis, Jr. and H. G. Dales, Amenability and weak amenability for Beurling and Lipschitz algebras, *Proc. London Math. Soc. (3)* **55** (1987), no. 2, 359–377.
[2] N. Dunford and J. T. Schwartz, Linear Operators, Part I, General Theory, John Wiley & Sons, Inc., New York, 1988.
[3] K. Esmaeili and H. Mahyar, Weighted composition operators between vector-valued Lipschitz function spaces, *Banach J. math. Anal.* **7** (2013), no. 1, 59–72.
[4] A. Jiménez-Vargas, M. Lacruz and M. Villegas-Vallecillos, Essential norm of composition operators on Banach spaces of Hölder functions, *Abstr. Appl. Anal.* (2011), Article ID 590853, 13 pages.
[5] H. Kamowitz and S. Scheinberg, Some properties of endomorphisms of Lipschitz algebras, *Studia Math.* **24** (1990), no. 3, 255–261.
[6] K. de Leeuw, Banach spaces of Lipschitz functions, *Studia Math.* **21** (1961/1962) 55–66.
[7] D. R. Sherbert, Banach algebras of Lipschitz functions, *Pacific J. Math.* **13** (1963) 1387–1399.
[8] D. R. Sherbert, The structure of ideals and point derivations in Banach algebras of Lipschitz functions, *Trans. Amer. Math. Soc.* **111** (1964) 240–272.
[9] N. Weaver, Lipschitz Algebras, World Scientific Publishing Co., Inc., River Edge, 1999.

(Azin Golbaharan) Department of Mathematics, Faculty of Mathematical Sciences and Computer, Kharazmi University, 50, Taleghani Ave., 15618, Tehran, Iran.
E-mail address: golbaharan_azin@yahoo.com

(Hakimeh Mahyar) Department of Mathematics, Faculty of Mathematical Sciences and Computer, Kharazmi University, 50, Taleghani Ave., 15618, Tehran, Iran.
E-mail address: mahyar@khu.ac.ir