Title:
A generalization of \(\oplus\)-cofinitely supplemented modules

Author(s):
B. Koşar and B. N. Türkmen
A GENERALIZATION OF ☐-COFINITELY SUPPLEMENTED MODULES

B. KOŞAR AND B. N. TÜRKMEN*

(Communicated by Omid Ali S. Karamzadeh)

Abstract. We say that a module \( M \) is a \( cns \)-module if, for every cofinite submodule \( N \) of \( M \), there exist submodules \( K \) and \( K' \) of \( M \) such that \( K \) is a supplement of \( N \), and \( K, K' \) are mutual supplements in \( M \). In this article, the various properties of \( cns \)-modules are given as a generalization of \( ☐-\)cofinitely supplemented modules. In particular, we prove that a \( \pi \)-projective module \( M \) is a \( cns \)-module if and only if \( M \) is \( ☐-\)cofinitely supplemented. Finally, we show that every free \( R \)-module is a \( cns \)-module if and only if \( R \) is semiperfect.

Keywords: Supplements, cofinite submodule, (☐-)cofinitely supplemented module.


1. Introduction

Throughout this paper, it is assumed that \( R \) is an associative ring with identity and all modules are unital right \( R \)-modules. Let \( R \) be such a ring and let \( M \) be an \( R \)-module. The notation \( K \subseteq M \) (\( K \subset M \)) means that \( K \) is a (proper) submodule of \( M \). A submodule \( N \) of \( M \) is called cofinite in \( M \) if the factor module \( M/N \) is finitely generated. A module \( M \) is called extending if every submodule is essential in a direct summand of \( M \) [3]. Here a submodule \( K \subseteq M \) is said to be essential in \( M \), denoted as \( K \leq M \), if \( K \cap N \neq 0 \) for every non-zero submodule \( N \subseteq M \). Dually a proper submodule \( S \) of \( M \) is called small (in \( M \)), denoted as \( S << M \), if \( M \neq S + L \) for every proper submodule \( L \) of \( M \) [12]. The Jacobson radical of \( M \) will be denoted by \( \text{Rad}(M) \). It is known that \( \text{Rad}(M) \) is the sum of all small submodules of \( M \).

A non-zero module \( M \) is said to be hollow if every proper submodule of \( M \) is small in \( M \), and it is said to be local if it is hollow and is finitely generated. A module \( M \) is local if and only if it is finitely generated and \( \text{Rad}(M) \) is maximal.
A generalization of $\oplus$-cofinitely supplemented modules

A ring $R$ is said to be local if $J$ is maximal, where $J$ is the Jacobson radical of $R$.

An $R$-module $M$ is called supplemented if every submodule of $M$ has a supplement in $M$. Here a submodule $K \subseteq M$ is said to be a supplement of $N$ in $M$ if $K$ is minimal with respect to $N + K = M$, or equivalently, $N + K = M$ and $N \cap K \ll K$ [12]. A supplement submodule $X$ of $M$ is then defined when $X$ is a supplement of some submodule of $M$. Every direct summand of a module $M$ is a supplement submodule of $M$, and supplemented modules are a generalization of semisimple modules. In addition, every factor module of a supplemented module is again supplemented. For a module $M$, two submodules $N$ and $K$ of $M$ are called mutual supplements if, $M = N + K$, $N \cap K \ll K$ and $N \cap K \ll N$ [3]. Alizade et al. [1] have defined cofinitely supplemented modules as a proper generalization of supplemented modules. They call a module $M$ cofinitely supplemented if every cofinite submodule $N$ of $M$ has a supplement in $M$, and give characterizations of these modules over any ring and commutative domain (see [1]).

A module $M$ is called lifting (or $D_1$-module) if, for every submodule $N$ of $M$, there exists a direct summand $K$ of $M$ such that $K \leq N$ and $\frac{N}{K} \ll \frac{M}{K}$. Mohamed and Müller has generalized the concept of lifting modules to $\oplus$-supplemented modules. $M$ is called $\oplus$-supplemented if every submodule $N$ of $M$ has a supplement that is a direct summand of $M$ [7]. Clearly every $\oplus$-supplemented module is supplemented, but a supplemented module need not be $\oplus$-supplemented in general (see [7, Lemma A.4 (2)]). It is shown in [7, Proposition A.7 and Proposition A.8] that if $R$ is a Dedekind domain, every supplemented $R$-module is $\oplus$-supplemented. Hollow modules are $\oplus$-supplemented.

In [4], Çağşıcı and Pancar call a module $M$ $\oplus$-cofinitely supplemented if every cofinite submodule of $M$ has a supplement that is a direct summand of $M$. They gave in the same paper some properties of these module. In particular, it is shown in [4, Theorem 2.9] that every free $R$-module is $\oplus$-cofinitely supplemented if and only if $R$ is semiperfect. Now we generalize these modules, and so we define cms-modules.

In this paper, we provide the some properties of cms-modules. Some examples are given to separate cms-modules and $\oplus$-cofinitely supplemented modules. We prove that a $\pi$-projective module $M$ is a cms-module if and only if $M$ is $\oplus$-cofinitely supplemented. In Proposition 2.5, we show that if $M$ is cofinitely supplemented and $f$-supplemented, then it is a cms-module. We obtain a new characterization of semiperfect rings by using this result. We give some conditions for factor modules (in particular, cofinite direct summands) of a cms-module to be a cms-module. We prove that a refinable module $M$ is $\oplus$-cofinitely supplemented if and only if $M$ is a cms-module if and only if it is cofinitely supplemented.
2. CMS-MODULES

In this section, we define the concept of cms-modules and give various properties of them.

**Definition 2.1.** Let $M$ be a module. Then $M$ is called a **cms-module** if, for every cofinite submodule $N$ of $M$, there exist submodules $K$ and $K'$ of $M$ such that $K$ is a supplement of $N$, and $K$, $K'$ are mutual supplements in $M$.

From the above definition it is clear that every supplemented module is a cms-module. But every cms-module is not always supplemented. For example, let $R$ (e.g. $\mathbb{Z}$) be a non-local Dedekind domain which is not a field and $Q$ be a quotient field of $R$. Consider the right $R$-module $M = Q^{(I)}$, where $I$ is any index set. Since $M$ has not any maximal submodule, $M$ is a unique cofinite submodule of $M$. So $M$ is a cms-module. Suppose that $M$ is supplemented. Then $Q$ is supplemented as a factor module of $M$. By [13], this implies that $R$ is local, a contradiction. Therefore $M$ is not supplemented. It is easy to see that every finitely generated cms-module is supplemented.

Resulting from all direct summands are mutual supplements to each other, every $\oplus$-cofinitely supplemented module is a cms-module. Under given definitions, we clearly have the following implication on modules:

\[\begin{array}{c}
\oplus \text{ – cofinitely supplemented modules} \\
cms \text{ – modules} \\
cofinitely supplemented modules
\end{array}\]

But we shall give example of a cms-module which is not $\oplus$-cofinitely supplemented.

**Example 2.2.** (See [6]) Let $F$ be any field and $R = F[[X,Y]]$ the ring of formal power series over $F$ indeterminates $X, Y$. Then $R$ is a local commutative Noetherian domain. Now suppose that $M$ is the Noetherian right $R$-module $J$. Therefore $M = XR + YR$. By [12, 42.6], since $R$ is a local ring, every submodule of $M$ is supplemented and so it is a cms-module. It follows from [6, Corollary 2.4] that $M$ is not $\oplus$-supplemented. Since $M$ is finitely generated, $M$ is not $\oplus$-cofinitely supplemented.

In [9, 1.4], a module $M$ is called uniserial if its lattice of submodules is a chain. $M$ is said to be serial if $M$ is a direct sum of uniserial modules. A ring $R$ is right (left) serial if the module $R_R$ ($R_R$) is serial. In [3, 29.10] a ring $R$ is...
artinian serial with $J^2 = 0$ if and only if every $R$-module is lifting if and only if every $R$-module is extending.

**Example 2.3.** (See [5]) Let $R$ be a local artinian ring with radical $W$ such that $W^2 = 0$, $Q = \frac{R}{W}$ is commutative, $\dim(QW) = 1$, and $\dim(W_Q) = 3$. Then $R$ is left serial but not right serial. Let $W = w_1R \oplus w_2R \oplus w_3R$. By [5, Proposition 4.9], there exist five isomorphism types of indecomposable $R$-modules defined in [5, Lemmas 4.1\&4.2], where $X_5 = R[0, w_1R] \oplus (w_2, w_3)R$ is an indecomposable $R$-module of length 5 which is not left local. Hence, $X_5$ is not $\oplus$-supplemented by [6, Lemma 3.1]. Since $X_5$ is 2-generated, it is not $\oplus$-cofinitely supplemented. Applying [12, 42.6], since $R$ is local, we obtain that $X_5$ is supplemented. Therefore $X_5$ is a cms-module.

A module $M$ is called $\pi$-projective if, for every two submodules $U, V$ of $M$ and identity homomorphism $i_M : M \rightarrow M$ with $M = U + V$, there exists $f \in \text{End}(M)$ with $\text{Im}(f) \subseteq U$ and $\text{Im}(i_M - f) \subseteq V$ [12, 41.13].

**Proposition 2.4.** Let $M$ be a $\pi$-projective module. If $M$ is a cms-module, then $M$ is a $\oplus$-cofinitely supplemented module.

**Proof.** Let $N$ be any cofinite submodule of $M$. By the hypothesis, there exist submodules $K$ and $K'$ of $M$ such that $K$ is a supplement of $N$, and $K, K'$ are mutual supplements in $M$. Since $M$ is a $\pi$-projective module, in accordance with [3, 20.9], $K \cap K' = 0$ and hence $M = K \oplus K'$. Therefore $M$ is a $\oplus$-cofinitely supplemented module. \qed

Recall from [12, 41.1] that a module $M$ is $f$-supplemented if every finitely generated submodule of $M$ has a supplement in $M$.

**Proposition 2.5.** Let $M$ be a cofinitely supplemented module.

1. If $M$ is $f$-supplemented, then it is cms.
2. If every proper cofinite submodule of $M$ is supplemented, then $M$ is a cms-module.

**Proof.** (1) For any cofinite submodule $U \subseteq M$, it follows from assumption that we can write $M = U + V$ and $U \cap V \ll V$ for some submodule $V \subseteq M$. Now

$$\frac{M}{V} \cong \frac{V}{U \cap V}$$

is finitely generated. Since $U \cap V$ is a small submodule of $V$, we obtain that $V$ is finitely generated. By (1), $V$ has a supplement in $M$, say $V'$. Then, $M = V + V'$ and $V \cap V' \ll V'$, by [12, 41.1(5)], we deduce that $V \cap V' \ll V$. Hence, $V$ and $V'$ are mutual supplements in $M$.

(2) Let $U$ be any cofinite submodule of $M$. Since $M$ is cofinitely supplemented module, there exists a submodule $V \subseteq M$ that $M = U + V$ and $U \cap V \ll V$. By the hypothesis, $U = (U \cap V) + T$ and $(U \cap V) \cap T = V \cap T \ll T$ for some submodule $T \subseteq U$. Now $M = U + V = (U \cap V) + T + V = V + T$. 


Note that $V \cap T \ll M$. Since $V$ is a supplement of $U$ in $M$, we have $V \cap T \ll V$ by [12, 41.1(5)]. Therefore $M$ is a cms-module.

We don’t know whether or not any factor module of a cms-module is a cms-module. But we prove that a factor module of a cms-module by a fully invariant submodule is a cms-module in the following theorem.

Recall from [12, 6.4] that a submodule $U$ of an $R$-module $M$ is called fully invariant if $f(U)$ is contained in $U$ for every $R$-endomorphism $f$ of $M$. A module $M$ is called duo, if every submodule of $M$ is fully invariant [8].

**Theorem 2.6.** Let $M$ be a cms-module and $N$ be a fully invariant submodule of $M$. Then $\frac{M}{N}$ is a cms-module.

**Proof.** Let $\frac{U}{N}$ be any cofinite submodule of $\frac{M}{N}$.

\[ \frac{M}{N} \cong \frac{U}{N} \]

is finitely generated. So $U$ is cofinite in $M$. Since $M$ is a cms-module, then there exist submodules $V$ and $V'$ of $M$ such that $V$ is a supplement of $U$, and $V$, $V'$ are mutual supplements in $M$. It is clear that $\frac{V+U}{N}$ is a supplement of $\frac{U}{N}$ in $\frac{M}{N}$. Since $V \cap V' \ll V'$, $V \cap V' \ll V$ and $N$ is a fully invariant submodule of $M$, then $\frac{V+U}{N} \cap \frac{V'+U}{N} \subseteq \frac{(V \cap V')+U}{N} \ll \frac{V+U}{N}$ and $\frac{V+U}{N} \cap \frac{V'+U}{N} \subseteq \frac{(V \cap V')+U}{N} \ll \frac{V'+U}{N}$. Thus $M$ is a cms-module. \(\square\)

Since $\text{Rad}(M)$ is a fully invariant submodule of a module $M$, we obtain the following corollary as an immediate consequence of Theorem 2.6.

**Corollary 2.7.** If $M$ is a cms-module, then every cofinite submodule of $\frac{M}{\text{Rad}(M)}$ is a direct summand.

**Proposition 2.8.** Let $0 \rightarrow N \rightarrow M \rightarrow K \rightarrow 0$ be a short exact sequence such that $N$ is small in a module $M'$, whenever $N \subseteq M'$. If $K$ is a cms-module, then $M$ is a cms-module.

**Proof.** Without loss of generality we will assume that $N \subseteq M$. Then, $\frac{M}{N} \cong K$ is a cms-module. Let $U$ be any cofinite submodule of $M$,

\[ \frac{M}{U+N} \cong \frac{M}{U+N} \]

and, so

\[ \frac{M}{U+N} \cong \frac{M}{U+N} \]

is finitely generated. Then $\frac{U+N}{N}$ is a cofinite submodule of $\frac{M}{N}$. Since $\frac{M}{N}$ is cms-module, then there exist submodules $\frac{T}{N}$ and $\frac{T'+N}{N}$ of $\frac{M}{N}$ such that $\frac{T}{N}$ is a supplement of $\frac{U+N}{N}$, and $\frac{T'+N}{N}$, $\frac{T'+N}{N}$ are mutual supplements in $M$. It is
clear that $M = U + N + T = U + T$ and $\frac{U+N}{N} \cap \frac{T}{T} = \frac{(U \cap T) + N}{N} \ll \frac{T}{T}$. By the hypothesis $N \ll T$. Note that $M = T + T'$. Then $U \cap T \ll T$ and $T \cap T' \ll T$. Again by the hypothesis, $N \ll T'$, from which it follows that $T \cap T' \ll T'$. Therefore $M$ is a cms-module.

Recall from [11, 1.11] that a module $M$ is said to be distributive if $(X + Y) \cap Z = (X \cap Z) + (Y \cap Z)$ for any submodules $X, Y,$ and $Z$ of $M$. This means that the submodule lattice $\text{Lat}(M)$ is distributive.

**Proposition 2.9.** Let $M$ be a distributive cms-module and $N$ be a cofinite direct summand of $M$. Then $N$ is a cms-module.

**Proof.** Let $L$ be any cofinite submodule of $N$. Then $\frac{N}{L}$ is finitely generated. Since $N$ is a direct summand of $M$, there exists a finitely generated submodule $N'$ of $M$ such that $M = N \oplus N'$. Then $N' \cong \frac{N}{L}$ is finitely generated. Furthermore $M = N + N' + L$ and $N \cap (N' + L) = L$. Since

$$\frac{(N' + L)}{L} \cong \frac{N'}{N \cap L} = \frac{N'}{N} \cong N'$$

is finitely generated, then $\frac{M}{L} = \frac{N}{L} + \frac{N' + L}{L}$ is finitely generated. Therefore $L$ is a cofinite submodule of $M$. Since $M$ is a cms-module, there exist submodules $L'$ and $K'$ of $M$ such that $L'$ is a supplement of $L$, and $L', K'$ are mutual supplements in $M$. Then we have $N = L + (N \cap L')$ and $N \cap (N \cap L') \ll L'$. Since $M$ is a distributive module, $L' = (N \cap L') \oplus (N' \cap L')$. It follows that $L \cap (N \cap L') \ll N \cap L'$. Since $M$ is a distributive module, $K' = (N \cap K') \oplus (N' \cap K')$. It follows that $N = (N \cap L') + (N \cap K')$. So we have $(N \cap L') \cap (N \cap K') \ll N \cap K'$ and $(N \cap L') \cap (N \cap K') \ll N \cap L'$ due to the inequality $(N \cap L') \cap (N \cap K') \leq L \cap K' \ll K'$. Therefore $N$ is a cms-module. $\square$

**Theorem 2.10.** Let $\{M_i\}_{i \in I}$ be a family of cms-modules and $M = \oplus_{i \in I} M_i$. If every cofinite submodule of $M$ is fully invariant, then $M$ is a cms-module.

**Proof.** Let $N$ be any cofinite submodule of $M$. Then $\frac{M}{N}$ is finitely generated. By the hypothesis, $N = \oplus_{i \in I} (N \cap M_i)$. Note that $\oplus_{i \in I} \frac{(N \cap M_i)}{N \cap M_i} = \frac{M}{N} \cong \frac{M}{N}$ is finitely generated. Then for every $i \in I$, $\frac{M_i}{N \cap M_i}$ is finitely generated. Since for every $i \in I$, $M_i$ is a cms-module, there exist submodules $K_i$ and $T_i$ of $M_i$ such that $K_i$ is a supplement of $N \cap M_i$, and $K_i$ and $T_i$ are mutual supplements in $M_i$. Let $\oplus_{i \in I} K_i = K$ and $\oplus_{i \in I} T_i = T$, and $M = \oplus_{i \in I} M_i = \oplus_{i \in I} (N \cap M_i) + \oplus_{i \in I} K_i = N + K$, and $N \cap K = \oplus_{i \in I} (N \cap M_i) \cap \oplus_{i \in I} K_i \subseteq \oplus_{i \in I} [(N \cap M_i) \cap K_i] = \oplus_{i \in I} (N \cap K_i) \ll K$. It follows that $M = K + T$, $K \cap T \ll K$ and $K \cap T \ll T$. Therefore $M$ is a cms-module. $\square$

**Corollary 2.11.** Let $\{M_i\}_{i \in I}$ be a family of cms-modules and $M = \oplus_{i \in I} M_i$. If $M$ is a duo module, then $M$ is a cms-module.
Lemma 2.12. Let $R$ be a ring with identity. Then the $R$-module $RR$ is a cms-module if and only if every free $R$-module is a cms-module.

Proof. ($\Rightarrow$) Let $M$ be a free $R$-module. Suppose that $RR$ is a cms-module. Since $R$ is $\pi$-projective, $RR$ is a $\oplus$- cofinitely supplemented module by Proposition 2.4. It follows that $M$ is $\oplus$- cofinitely supplemented module by [4, Lemma 2.8]. So $M$ is a cms-module.

($\Leftarrow$) is obvious. \qed

For modules $M$ and $P$, let $f : P \to M$ be an epimorphism. $f$ is called cover if $\ker(f)$ is small in $P$. A projective module $P$ together with a cover $f : P \to M$ is called a projective cover of $M$. By [2, Theorem 2.1], rings whose (finitely generated) modules have a projective cover are (semi)perfect.

Theorem 2.13. Let $R$ be a ring with identity. Then the following statements are equivalent.

1. $R$ is semiperfect;
2. $RR$ is $\oplus$- cofinitely supplemented;
3. every free $R$-module is $\oplus$- cofinitely supplemented;
4. $RR$ is a cms-module;
5. every free $R$-module is a cms-module.
6. every finitely generated $R$-module is a cms-module.

Proof. (1) $\iff$ (2) $\iff$ (3) It follows from [4, Theorem 2.9].

(3) $\iff$ (4) $\iff$ (5) By Lemma 2.12 and Proposition 2.4.

(1) $\Rightarrow$ (6) Let $R$ be a semiperfect ring. By [12, 42.6], every finitely generated $R$-module is supplemented. Thus every finitely generated $R$-module is a cms-module.

(6) $\Leftarrow$ (1) Suppose that every finitely generated $R$-module is a cms-module. In particular $RR$ is a cms-module. Since $RR$ is finitely generated, then $RR$ is supplemented. By [12, 42.6], $R$ is semiperfect. \qed

Recall from [12, 21.4] that a submodule $N$ of a module $M$ is called radical if $N$ has no maximal submodule, that is, $N = \text{Rad}(N)$. For a module $M$, $P(M)$ will indicate the sum of all radical submodules of $M$. If $P(M) = 0$, $M$ is called reduced. Note that $P(M)$ is the largest radical submodule of $M$.

Lemma 2.14. Let $R$ be a Dedekind domain and $M$ be an $R$-module. Then $P(M)$ is a cms-module.

Proof. Let $R$ be a Dedekind domain, and so $R$ is noetherian. Here, $P(M)$ denotes the divisible part of $M$. Then $P(M)$ is injective by [10, proposition 2.10], hence $M = P(M) \oplus N$ for some submodule $N$ of $M$. In this case $N$ is called the reduced part of $M$. By [1, Lemma 4.4], $P(M)$ is the only cofinite submodule of $P(M)$. Thus $P(M)$ is a cms-module. \qed
**Proposition 2.15.** Let $R$ be a Dedekind domain, $M$ be a duo $R$-module and $N$ be the reduced part of $M$. Then $M$ is a cms-module if and only if $N$ is a cms-module.

**Proof.** ($\Rightarrow$) Since $P(M)$ is a fully invariant submodule, then $\frac{M}{P(M)} \cong N$ is a cms-module by Theorem 2.6.

($\Leftarrow$) It is clear by Corollary 2.11 and Lemma 2.14. \qed

In [3, 11.26], an $R$-module $M$ is called refinable if for any submodules $U, V \subseteq M$ with $M = U + V$, there exists a direct summand $U'$ of $M$ with $U' \subseteq U$ and $M = U' + V$. Every finitely generated regular module is refinable. Note that every direct summand of a refinable module is refinable.

**Theorem 2.16.** Let $M$ be a refinable module. Then the following statements are equivalent.

1. $M$ is $\oplus$-cofinitely supplemented;
2. $M$ is a cms-module;
3. $M$ is cofinitely supplemented.

**Proof.** (1) $\Rightarrow$ (2) are obvious.

(3) $\Rightarrow$ (1) Let $N$ be any cofinite submodule of $M$. Since $M$ is a cofinitely supplemented module, then there exists a submodule $K$ of $M$ such that $M = N + K$ and $N \cap K \ll K$. So we have $N \cap L \ll L$. Thus $M$ is a $\oplus$-cofinitely supplemented module. \qed

**Corollary 2.17.** Let $M$ be a finitely generated refinable module. Then the following statements are equivalent.

1. $M$ is $\oplus$-supplemented;
2. $M$ is $\oplus$-cofinitely supplemented;
3. $M$ is a cms-module;
4. $M$ is cofinitely supplemented;
5. $M$ is supplemented;
6. every maximal submodule of $M$ has a supplement.

**Corollary 2.18.** Let $M$ be a refinable module. $M = \oplus_{i \in I} M_i$. Suppose that for every submodule $N$ of $M$ there is a cofinite submodule $L$ of $M$ such that $N = L + T$ or $L = N + T$ for some $T \ll M$. Then $M$ is a cms-module if and only if $M_i$ is a cms-module.

Finally, we have the following fact.

**Corollary 2.19.** Consider the following statements for a ring $R$.

1. $R$ is right perfect.
2. Every right $R$-module is cms.
3. $R$ is semiperfect.
Proof. (1) ⇒ (2) Since every module over a right perfect ring is supplemented, it is clear.

(2) ⇒ (3) It follows from Theorem 2.13.

Acknowledgments

We would like to thank the referee for the valuable suggestions and comments which improved the revision of the paper.

References


(Berna Koşar) Ondokuz Mayıs University, Samsun, Turkey.
E-mail address: bernako@omu.edu.tr

(Burcu Nişancı Türkmen) Amasya University, Faculty of Art and Science, İpekköy, Amasya, Turkey.
E-mail address: burcunisancie@hotmail.com