ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the Iranian Mathematical Society

Vol. 42 (2016), No. 1, pp. 101–112

Title:

Amenability of groups and semigroups characterized by configuration

Author(s):

A. Tavakoli and A. Rejali

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 1, pp. 101–112
Online ISSN: 1735-8515

AMENABILITY OF GROUPS AND SEMIGROUPS CHARACTERIZED BY CONFIGURATION

A. TAVAKOLI* AND A. REJALI

(Communicated by Hamid Reza Ebrahimi Vishki)

Abstract. In 2005, Abdollahi and Rejali, studied the relations between paradoxical decompositions and configurations for semigroups. In the present paper, we introduce another concept of amenability on semigroups and groups which includes amenability of semigroups and inner-amenability of groups. We have the previous known results to semigroups and groups satisfying this concept.
Keywords: Amenability, configuration, paradoxical decomposition, semigroup
MSC(2010): Primary: 22A05; Secondary: 43A07.

1. Introduction

The notion of a configuration for groups was first introduced by Rosenblatt and Willis in [6], but here, the definition is changed to another form.
Let G be a finitely generated group and F be a non-empty subset of the set $S(G)$ of all bijective maps on G. Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be a sequence in F such that the subgroup $< F >$ generated by F in $S(G)$, is equal to $< \varphi_1, \ldots, \varphi_n >$ and let $\mathcal{E} = \{E_1, \ldots, E_m\}$ be a partition of G. An $(n+1)$-tuple $C = (c_0, \ldots, c_n)$, where $c_i \in \{1, \ldots, m\}$ for each $i \in \{0, 1, \ldots, n\}$, is called an F-configuration corresponding to the configuration pair $(\varphi, \mathcal{E})$, if there exists an element $x \in G$ with $x \in E_{c_0}$ such that $\varphi_i(x) \in E_{c_i}$, for each $i \in \{1, \ldots, n\}$. The set of all F-configurations corresponding to the configuration pair $(\varphi, \mathcal{E})$ will be denoted by $Con_F(\varphi, \mathcal{E})$.
Let $x_0(C) = E_{c_0} \cap \varphi_1^{-1}(E_{c_1}) \cap \ldots \cap \varphi_n^{-1}(E_{c_n})$ and $x_j(C) = \varphi_j(x_0(C))$, for $C \in Con_F(\varphi, \mathcal{E})$. Then the F-configuration equation corresponding to the configuration pair $(\varphi, \mathcal{E})$ is the system of equations

$$\sum\{f_C \mid \quad x_0(C) \subseteq E_i\} = \sum\{f_C \mid \quad x_j(C) \subseteq E_i\}, \tag{1.1}$$

Article electronically published on February 22, 2016.
Received: 8 February 2014, Accepted: 13 November 2015.
*Corresponding author.

where f_C is the variable corresponding to the configuration C. This system of equations will be denoted by $Eq_F(\varphi, \mathcal{E})$. In this case, this equation system is equivalent to a matrix equation as

$$AX = 0, \tag{1.2}$$

where A is an $nm \times |Con_F(\varphi, \mathcal{E})|$ matrix whose entries are 0, 1 or -1 and X is the vector $[f_C]$, where C runs over $Con_F(\varphi, \mathcal{E})$.
A solution $[f_C]$ to $Eq_F(\varphi, \mathcal{E})$ satisfying $\sum_C \{f_C \mid C \in Con_F(\varphi, \mathcal{E})\} = 1$ and $f_C \geq 0$, for all $C \in Con_F(\varphi, \mathcal{E})$ will be called a normalized solution of the equations system (1.1). The corresponding matrix form whose solution is normalized, has the form $AX = B$, where A is an $(nm+1) \times |Con_F(\varphi, \mathcal{E})|$ matrix whose entries are 0, 1 or -1 and all entries of the last row of A are 1. X is the vector $[f_C]$ and B is the vector whose last entry is 1 and all others are 0. It is well known that, if $A = [a_{i,j}]$, then $a_{i,j} = 1$ [resp. $a_{i,j} = -1$] if and only if $x_i(C) \subseteq E_j$ and $x_0(C) \nsubseteq E_j$ [resp. $x_i(C) \nsubseteq E_j$ and $x_0(C) \subseteq E_j$], for some $C \in Con_F(\varphi, \mathcal{E})$; otherwise $a_{i,j} = 0$.
By a non-zero solution of $Eq_F(\varphi, \mathcal{E})$, we mean a solution $\{f_C \mid C \in Con_F(\varphi, \mathcal{E})\}$ of $Eq_F(\varphi, \mathcal{E})$ such that $f_C \neq 0$ for some $C \in Con_F(\varphi, \mathcal{E})$. We show that (see proposition 2.1) the equation in matrix form has a non-zero solution if and only if the latter has a normalized solution. It is easy to see that a matrix equation $AX = 0$ has a non-zero solution if and only if rank(A) is less than the number of columns of A. Therefore the matrix equation (1.2) has no non-zero solution if and only if $rank(A) \leq |Con_F(\varphi, \mathcal{E})|$.
The relation between amenability and configuration of a group was studied in [6] and [7]. Here, we introduce the concept of F-amenability of a group.

Definition 1.1. A group G is called F-amenable, if there exist an F-invariant mean M on $\ell_\infty(G)$ that is $M(f \circ \varphi) = M(f)$, for all $f \in \ell_\infty(G)$ and $\varphi \in F$, where $\ell_\infty(G)$ denotes the set of all real valued bounded functions on G.

Now let G be a finitely generated group and $L(G) = \{\lambda_x : x \in G\}$, where $\lambda_x : G \to G$ is the left translation $y \mapsto xy$ for each $y \in G$, and $I(G) = \{I_x : x \in G\}$ where $I_x : G \to G$ is the inner automorphism $y \mapsto x^{-1}yx$. Then, according to our terminology, G is $L(G)$-amenable [$I(G)$-amenable] if and only if G is amenable [resp. inner amenable]. In general, inner amenability is much weaker than amenability. So, F-amenability does not imply amenability.
The configuration which introduced in [6] can be obtained as an important special case of our notion. In fact, Rosenblatt and Willis studied

$$Con(G) = \{Con_F(\varphi, \mathcal{E}) | F \text{ is a finite subset of } L(G) \text{ s.t. } \lambda(G) =< F >\}.$$

Remark 1.2. Let $F =< \varphi_1, \ldots, \varphi_n >$, for some $\varphi_i \in S(G)$. Then each $\varphi \in F$ is a finite product of φ_j and φ_j^{-1}. Let M be a $\{\varphi_1, \ldots, \varphi_n\}$-invariant mean on $\ell_\infty(G)$. Then

$$M(f) = M((f \circ \varphi_j^{-1}) \circ \varphi_j) = M(f \circ \varphi_j^{-1}),$$

for all $f \in \ell_\infty(G)$ and $j \in \{1, 2, \ldots, n\}$. Therefore M is an F-invariant mean on $\ell_\infty(G)$.
Now suppose that F is a non-empty subset of $S(G)$, not necessarily finite. We have the following two facts.
(1) if M is an F-invariant mean on $\ell_\infty(G)$, then M is an $< F >$-invariant mean on G.
(2) if $F_1 \subseteq F_2$ are non-empty subsets of $S(G)$, then F_2-amenability of G implies F_1-amenability of G.

Lemma 1.3. *Let F be a non-empty subset of $S(G)$, not necessarily finite. The following statements are equivalent.*

(1) *G is F-amenable.*
(2) *G is $< \varphi_1, \ldots, \varphi_n >$-amenable, for all finite subsets $\{\varphi_1, \ldots, \varphi_n\}$ of F.*
(3) *G is $\{\varphi_1, \ldots, \varphi_n\}$-amenable, for all finite subsets $\{\varphi_1, \ldots, \varphi_n\}$ of F.*

Proof. Due to the Remark 1.2, it is sufficient to prove (3)⇒(1).
Let T be the family of all finite non-empty subsets of F. Then for every $C \in \mathrm{T}$, there exists a C-invariant mean M_C on $\ell_\infty(G)$. If T is partially ordered by set inclusion, then, every $M \in w^* - cl\{M_C\}$ is an F-invariant mean on $\ell_\infty(G)$, where $w^* - cl$ means the weak-$*$ closure. □

In [6] it is proved that a finitely generated group G is amenable if and only if each configuration equation associated to a configuration pair in $Con(G)$ has a normalized solution. The link between amenability and normalized solution is seen in [2] and certain group properties which can be characterized by configurations is also studied. In [2] it is asked whether the normalized solution can be replaced by a non-zero solution in the latter. In Section 2 we not only give a positive answer to this question, but also we generalize it for F-amenability.

Definition 1.4. Let $\{A_1, \ldots, A_n; B_1, \ldots, B_m\}$ be a partition of G such that there exist two subsets $\{\varphi_1, \ldots, \varphi_n\}$ and $\{\psi_1, \ldots, \psi_m\}$ of F with the following property:

$$\begin{aligned} G &= A_1 \cup A_2 \cup \ldots \cup A_n \cup B_1 \cup B_2 \cup \ldots B_m \\ &= \varphi_1(A_1) \cup \varphi_2(A_2) \cup \ldots \cup \varphi_n(A_n) \\ &= \psi_1(B_1) \cup \psi_2(B_2) \cup \ldots \cup \psi_m(B_m). \end{aligned}$$

Then we say that G has an F-paradoxical decomposition $(\varphi_i, \psi_j; A_i, B_j)$. In this case, the F-Tarski number of a group G is the minimum of $m+n$, over all possible F-paradoxical decompositions of G; this will be denoted by $\tau_F(G)$. If G has no F-paradoxical decomposition, we put $\tau_F(G) = \infty$.

In Section 3, we study the relation between non-F-amenability and having an F-paradoxical decomposition for a group.
A dynamical system is a triple (G, X, α), where $\alpha : G \to S(X)$ is an action

of a group G on a set X and $S(X)$ is the set of all bijection self-maps of X. The dynamical system (G, X, α) is amenable if there exists a finitely additive probability measure μ defined on the power set $P(X)$ of the space X which is α-invariant, i.e. $\mu(\alpha_g(A)) = \mu(A)$, for all $A \subset X$ and $g \in G$. We know that the dynamical system (G, X, α) is amenable if and only if X has no paradoxical decomposition (see [4]). Let $F = \{\alpha_g | g \in G\}$ and $X = G$. Then the dynamical system (G, X, α) is amenable if and only if G if F-amenable.

2. F-Amenability of Groups

Throughout this section G is a finitely generated group and F is a non-empty subset of all bijective maps on G such that $< F >=< \varphi_1, \dots, \varphi_n >$, where $\varphi_i \in F$, for $i = 1, 2, \dots, n$.

Proposition 2.1. *The following statements are equivalent.*

(1) *G is F-amenable.*
(2) *Each F-configuration equation $Eq_F(\varphi, \mathcal{E})$ has a normalized solution.*
(3) *Each F-configuration equation $Eq_F(\varphi, \mathcal{E})$ has a non-zero solution.*

Proof. (1)$\Rightarrow$ (2) Let M be an F-invariant mean on $\ell_\infty(G)$. Then $f_C = M(\chi_{x_0(C)})$, for $C \in Con_F(\varphi, \mathcal{E})$, is a normalized solution of $Eq_F(\varphi, \mathcal{E})$.
(2)$\Rightarrow$(1) Let (f_C) be a normalized solution of $Eq_F(\varphi, \mathcal{E})$.
Choose $x_C \in x_0(C)$ and define:

$$f_{(\varphi,\mathcal{E})}(x) = \begin{cases} f_C & \text{if } x = x_C, \\ 0 & \text{otherwise} . \end{cases}$$

Then each $M \in w^* - cl\{\hat{f}_{(\varphi,\mathcal{E})}\}$ satisfies $M(f \circ \varphi) = M(f)$, for all $f \in \ell_\infty(G)$ and $\varphi \in F$.
(3)$\Rightarrow$(2) Let $f \in \ell_1(G)$ be a non-zero solution of $Eq_F(\varphi, \mathcal{E})$. Define $\Phi \in \ell_\infty(G)^*$ by $\Phi(h) = \sum_{x \in G} f(x)h(x)$, for $h \in \ell_\infty(G)$. There exist positive linear functionals Φ^+ and Φ^- such that $\Phi = \Phi^+ - \Phi^-$ and $\|\Phi\| = \|\Phi^+\| + \|\Phi^-\|$. Since $\|\Phi\| = \|f\|_1 \neq 0$, so we can assume $\Phi^+ \neq 0$, say. By definition,

$$\Phi^+(g) = \sup\{\Phi(h) : 0 \leq h \leq g\},$$

for any non-negative function g. Furthermore,

$$\begin{aligned} \Phi(\chi_{E_i} o \varphi_j) = \Phi(\chi_{\varphi_j^{-1}(E_i)}) &= \sum_C \{\Phi(\chi_{x_0(C)}) : x_j(C) \subseteq E_i\} \\ &= \sum_C \{f_C : x_j(C) \subseteq E_i\} \\ &= \sum_C \{f_C : x_0(C) \subseteq E_i\} = \Phi(\chi_{E_i}), \end{aligned}$$

for all i and j. Thus $\Phi(h \circ \varphi_j) = \Phi(h)$, for all $h \geq 0$. Therefore:

$$\begin{aligned} \Phi^+(\chi_{\varphi_j^{-1}(E_i)}) &= \sup\{\Phi(h \circ \varphi_j) : 0 \leq h \circ \varphi_j \leq \chi_{E_i} \circ \varphi_j\} \\ &= \sup\{\Phi(h) : 0 \leq h \leq \chi_{E_i}\} = \Phi^+(\chi_{E_i}). \end{aligned}$$

Let $k_C = \Phi^+(\chi_{x_0(C)})/\|\Phi^+\|$, then (k_C) is a normalized solution of $Eq_F(\varphi, \mathcal{E})$.
(2)$\Rightarrow$(3) This is trivial. $\square$

Corollary 2.2. *Let G_1 and G_2 be finitely generated groups such that $Con_{F_1}(G_1) = Con_{F_2}(G_2)$. Then G_1 is F_1-amenable if and only if G_2 is F_2-amenable.*

3. F-Paradoxical Decomposition of Groups

In this section, we generalize Tarski's theorem on amenability for F-amenability of groups. For the special case, set $F = L(G)$.
Let F be a subgroup of $S(G)$ under composition operation and $A, B \subseteq G$. So A and B are F-equidecomposable if there exist partitions $\{A_1, \ldots, A_m\}$ and $\{B_1, \ldots, B_m\}$ of A and B, respectively, and elements $\varphi_i \in F$ such that $\varphi_i(A_i) = B_i$ for all $i = 1, \ldots, m$. If A and B are F-equidecomposable, then we write $A \cong B$. We say that $A \leq B$, if $A \cong C$ for some subset C of B. It is routine to show that "$\cong$" is an equivalence relation on power set $P(G)$. Also a standard Cantor-Bernstein argument shows that $A \leq B$ and $B \leq A$ implies $A \cong B$.
Let $S_{\mathbb{N}}$ be the set of all bijective maps on $\mathbb{N}$. Define $(\varphi, p)(x, n) = (\varphi(x), p(n))$, for $\varphi \in F$ and $p \in S_{\mathbb{N}}$. Let

$$\mathcal{N} = \{C \subseteq G \times \mathbb{N} : C \subseteq B \times F \text{ for some } B \subseteq G \text{ and finite set } F \subseteq \mathbb{N}\}.$$

Then each $N \in \mathcal{N}$ can be written uniquely in the form $N = \bigcup_{i=1}^n C_i \times \{j_i\}$, where $1 \leq j_1 < j_2 < \ldots < j_n$ and $\emptyset \neq C_i \subseteq G$.
Let $N_1 = \bigcup_{i=1}^n C_i \times \{j_i\}$ and $N_2 = \bigcup_{i=1}^n D_i \times \{k_i\}$ be elements of $\mathcal{N}$. Then $N_1 \cong N_2$ if and only if there exist $\varphi_i \in F$ and $p_i \in S_{\mathbb{N}}$ such that $\varphi_i(C_i) = D_i$ and $p_i(j_i) = k_i$, for $i \in \{1, 2, \ldots n\}$. Define $\sum = \frac{\mathcal{N}}{\cong} = \{N^\sim : N \in \mathcal{N}\}$, where $N^\sim$ is the equivalence class of N. Choose $h \in F \times S_{\mathbb{N}}$ such that $h(N_1) \cap N_2 = \emptyset$. Then $\sum$ is an abelian semigroup under addition operation $N_1^\sim + N_2^\sim := (h(N_1) \cup N_2)^\sim$.
Define $\alpha = (G \times \{1\})^\sim$, so $2\alpha = \alpha + \alpha = (G \times \{1\} \cup G \times \{2\})^\sim$.
In the following, we show that, G is F-amenable if and only if $\alpha \neq 2\alpha$. A finitely additive probability measure μ of the power set $P(G)$ is called F-invariant, if $\mu(\phi(A)) = \mu(A)$ for all $A \subseteq G$ and $\phi \in F$.

Lemma 3.1. *The following statements are equivalent.*

(1) *G is F-amenable.*
(2) *There exist an $F \times S_{\mathbb{N}}$-invariant measure μ on $\mathcal{N}$ such that $\mu(G \times \{1\}) = 1$.*
(3) *There exist a homomorphism $f : \sum \to [0, \infty)$ such that $f(\alpha) = 1$.*

(4) $\alpha \neq 2\alpha$.

Proof. (1)⇒(2) let ν be an F-invariant measure on $P(G)$. Define $\mu(N) = \sum_{i=1}^{n} \nu(C_i)$, for each $N = \cup_{i=1}^{n} C_i \times \{j_i\}$ in $\mathcal{N}$. Since $\nu(G) = 1$, we have $\mu(G \times \{1\}) = 1$ and

$$\mu(\varphi \times p(N)) = \mu(\bigcup_{i=1}^{n} \varphi(C_i) \times \{p(j_i)\}) = \sum_{i=1}^{n} \nu(\varphi(C_i)) = \sum_{i=1}^{n} \nu(C_i) = \mu(N).$$

Hence μ is an $F \times S_{\mathbb{N}}$-invariant measure on $\mathcal{N}$.
(ii)⇒(i) Let μ be an $F \times S_{\mathbb{N}}$-invariant measure on $\mathcal{N}$. Then $\nu(A) = \mu(A \times \{1\})$ is an F-invariant measure on $P(G)$. Thus G is F-amenable.
(3)⇒(2) Let $\nu(A) = f(A \times \{1\})^{\sim}$, for $A \subseteq G$. Then

$$\nu(G) = f(G \times \{1\})^{\sim} = f(\alpha) = 1,$$

and

$$\nu(A_1 \cup A_2) = f((A_1 \times \{1\}) \cup (A_2 \times \{1\}))^{\sim} = \nu(A_1) + \nu(A_2),$$

for $A_1, A_2 \subseteq G$ such that $A_1 \cap A_2 = \emptyset$.
(4)⇒(3) Let $T = \{n\alpha : n \in \mathbb{N}\}$ and $F : T \to [0, \infty)$ defined by $F(n\alpha) = n$. Then by a similar argument as in [5], p. 119, $\alpha \neq 2\alpha$ if and only if $k\alpha \neq l\alpha$ whenever $k \neq l$. T is a sub-semigroup of the abelian semigroup $\sum$ and $F(\alpha) = 1$; also $s \leq t$ in T (i.e. $s = t$ or there exists $w \in T$ such that $s + w = t$) implies $F(s) \leq F(t)$; thus F can be extended to a homomorphism $f : \sum \to [0, \infty)$ so that $f(\alpha) = 1$ by [5], p. 117.
(1)⇒(3) Let ν be an F-invariant measure in G. Define $f(N^{\sim}) = \sum_{i=1}^{n} \nu(C_i)$, for $N = \cup_{i=1}^{n} C_i \times \{j_i\}$. Let $N_1 = \cup_{i=1}^{n} C_i \times \{j_i\}$ and $N_2 = \cup_{i=1}^{n} D_i \times \{k_i\}$ and $N_1^{\sim} = N_2^{\sim}$. Then $N_1^{\sim} \cong N_2^{\sim}$, so there exist $\varphi_i \in F$ and $p_i \in S_{\mathbb{N}}$ such that $\varphi_i(C_i) = D_i$ and $p_i(j_i) = k_i$. Hence $f(N_1^{\sim}) = \sum_{i=1}^{n} \nu(C_i) = \sum_{i=1}^{n} \nu(\varphi(C_i)) = f(N_2^{\sim})$, so f is well-defined.
Let $h = \varphi \times p \in F \times S_{\mathbb{N}}$, such that $h(N_1) \cap N_2 = \emptyset$. Then:

$$\begin{aligned} f(N_1^{\sim} + N_2^{\sim}) &= f(\bigcup_{i=1}^{n} \varphi(C_i) \times \{p(j_i)\} \cup D_i \times \{k_i\}) \\ &= \sum_{i=1}^{n} \nu(\varphi(C_i)) + \sum_{i=1}^{n} \nu(D_i) = \sum_{i=1}^{n} \nu(C_i) + \sum_{i=1}^{n} \nu(D_i) \\ &= f(N_1^{\sim}) + f(N_2^{\sim}). \end{aligned}$$

So f is a homomorphism. Clearly, $f(\alpha) = f((G \times \{1\})^{\sim}) = \nu(G) = 1$.
(3)⇒(4) Since $f(\alpha) = 1$, so $f(2\alpha) = 2$. Thus $\alpha \neq 2\alpha$. Hence the proof is complete. □

We now state the main result of this section.

Theorem 3.2. *The following statements are equivalent.*

(1) *G is F-amenable.*

(2) *There exist no F-paradoxical decomposition for G.*

Proof. (1)$\Rightarrow$(2) Suppose not! Let ν be an F-invariant measure for G and $(\varphi_i, \psi_j; A_i, B_j)$ be an F-paradoxical decomposition for G. Then:

$$1 = \nu(G) = \nu(\bigcup_{i=1}^{n} \varphi_i(A_i)) = \sum_{i=1}^{n} \nu(\varphi_i(A_i)) = \sum_{i=1}^{n} \nu(A_i).$$

Similarly, $\sum_{j=1}^{m} \nu(B_j) = 1$. Hence,

$$1 = \nu(G) = \nu(\bigcup_{i=1}^{n} A_i) + \nu(\bigcup_{j=1}^{m} B_j) = 1 + 1 = 2,$$

which is a contradiction.
(2)$\Rightarrow$(1) Suppose not! so by Lemma 3.1, $\alpha = 2\alpha$. Then $G \times \{1\} \cong (G \times \{1\}) \cup (G \times \{2\})$. Thus there exist a partition $\{A_1 \times \{1\}, \ldots, A_n \times \{1\}; B_1 \times \{1\}, \ldots, B_m \times \{1\}\}$ of $G \times \{1\}$ and $(\varphi_i, p_i), (\psi_j, q_j) \in F \times S_{\mathbb{N}}$ such that $p_i(1) = 1$ and $q_j(1) = 2$ for all i and j, so that:

$$G \times \{1\} \bigcup G \times \{2\} = (\bigcup_{i=1}^{n} \varphi_i \times p_i(A_i \times \{1\})) \bigcup (\bigcup_{j=1}^{m} \psi_j \times q_j(B_j \times \{1\})).$$

Thus $G \times \{1\} = \cup_{i=1}^{n} \varphi_i(A_i) \times \{1\}$ and $G \times \{2\} = \cup_{j=1}^{m} \psi_j(B_j) \times \{2\}$. Hence $G = \cup \varphi_i(A_i) = \cup \psi_j(B_j)$. So G has an F-paradoxical decomposition, which is a contradiction. $\square$

Similar to [7], we are interested to construct an F-paradoxical decomposition for non-F-amenable groups by using F-configuration equations.
Let $(\varphi_i, \psi_j; A_i, B_j)$ be an F-paradoxical decomposition of G and $f \in \ell_1^+(G)$. Then:

$$\begin{aligned}
\|f\|_1 &= \sum_{C} \{f_C : C \in Con_F(\varphi, \mathcal{E})\} \\
&= \sum_{C} \sum_{i=1}^{n} \{f_C : x_0(C) \subseteq A_i\} + \sum_{C} \sum_{j=1}^{m} \{f_C : x_0(C) \subseteq B_j\} \\
&= \sum_{C} \sum_{i=1}^{n} \{f_C : x_i(C) \subseteq A_i\} + \sum_{C} \sum_{j=1}^{m} \{f_C : x_j(C) \subseteq B_j\} \\
&= 2 \sum_{C} \{f_C : C \in Con_F(\varphi, \mathcal{E})\} = 2\|f\|_1,
\end{aligned}$$

where $\varphi = (\varphi_1, \ldots, \varphi_n; \psi_1, \ldots, \psi_m)$ and $\mathcal{E} = \{A_1, \ldots, A_n; B_1, \ldots, B_m\}$. Therefore $Eq_F(\varphi, \mathcal{E})$ has no non-zero solution.
Suppose $Eq_F(\varphi, \mathcal{E})$ has no non-zero solution. Suppose $Con_F(\varphi, \mathcal{E}) = \{D_1, \ldots, D_s\}$ such that $E_1 = \cup_{i=1}^{r_1} x_0(D_i), E_2 = \cup_{i=r_1+1}^{r_1+r_2} x_0(D_i)$ and so on. Define $\mathcal{E}' = \{E_i' : i = 1, \ldots, s\}$ where $E_i' = x_0(D_i)$ for each i. Then $Eq_F(\varphi, \mathcal{E}')$

has a non-zero solution. Similarly, if $Eq_F(\varphi, \mathcal{E}')$ has a non-zero solution then $Eq_F(\varphi, \mathcal{E})$ has a non-zero solution.

Question 3.3. *Let $Eq_F(\varphi, \mathcal{E})$ be a system of equations having no non-zero solution for some configuration pair $(\varphi, \mathcal{E})$. How can we "explicitly" construct an F-paradoxical decomposition from $Eq_F(\varphi, \mathcal{E})$?*

It is to be noted that $G =< g_1, g_2, \ldots, g_l >$ is non-amenable if and only if the equation $|g_i^{-1}E_j \cap X| = |E_i \cap X|$, for $1 \leq i \leq l, 1 \leq j \leq m$, has no non-empty finite solution X in G, for some partition $\mathcal{E} = \{E_1, \ldots, E_m\}$.

Example 3.4. [6]. Let $G =< g_1, g_2 >$ be the free group eith two (free) generators g_1, g_2 and E_i be the set of all reduced words starting g_i, for $i = 1, 2$, and $E_3 = G-(E_1 \cup E_2)$. Then $Eq(\varphi, \mathcal{E})$ has no non-zero solution. In comparison to the above notations, let

$$(\varphi_i) = (1, \lambda_{g_1}, \lambda_{g_1}), (\psi_j) = (1, 1, 1, \lambda_{g_2}, \lambda_{g_2})$$

and

$$(A_i) = (E_1', E_2', E_5'), (B_j) = (E_2', E_3', A, E_7', B),$$

for some $A \subseteq E_6'$ and $B = E_6' - A$. Then $(\varphi_i, \psi_j; A_i, B_j)$ is a paradoxical decomposition of G.

4. F-Amenability of Semigroups

In this section, a new type of amenability for semigroups is introduced. Also the notion of an F-paradoxical decomposition for semigroups which was asked by Paterson in special case in [5] p. 120, is defined. We find the relation between the existence of F-paradoxical decompositions and non-F-amenability for semigroups. The definition is almost similar to that of groups, we bring it for completeness.

Let S be a discrete semigroup and $A \subseteq S$. For any map $f : S \to S$ (not necessarily invertible), recall that $f^{-1}(A) = \{t \in S : f(t) \in A\}$. Let $\varphi = (\varphi_1, \ldots, \varphi_n)$ be an n-tuple of the functions (not necessarily invertible) on S and $\mathcal{E}_0 = \{E_1, \ldots, E_m\}$ be a partition of S. An $(n+1)$-tuple $C = (c_0, \ldots, c_n)$, where $c_i \in \{1, \ldots, m\}$ for each $i \in \{0, 1, \ldots, n\}$, is called a configuration corresponding to the configuration pair $(\varphi, \mathcal{E}_0)$, if there exists an element $x \in S$ with $x \in E_{c_0}$ such that $\varphi_i(x) \in E_{c_i}$, for each $i \in \{1, \ldots, n\}$. The set of all configurations corresponding to the configuration pair $(\varphi, \mathcal{E}_0)$ will be denoted by $Con(\varphi, \mathcal{E}_0)$. Let $\mathcal{E}_i = \{\varphi_i^{-1}(E_j) : j \in \{1, \ldots, m\}\}$, for each $i \in \{1, \ldots, n\}$. Then $\mathcal{E}_i$ is a partition of S for each $i = 1, 2, \ldots, n$. (We remove empty elements from these collections.)
Let $x_0(C) = E_{c_0} \cap \varphi_1^{-1}(E_{c_1}) \cap \ldots \cap \varphi_n^{-1}(E_{c_n})$ and $x_j(C) = \varphi_j(x_0(C))$.
Let F be a non-empty subset of the set of all maps S^S on S. Also for an n-tuples $\varphi = (\varphi_1, \ldots, \varphi_n)$ in F, let the semigroup $< F >$ is equal to $< \varphi_1, \ldots, \varphi_n >$. We

denote a configuration corresponding to the pair $(\varphi, \mathcal{E})$ by $Con_F(\varphi, \mathcal{E})$. Then the F-configuration equations corresponding to the configuration pair $(\varphi, \mathcal{E})$ are defined similarly to the previous case. (equations (1.1))
A semigroup S is called F-amenable, if there exists an F-invariant mean M on $\ell_\infty(S)$, that is $M(f \circ \varphi) = M(f)$, for all $f \in \ell_\infty(S)$ and $\varphi \in F$, where $\ell_\infty(S)$ denotes the set of all real valued bounded functions on S.

Adler and Hamilton, [3], showed that S is left amenable if and only if S satisfies the following left invariant condition:
for any sequence $(s_1, \ldots, s_n)$ in S and for all sequences $(A_1, \ldots, A_n)$ of subsets in S there exists a non-empty finite set $X \subseteq S$ such that $|s_i^{-1} A_i \cap X| = |A_i \cap X|$ for all $i \in \{1, 2, \ldots, n\}$.
We prove that S is F-amenable if and only if S satisfies the F-invariant condition.

Definition 4.1. Let $\{A_1, \ldots, A_n; B_1, \ldots, B_m\}$ be a partition of semigroup S and there exist two subsets $\{\varphi_1, \ldots, \varphi_n\}$ and $\{\psi_1, \ldots, \psi_m\}$ of F such that the sets $\{\varphi_1^{-1}(A_1), \ldots, \varphi_n^{-1}(A_n)\}$ and $\{\psi_1^{-1}(B_1), \ldots, \psi_m^{-1}(B_m)\}$ are two partitions of S. Then we say that S admits an F-paradoxical decomposition $(\varphi_i, \psi_j; A_i, B_j)$. In this case, the F-Tarski number of a semigroup S is the minimum of $m + n$, over all possible F-paradoxical decompositions of S.

We show that the F-Tarski number for semigroups can be 2; however the corresponding number for groups is at least 4. At first, by a similar argument as in Proposition 2.1, the following proposition is immediate.

Proposition 4.2. *The following statements are equivalent.*

1. *S is F-amenable.*

2. *Each F-configuration equation $Eq_F(\varphi, \mathcal{E})$ has a normalized solution.*

3. *Each F-configuration equation $Eq_F(\varphi, \mathcal{E})$ has a non-zero solution.*

Lemma 4.3. *The following statements are equivalent.*

1. *S is F-amenable.*

2. *For any sequence $(\varphi_1, \ldots, \varphi_k)$ in F and for all sequences $(A_1, \ldots, A_k)$ of subsets in S, there exist a finite non-empty subset $X \subseteq S$ such that*
$$|\varphi_i^{-1}(A_i) \cap X| = |A_i \cap X|, \text{ for all } i = 1, \ldots, k.$$

3. *For any sequence $(\varphi_1, \ldots, \varphi_n)$ in F and for each partition $\{E_1, \ldots, E_m\}$ of S, there exist a non-empty finite subset $X \subseteq S$ such that,*
$$|\varphi_i^{-1}(E_j) \cap X| = |E_j \cap X|, \text{ for all } i, j.$$

Proof. (2)⇒(3) Let $(\varphi_1, \ldots, \varphi_n)$ be a sequence in F and $\{E_1, \ldots, E_m\}$ be a partition of S. Put

$$A_j = E_j, A_{m+j} = E_j, \ldots, A_{(n-1)m+j} = E_j \text{ for all } j = 1, \ldots, m.$$

Put also,

$$\varphi'_j = \varphi_1, \varphi'_{m+j} = \varphi_2, \ldots, \varphi'_{(n-1)m+j} = \varphi_n \text{ for all } j = 1, \ldots, m.$$

Then for $(\varphi'_1, \ldots, \varphi'_{mn})$ and $(A_1, \ldots, A_{mn})$, there exists a non-empty subset $X \subseteq S$ such that,

$$|\varphi_i^{-1}(E_j) \cap X| = |E_j \cap X|,$$

for $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$.

(3)⇒(2) Let $(\varphi_1, \ldots, \varphi_k)$ be a sequence in F and $(A_1, \ldots, A_k)$ be a sequence of subsets in S. Let $\mathcal{E}_i = \{A_i, A_i^c\}$, for $i = 1, \ldots, k$ and $\mathcal{E}$ be the family of all n-tuple intersections on $\mathcal{E}_i$. Clearly, the cardinality of $\mathcal{E}$ is 2^n and it is a partition of S. By (3), There exists a finite, non-empty subset $X \subseteq S$ such that $|\varphi_i^{-1}(E) \cap X| = |E \cap X|$ for all $i = 1, \ldots, k$ and $E \in \mathcal{E}$. Then one can show easily that $|\varphi_i^{-1}(A_i) \cap X| = |A_i \cap X|$, for all $i = 1, \ldots, k$.

For example, if $k = 1$, then $\mathcal{E} = \{A_1, A_1^c\}$ and there exists a finite, non-empty subset $X \subseteq S$ such that $|\varphi_1^{-1}(A_1) \cap X| = |A_1 \cap X|$. Also, if $k = 2$, then $\mathcal{E} = \{A_1 \cap A_2, A_1 \cap A_2^c, A_1^c \cap A_2, A_1^c \cap A_2^c\}$. Hence, there exists a finite non-empty subset $X \subseteq S$ such that

$$|\varphi_i^{-1}(E) \cap X| = |E \cap X|$$

for all $i \in \{1, 2\}$ and $E \in \mathcal{E}$. Now we have:

$$\begin{aligned} |\varphi_1^{-1}(A_1) \cap X| &= |\varphi_1^{-1}(A_1 \cap A_2) \cap X| + |\varphi_1^{-1}(A_1 \cap A_2^c) \cap X| \\ &= |(A_1 \cap A_2) \cap X| + |(A_1 \cap A_2^c) \cap X| = |A_1 \cap X|. \end{aligned}$$

Similarly, $|\varphi_2^{-1}(A_2) \cap X| = |A_2 \cap X|$.

This completes the proof of (2).

(3)⇒(1) Suppose $\mathcal{E} = \{E_1, \ldots, E_m\}$ is a partition of S and $\varphi = (\varphi_1, \ldots, \varphi_n)$ is a sequence in F. Then there exists a non-empty finite subset $X \subseteq S$ such that

$$|\varphi_i^{-1}(E_j) \cap X| = |E_j \cap X|, \text{ for all } i, j.$$

Let

$$f_C = \frac{1}{|X|}|X \cap x_0(C)|, \text{ for all } C \in Con_F(\varphi, \mathcal{E}).$$

Therefore, $[f_C]$ is a normalized solution. In fact:

$$\begin{aligned} \sum\{f_C : x_i(C) \subseteq E_j\} &= \frac{1}{|X|}|\varphi_i^{-1}(E_j) \cap X| \\ &= \frac{1}{|X|}|E_j \cap X| = \sum\{f_C : x_0(C) \subseteq E_j\}. \end{aligned}$$

Hence, S is F-amenable.

(1)⇒(2) See [3]. □

The condition (ii) of Lemma 4.3, is called F-invariant condition of semigroup S. In the following, we extend F-paradoxical decomposition for semigroups which was asked in [5] p. 120.
Now, suppose that the identity function $I : S \to S$ belongs to F and $A, B \subseteq S$; then we call A and B are F-equidecomposable and write $A \cong B$, if there exist partitions $\{A_1, \dots, A_n\}$ of A and $\{B_1, \dots, B_n\}$ of B, and elements φ_i, ψ_i in F such that $\varphi_i^{-1}(A_i) = B_i$ and $\psi_i^{-1}(B_i) = A_i$ for all $i \in \{1, 2, \dots, n\}$. It is clear that the relation "$\cong$" is an equivalence relation on the power set $P(S)$.
We say also that a finitely additive probability measure μ of the power set $P(S)$ is an F-invariant measure if $\mu(\varphi^{-1}(E)) = \mu(E)$ for all $\varphi \in F$ and $E \subseteq S$. By an argument as in Lemma 3.1, one can show that S is F-amenable if and only if $\alpha \neq 2\alpha$, where $\alpha = (S \times \{1\})^{\sim}$.

Lemma 4.4. *The following statements are equivalent.*

(1) *S is not F-amenable.*

(2) *S admits an F-paradoxical decomposition.*

Proof. (2)⇒(1) Let $(\varphi_i, \psi_j; A_i, B_j)$ be an F-paradoxical decomposition of S and suppose that M is an F-invariant mean on $\ell_\infty(S)$. Then

$$1 = M(1) = \sum_{i=1}^{n} M(\chi_{A_i} \circ \varphi_i) = \sum_{i=1}^{n} M(\chi_{A_i}).$$

Similarly, $\sum_{j=1}^{m} M(\chi_{B_j}) = 1$. Since $\{A_1, \dots, A_n; B_1, \dots, B_m\}$ is a partition of S, we deduce that $1 = \sum_{i=1}^{n} M(\chi_{A_i}) + \sum_{j=1}^{m} M(\chi_{B_j}) = 2$, which is a contradiction.
(1)⇒(2) Use a similar argument as in theorem 3.2. □

Remark 4.5. Let $(s_i, t_j; A_i, B_j)$ be an F-paradoxical decomposition of semigroup S so that $|s_i^{-1}A_i \cap X| = |A_i \cap X|$ and $|t_j^{-1}B_j \cap X| = |B_j \cap X|$, for all i, j, for some non-empty subset $X \subseteq S$. Then:

$$|X| = \sum_i |A_i \cap X| + \sum_j |B_j \cap X| = \sum_i |s_i^{-1}A_i \cap X| + \sum_j |t_j^{-1}B_j \cap X| = 2|X|,$$

hence, X is empty.

Since the existence of F-invariant mean is independent of generating sequence of F, the following statement is immediate.

Corollary 4.6. *Let $F =< \varphi_1, \dots, \varphi_n >$; the following statements are equivalent.*

(1) *S is F-amenable.*

(2) *For any partition* $\{E_1, \ldots, E_m\}$*, there exist a non-empty finite subset* $X \subseteq S$ *such that,*

$$|\varphi_i^{-1}(E_j) \cap X| = |E_j \cap X|, \text{ for all } i, j.$$

(3) *For any partition* $\mathcal{E} = \{E_1, \ldots, E_m\}$ *of* S*, there exist a non-empty finite subset* $X \subseteq S$ *such that*

$$|\varphi_i^{-1}(x_0(C)) \cap X| = |x_0(C) \cap X|$$

for all $i \in \{1, \ldots, n\}$ *and* $C \in Con_F(\varphi, \mathcal{E})$*, where* $\varphi = (\varphi_1, \ldots, \varphi_n)$*.*

Example 4.7. (1) Let $S = (\mathbb{N}, \cdot)$ and $x \cdot y = x$ for $x, y \in S$. Then ${}_xf = f(x)1$ for $f \in \ell_\infty(S)$. So S is not left-amenable and $S = E_1 \cup E_2 = g_1^{-1}E_1 = g_2^{-1}E_2$, where $E_1 = 2\mathbb{N}$, $E_2 = 2\mathbb{N} + 1$, $g_1 = 2$ and $g_2 = 3$. Hence S has a paradoxical decomposition of Tarski number 2, see [4].
(2) Let $S = (\mathbb{N}, \circ)$ and $x \circ y = y$, for $x, y \in S$. Then ${}_xf = f$, for $f \in \ell_\infty(S)$. Then S is left-amenable and $g^{-1}E = E$, for all $g \in S$ and $E \subseteq S$. Hence S has no paradoxical decompositions.

Acknowledgments

The authors would like to thank the referee for his very careful reading and his invaluable comments. They also thank the Banach Algebra Center of Excellence for Mathematics at the University of Isfahan.

References

[1] A. Abdollahi and A. Rejali, Paradoxical decomposition of semigroups, *Semigroup Forum* **71** (2005), no. 3, 471–480.
[2] A. Abdollahi, A. Rejali and G. A. Willis, Group properties characterised by configurations, *Illinois J. Math.* **48** (2004), no. 3, 861–873.
[3] A. Adler and J. Hamilton, Invariant means via the ultrapower, *Math. Ann.* **202** (1973) 71–76.
[4] T. G. Ceccherini Silberstein, Around amenability, Pontryagin Conference, **8**, Algebra (Moscow, 1998), *J. Math. Sci.* (New York) **106** (2001), no. 4, 3145–3163.
[5] A. L. T. Paterson, Amenability, Mathematical Surveys and Monographs, 29, Amer. Math. Soc., Providence, 1988.
[6] J. M. Rosenblatt and G. A. Willis, Weak convergence is not strong convergence for amenable groups, *Canad. Math. Bull.* **44** (2001), no. 2, 231–241.
[7] A. Yousofzadeh, A. Tavakoli and A. Rejali, On configuration graph and paradoxical decomposition, *J. Algebra Appl.* **13** (2014), no. 2, 11 pages.

(Ali Tavakoli) Department of Mathematics, Meymeh Branch, Islamic Azad University, Meymeh, Iran.
E-mail address: at4300125@gmail.com

(Ali Rejali) Department of Mathematics, University of Isfahan, Isfahan, Iran.
E-mail address: rejali@sci.ui.ac.ir