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Author(s):

L. Liu and C. Chen

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ON NONLOCAL ELLIPTIC SYSTEM OF p-KIRCHHOFF-TYPE IN $\mathbb{R}^N$

L. LIU * AND C. CHEN

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Abstract. Using Nehari manifold methods and Mountain pass theorem, the existence of nontrivial and radially symmetric solutions for a class of p-Kirchhoff-type system are established.
Keywords: p-Kirchhoff-type elliptic system, Nehari manifold, Mountain pass theorem.
MSC(2010): Primary: 35J50; Secondary: 35J75, 35J92.

1. Introduction

In this paper, we study the existence of nontrivial solutions to the following p-Kirchhoff-type elliptic system

$$\begin{cases} \left(a+c(\int_{\mathbb{R}^N}(|\nabla u|^p+b|u|^p)dx)^\tau\right)(-\Delta_p u+b|u|^{p-2}u) \\ =\dfrac{1}{d}F_u(u,v)+\lambda|u|^{q-2}u,\ x\in\mathbb{R}^N, \\ \left(a+c(\int_{\mathbb{R}^N}(|\nabla v|^p+b|v|^p)dx)^\tau\right)(-\Delta_p v+b|v|^{p-2}v) \\ =\dfrac{1}{d}F_v(u,v)+\mu|v|^{q-2}v,\ x\in\mathbb{R}^N, \\ u,v\in W^{1,p}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $a,b>0,c,\tau\geq\lambda,\mu\in\mathbb{R}^N, 1<p<N, p<q<d<p^*=\frac{pN}{N-p}, F(u,v)\in C^1(\mathbb{R}^2), F_u=\frac{\partial F}{\partial u}, F_v=\frac{\partial F}{\partial v}$ and $\Delta_p u=div(|\nabla u|^{p-2}\nabla u)$.

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*Corresponding author.

System (1.1) is a generalization of a model introduced by Kirchhoff [10]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho_{tt} - (\frac{P_0}{h} + \frac{E}{2L}\int_0^L u_x^2 dx)u_{xx} = 0,\ 0 < x < L,\ t > 0. \tag{1.2}$$

which takes into account the changes in length of string produced by transverse vibration. The parameters in (1.2) have the following meaning: L is the length of the string, h is the area of cross-section, E is the Young modulus of material, ρ is the mass density and P_0 is the initial tension.

The equation

$$\rho_{tt} - M(\|\nabla u\|_2^2)\Delta u = f(x,u),\ x \in \Omega,\ t > 0, \tag{1.3}$$

generalizes equation (1.2), where $M : \mathbb{R}^+ \to \mathbb{R}$ is a given function, Ω is a domain of $\mathbb{R}^N$. The stationary counterpart of (1.3) is Kirchhoff-type elliptic equation

$$-M(\|\nabla u\|_2^2)\Delta u = f(x,u),\ x \in \Omega,\ t > 0, \tag{1.4}$$

Some classical and interesting results of Kirchhoff-type elliptic equation can be found, for example, in [1,4,12,14,18,19].

In this paper, we investigate the existence of nontrivial and radially symmetric solutions for system (1.1). In particular, we are interested in the nonlinear term $F(u,v)$ including the two cases: $F(u,v) = |u|^\alpha |v|^\beta$ with $\alpha+\beta = d$, and $F(u,v) = (u^2+v^2)^{d/2}$, where $p(\tau+1) < d < p*$ and the asymptotic behavior of $F(u,v)$ is different as $u^2+v^2 \to \infty$. Furthermore, we assume $p < q < d$ in system (1.1), that is, the nonlinear term $\lambda|u|^q + \mu|v|^q$ is a lower degree perturbation of $F(u,v)$.

For $F(u,v) = |u|^\alpha |v|^\beta$, the authors in [2,3,5–8,11,13] considered the existence of solution for (1.1) with $c = 0$. Under the assumption $p(\tau+1) < q < d < p^*$, the authors in [9,20,21] studied the existence of solutions for Kirchhoff-type elliptic equation. Clearly, it is an interesting problem for the existence of solution for system (1.1) with $p < q < p(\tau+1) < d < p^*$. It is noticeable that the Mountain Pass Theorem, Fountain Theorem, Ekeland's variational principle, and the other variational methods have been used to get the existence of solutions in the above references. But, we know that the Nehari manifold and fibrering maps methods are useful to prove the existence of at least two solutions for (1.4) with a concave-convex term, see [19, 20] and the references therein.

For the nonlinearity $f(u)$, problem (1.1) is not compact, that is, the minimizing sequences are bounded, but not pre-compact in $W^{1,p}(\mathbb{R}^N)$. To overcome this diffculty, motivated by [17], we will use the Nehari manifold and the fibering maps method and Mountain Pass Theorem to study the existence of solutions for system (1.1).

In order to state our main results, we first introduce some Sobolev spaces and norms. Let $C_0^\infty(\mathbb{R}^N)$ denote the collection of smooth functions with compact support and

$$C_{0,r}^\infty(\mathbb{R}^N) = \{u \in C_0^\infty(\mathbb{R}^N) | u(x) = u(|x|), x \in \mathbb{R}^N\}. \tag{1.5}$$

Let $X = \mathfrak{D}_r^{1,p}(\mathbb{R}^N)$ be the completion of $C_{0,r}^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_X = (\int_{\mathbb{R}^N} |\nabla u|^p dx)^{1/p}. \tag{1.6}$$

We will work on the space $Y = X \cap L^p(\mathbb{R}^N)$, which is a subspace of the Sobolev space $W^{1,p}(\mathbb{R}^N)$, endowed with the norm

$$\|u\|_Y = (\int_{\mathbb{R}^N} (|\nabla u|^p + b|u|^p) dx)^{1/p}. \tag{1.7}$$

with the constant $b > 0$. It is well known that the embedding $Y \to L^m(\mathbb{R}^N)$ for $p \le m \le p^*$ is continuous, and there exists a constant $c_m > 0$ such that

$$\|u\|_m \le c_m \|u\|_Y, \ \forall u \in Y, \tag{1.8}$$

where $\| \cdot \|$ is the usual norm of $L^m(\mathbb{R}^N)$.

Obviously, under the norm $\| \cdot \|_Y$, Y is a Banach space. For the product space $E = Y \times Y$, the norm of $(u, v) \in E$ is defined by

$$\|(u, v\|_E = \|u\|_X + \|u\|_Y. \tag{1.9}$$

Definition 1.1. A pair of functions $(u, v) \in E$ is said to be a (weak) solution of (1.1) if for any $(\varphi, \psi) \in E$, there holds

$$\begin{aligned}
&(a + c\|u\|_Y^{p\tau}) \int_{\mathbb{R}^N} (|\nabla u|^{p-2} \nabla u \nabla \varphi + b|u|^{p-2} u\varphi) dx + \\
&(a + c\|v\|_Y^{p\tau}) \int_{\mathbb{R}^N} (|\nabla v|^{p-2} \nabla v \nabla \psi + b|v|^{p-2} v\psi) dx \\
&= \frac{1}{d} \int_{\mathbb{R}^N} (F_u \varphi + F_v \psi) dx + \int_{\mathbb{R}^N} (\lambda |u|^{q-2} u\varphi + \mu |v|^{q-2} v\psi) dx.
\end{aligned} \tag{1.10}$$

Let $J(u, v) : E \to \mathbb{R}$ be the energy functional associated with system (1.1) defined by

$$\begin{aligned}
J(u, v) = &\frac{a}{p} A(u, v) + \frac{c}{p}(\tau + 1) B(u, v) - \frac{1}{d} \int_{\mathbb{R}^N} F(u, v) dx \\
&- \frac{1}{q} \int_{\mathbb{R}^N} (\lambda |u|^q + \mu |v|^q) dx.
\end{aligned} \tag{1.11}$$

Here and in the sequel, we denote

$$A(u, v) = \|u\|_Y^p + \|v\|_Y^p, \quad B(u, v) = \|u\|_Y^{p(\tau+1)} + \|v\|_Y^{p(\tau+1)}. \tag{1.12}$$

It is easy to see that the functional $J \in C^1(E, \mathbb{R})$ and its Gateaux derivative is given by

$$J'(u,v)(\varphi,\psi) = (a + c\|u\|_Y^{p\tau}) \int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \nabla\varphi + b|u|^{p-2}u\varphi)dx$$
$$- \frac{1}{d}\int_{\mathbb{R}^N} (F_u\varphi + F_v\psi)dx + (a + c\|v\|_Y^{p\tau}) \int_{\mathbb{R}^N} (|\nabla v|^{p-2}\nabla v \nabla\psi + b|v|^{p-2}v\psi)dx$$
$$- \int_{\mathbb{R}^N} (\lambda|u|^{q-2}u\varphi + \mu|v|^{q-2}v\psi)dx.$$

Throughout this paper, we make the following assumptions:

(H) Let $F(u,v) \in C^1(\mathbb{R}^2)$ be positively homogeneous of degree $d \in (p, p^*)$, that is, $F(tu,tv) = t^d F(u,v)(t > 0)$ for any $(u,v) \in \mathbb{R}^2$. Also, assume $F(u,0) = F(0,v) = F_u(u,0) = F_v(0,v) = 0$ and $F(u,v) > 0$ for any $(u,v) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Furthermore, there exists a constant $k_1 > 0$ such that

$$0 \le F(u,v) \le k_1(|u|^d + |v|^d),\ \forall (u,v) \in \mathbb{R}^2,$$

and

$$|F_u(u,v)| \le k_1(|u|^{d-1} + |v|^{d-1}),$$
$$|F_v(u,v)| \le k_1(|u|^{d-1} + |v|^{d-1}),\ \forall (u,v) \in \mathbb{R}^2,$$

with $p(\tau + 1) < d < p^*$.

Remark 1.1. By the hypothesis (H), we have the so-called Euler identity

$$F_u(u,v)u + F_v(u,v)v = dF(u,v),\ \forall (u,v) \in \mathbb{R}^2.$$

Clearly, the functions $F(u,v) = |u|^\alpha |v|^\beta$ with $\alpha + \beta = d$ and $F(u,v) = (u^2 + v^2)^{d/2}$ satisfy (H).

Here are the main results of this paper.

Theorem 1.2. *Let (H) and $1 < p(\tau+1) \le q < d < p^*$ hold. Then, for any $a, b > 0, c, \tau \ge 0$ and $\lambda, \mu \in \mathbb{R}$, the system (1.1) admits at least a pair of positive ground state solution $(u,v) \in E$ with $J(u,v) > 0$.*

Theorem 1.3. *Let (H) and $1 < p < q < p(\tau+1) < d < p^*$ hold. Then, for any $a, b > 0, c, \tau \ge 0$ and $\lambda, \mu \in \mathbb{R}$, the system (1.1) admits at least a pair of positive ground state solution $(u,v) \in E$ with $J(u,v) > 0$.*

Remark 1.4. For the problem (1.1) with $p = 2, \tau = 1, \lambda = \mu$, Li et al. in [12] proved that there exists at least one positive solution for any $c \in [0, c_0)$ with some small $c_0 > 0$.

Remark 1.5. For the problem (1.1) with $c = 0$, if the perturbation terms $\lambda|u|^{q-2}u$ and $\mu|v|^{q-2}v$ are replaced by $f(x)$ and $g(x)$ respectively, such that $\|f\|_{p'}$ and $\|g\|_{p'}$ are small, the authors in [6] proved that problem (1.1) has at least one nontrivial solution $(u,v) \in E$ with $J(u,v) < 0$.

This paper is organized as follows. In Section 2, we set up the variational framework and derive some Lemmas, we shall discuss the proof of Theorem 1.2 in Section 3. The proof of Theorem 1.3 is given in Section 4.

2. Preliminaries

For the space Y, one can give a pointwise estimate for function in Y.

Lemma 2.1. *[15]. There exists a constant $C = C(p, N) > 0$ such that for every $u \in Y$,*

$$|u(x)| \le C|x|^{(p-N)/p}\|\nabla u\|_p, \ \forall x \in \mathbb{R}^N \setminus \{0\}. \tag{2.1}$$

Lemma 2.2. *Let $m \in (p, p^*)$. Then the embedding $Y \to L^m(\mathbb{R}^N)$ is compact.*

Proof. Let $\{u_k\}$ be a bounded sequence in Y. Without loss of a generality, we assume that $u_k \rightharpoonup 0$ in Y and $\|u_k\|_Y \le M, \forall k \in \mathbb{N}$ with some $M > 0$. For our purpose, it is enough to show that $u_k \to 0$ in $L_m(\mathbb{R}^N)$. By the Sobolev-Rellich embedding theorem in a bounded domain, we can assume $u_k \to 0$ in $L^m_{loc}(\mathbb{R}^N)$ and $u_k(x) \to 0$ a.e. in $\mathbb{R}^N$ as $k \to \infty$. Let B be the unit sphere in $\mathbb{R}^N$ with the center at the origin and $B^c = \mathbb{R}^N \setminus B$. Then, as $k \to \infty$, we have

$$\int_{\mathbb{R}^N} |u_k(x)|^m dx = \int_B |u_k(x)|^m dx + \int_{B^c} |u_k(x)|^m dx = \int_{B^c} |u_k(x)|^m dx + o(1).$$

Fix $q > p^* = \frac{pN}{N-}p$. By Lemma 2.1, we have

$$|u_k(x)|^q \le C_1|x|^{-q(N-p)/p}, \ x \in \mathbb{R}^N \ and \ \forall k \in \mathbb{N}, \tag{2.2}$$

where $C_1 > 0$, independent of k. Since $q > p^*$, we get $|x|^{-q(N-p)/p} \in L^1(B^c)$ and then, by Lebesgue dominated convergence theorem,

$$\int_{B^c} |u_k(x)|^q dx \to 0, \ as \ k \to \infty. \tag{2.3}$$

Let $s \in (0, 1)$ be so that $m = ps + (1 - s)q$. Then, it follows from the Hölder inequality that

$$\int_{B^c} |u_k|^m dx = \int_{B^c} |u_k|^{ps}|u_k|^{q(1-s)} dx \le (\int_{B^c} |u_k|^p dx)^s (\int_{B^c} |u|^q dx)^{1-s}.$$

Since $\{\int_{B^c} |u_k|^p dx\}$ is bounded and $\{\int_{B^c} |u_k|^q dx\} \to 0$, we have $\{\int_{B^c} |u_k|^m dx\}$ $\to 0$ as $k \to \infty$. Moreover, we obtain from (2.3) that

$$\int_{\mathbb{R}^N} |u_k|^m dx \to 0 \ as \ k \to \infty \tag{2.4}$$

and the embedding $Y \hookrightarrow L^m(\mathbb{R}^N)$ is compact. This completes the proof of Lemma 2.2. □

To prove the existence of solution for the system (1.1), we introduce the Nehari manifold

$$\mathcal{N} = \{(u,v) \in E \setminus \{0,0\} | J'(u,v)(u,v) = 0\} \tag{2.5}$$

that is, $(u,v) \in \mathcal{N}$ if and only if $(u,v) \neq (0,0)$ and

$$aA(u,v) + cB(u,v) = \int_{\mathbb{R}^N} (F(u,v) + \lambda|u|^q + \mu|v|^q)dx. \tag{2.6}$$

Furthermore, we define the fibering maps $\phi(t) = J(tu,tv)$ for $t > 0$. Clearly, $(u,v) \in \mathcal{N}$ if and only if $\phi'(1) = 0$ and, more generally, $(tu,tv) \in \mathcal{N}$ if and only if $\phi'(t) = 0$, that is, elements in $\mathcal{N}$ correspond to stationary points of fibering maps $\phi(t)$. By definition, we have

$$\begin{aligned}\phi(t) = J(tu,tv) &= \frac{a}{p}(\|tu\|_Y^p + (\|tv\|_Y^p) + \frac{c}{p(\tau+1)}(\|tu\|_Y^{p(\tau+1)} + \|tv\|_Y^{p(\tau+1)}) \\ &- \frac{t^d}{d}\int_{\mathbb{R}^N} F(u,v)dx - \frac{t^q}{q}\int_{\mathbb{R}^N}(\lambda|u|^q + \mu|v|^q)dx,\end{aligned}$$

and

$$\begin{aligned}\phi'(t) = at^{p-1}A(u,v) &+ ct^{p(\tau+1)-1}B(u,v) - t^{d-1}\int_{\mathbb{R}^N} F(u,v)dx \\ &-t^{q-1}\int_{\mathbb{R}^N}(\lambda|u|^q + \mu|v|^q)dx.\end{aligned}$$

Notice that, if $(u,v) \in \mathcal{N}$, then

$$\begin{aligned}J(u,v) &= a(\frac{1}{p} - \frac{1}{q})A(u,v) + c(\frac{1}{p(\tau+1)} - \frac{1}{q})B(u,v) \\ &+ (\frac{1}{q} - \frac{1}{d}\int_{\mathbb{R}^N} F(u,v)dx \\ &= a(\frac{1}{p} - \frac{1}{d})A(u,v) + c(\frac{1}{p(\tau+1)} - \frac{1}{d})B(u,v) \\ &+ (\frac{1}{d} - \frac{1}{q})\int_{\mathbb{R}^N}(\lambda|u|^q + \mu|u|^q)dx\end{aligned} \tag{2.7}$$

In the following, we derive some properties for the Nehari manifold $\mathcal{N}$.

Lemma 2.3. *Let $p < q < d$ and (H). Then, the Nehari manifold $\mathcal{N} \neq \emptyset$.*

Proof. Let $(u,v) \in E, (u,v) \neq (0,0)$. Consider the following function for $t > 0$:

$$\begin{aligned}\gamma(t) = J'(tu,tv)(tu,tv) &= a(\|tu\|_Y^p + \|tv\|_Y^p) + c(\|tu\|_Y^{p\tau+1} + \|tv\|_Y^{p\tau+1}) \\ &- t^d\int_{\mathbb{R}^N} F(u,v)dx - t^q\int_{\mathbb{R}^N}(\lambda|u|^q + \mu|v|^q)dx.\end{aligned}$$

Since $p < q < d$, it follows that $\gamma(t) > 0$ for small $t > 0$ and $\gamma(t) \to \infty$ as $t \to \infty$. Then there exists $t_1 > 0$ such that $\gamma(t_1) = 0$. Obviously, $(t_1u, t_1v) \neq (0,0)$. We conclude that $(t_1u, t_1v) \in \mathcal{N}$ and $\mathcal{N} \neq \emptyset$. ∎

Lemma 2.4. *Let (H) and $p(\tau+1) \leq q < d$ hold. Then, the functional J is coercive and bounded below on $\mathcal{N}$ and satisfies*

$$d = \inf_{(u,v)\in\mathcal{N}} J(u,v) > 0. \tag{2.8}$$

Proof. Let $(u,v) \in \mathcal{N}$. Then it follows from (1.8) and (2.6) that

$$\begin{aligned} aA(u,v) + cB(u,v) &= \int_{\mathbb{R}^N} (F(u,v) + \lambda|u|^q + \mu|v|^q)dx \\ &\leq C_0(\|u\|_Y^d + \|v\|_Y^d + \|u\|_Y^q + \|v\|_Y^q) \end{aligned} \tag{2.9}$$

where $C_0 = \max\{c_d k_1, c_q \max\{|\lambda|, |\mu|\}\}$ and c_m is the constant in (1.8). Notice that for any $m \geq 1$,

$$2^{1-m}\|(u,v)\|_E^m \leq \|u\|_Y^m + \|v\|_Y^m \leq \|(u,v)\|_E^m. \tag{2.10}$$

Then it follows from (2.9) that

$$2^{1-p}a \leq C_0(\|(u,v)\|_E^{d-p} + \|(u,v)\|_E^{q-p}). \tag{2.11}$$

If $\|(u,v)\|_E \leq 1$, (2.11) gives $a \leq 2^p C_0\|(u,v)\|_E^{q-p}$. So we have

$$\|(u,v)\|_E \geq \min\{1, (2^p C_0 a^{-1})^{\frac{1}{p-q}} \equiv C_1, \ \forall (u,v) \in \mathcal{N}. \tag{2.12}$$

Therefore, if $(u,v) \in \mathcal{N}$, we have from (2.7) and (2.12) that

$$J(u,v) \geq p_1\|u\|_X^p + p_2\|u\|_X^{p(\tau+1)} \geq C_2 \tag{2.13}$$

where

$$\begin{aligned} &p_1 = a(\frac{1}{p} - \frac{1}{q}) > 0, \\ &p_2 = c(\frac{1}{p(\tau+1)} - \frac{1}{q}) \geq 0, \\ &C_2 \equiv p_1 C_1^p + p_2 C_1^{p(\tau+1)} > 0. \end{aligned} \tag{2.14}$$

This shows that J is coercive and bounded below on $\mathcal{N}$ and $d \geq C_2 > 0$. This completes the proof of Lemma 2.4 ☐

Lemma 2.5. *Let (H) and $p(\tau+1) \leq q < d$ hold. Then, there exists $(u,v) \in \mathcal{N}$ such that $J(u,v) = d$ and $u, v \geq 0$ a.e. in $\mathbb{R}^N$.*

Proof. Let $\{(u_n, v_n)\}$ be a minimizing sequence for d in $\mathcal{N}$. The fact $J(u_n, v_n) = J(|u|_n, |v_n|)$ implies that $\{(|u_n|, |v_n|)\}$ is also a minimizing sequence, so that we can assume from beginning that $u_n, v_n \leq 0$ a.e. in $\mathbb{R}^N$. Since J is coercive and bounded below on $\mathcal{N}$, the sequence $\{(u_n, v_n)\}$ is bounded in E. We can assume that, up to a subsequence, $(u_n, v_n) \rightharpoonup (u,v)$ in E. By Lemma 2.2, we have $u_n \to u, v_n \to v$ in $L^d(\mathbb{R}^N) \cap \ L^q(\mathbb{R}^N)$, and, again up to a subsequence, $u_n(x) \to u(x), v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$. So that $u(x), v(x) \geq 0$ a.e. in $\mathbb{R}^N$ and

$(u, v) \in Y$. We now prove that $(u, v) \in \mathcal{N}$ and $J(u, v) = d$.
Since $(u_n, v_n) \in \mathcal{N}$, then,

$$(2.15)\quad aA(u_n, v_n) + cB(u_n, v_n) = \int_{\mathbb{R}^N} (F(u_n, v_n) + \lambda|u_n|^q + \mu|v_n|^q)dx.$$

By the weakly lower semi-continuity of norms, we have from (2.12) and (2.15) that

$$(2.16)\quad 0 < C_3 \le aA(u, v) + cB(u, v) \le \int_{\mathbb{R}^N} (F(u, v) + \lambda|u|^q + \mu|v|^q)dx.$$

This implies that $(u, v) \neq (0, 0)$. If the equality in (2.16) holds, then $(u, v) \in \mathcal{N}$. So, arguing by contradiction, we assume that

$$(2.17)\quad aA(u, v) + cB(u, v) < \int_{\mathbb{R}^N} (F(u, v) + \lambda|u|^q + \mu|v|^q)dx.$$

Let $\gamma(t) = J'(tu, tv)(tu, tv)$. Clearly, $\gamma(t) > 0$ for small $t > 0$ and $\gamma(1) < 0$. So that there exists $t \in (0, 1)$ such that $\gamma(t) = 0$ and $(tu, tv) \in \mathcal{N}$. Then we have from (2.11) and the weakly lower semi-continuity of norms that

$$\begin{aligned} d &\le J(tu, tv) = p_1(\|tu\|_Y^p + \|tv\|_Y^p) + p_2(\|tu\|_Y^{p(\tau+1)} + \|tv\|_Y^{p(\tau+1)}) \\ &\quad + p_3 t^d \int_{\mathbb{R}^N} F(u, v)dx \\ &< p_1(\|u\|_Y^p + \|v\|_Y^p) + p_2(\|u\|_Y^{p(\tau+1)} + \|v\|_Y^{p(\tau+1)}) + p_3 \int_{\mathbb{R}^N} F(u, v)dx \\ &\le p_1 \liminf_{n\to\infty}(p_1 A(u_n, v_n) + p_2 B(u_n, v_n) + p_3 \int_{\mathbb{R}^N} F(u_n, v_n)dx) \\ &= \liminf_{n\to\infty} J(u_n, v_n) = d, \end{aligned}$$

where p_1, p_2 are given in (2.14) and $p_3 = q^{-1} - d^{-1} > 0$.

This contradiction proves that the equality in (2.16) holds and then $(u, v) \in \mathcal{N}$. Again, the application of the weakly lower semi-continuity of norms, we get $J(u, v) \le \liminf_{n\to\infty} J(u_n, v_n) = d$. On the other hand, for every $(u, v) \in \mathcal{N}, J(u, v) \ge d$. So $J(u, v) = d$ and Lemma 2.5 is proved. □

3. **Proof of Theorem 1.2**

We can now prove the main result of this paper by use of lemmas in Section 2.

Proof of Theorem 1.2 Clearly, it is enough to prove that u is a critical point for J in X, that is, $J'(u, v)(\varphi, \psi) = 0$ for all $(\varphi, \psi) \in E$ and thus $J'(u, v) = 0$ in E^*, where (u, v) is in the position of Lemma 2.5.

For every $(\varphi,\psi)\in E$, we choose $\varepsilon>0$ such that $(u+s\varphi,v+s\psi)\neq(0,0)$ for all $s\in(-\varepsilon,\varepsilon)$. Define a function $\omega(s,t):(-\varepsilon,\varepsilon)\times(0,\infty)\to\mathbb{R}$ by

$$\begin{aligned}\omega(s,t)&=J'(t(u+s\varphi),t(v+s\psi))(t(u+s\varphi),t(v+s\psi))\\&=c(\|t(u+s\varphi)\|_Y^{p(\tau+1)}+\|t(v+s\psi)\|_Y^{p(\tau+1)})+a(\|t(u+s\varphi)\|_Y^p\\&\quad+\|t(v+s\psi)\|_Y^p)-t^d\int_{\mathbb{R}^N}F(u+s\varphi,v+s\psi)dx-t^q(\lambda\|u+s\varphi\|_q^q+\mu\|v+s\psi\|_q^q).\end{aligned}$$

Then

$$\begin{aligned}\omega(0,1)&=aA(u,v)+cB(u,v)-\int_{\mathbb{R}^N}F(u,v)dx\\&\quad-\lambda\|u\|_q^q-\mu\|v\|_q^q=0\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}\frac{\partial\omega}{\partial t}(0,1)&=paA(u,v)+pc(\tau+1)B(u,v)\\&\quad-d\int_{\mathbb{R}^N}F(u,v)dx-q(\lambda\|u\|_q^q+\mu\|v\|_q^q)\\&=(p-q)aA(u,v)+c(p(\tau+1)-q)B(u,v)\\&\quad+(q-d)\int_{\mathbb{R}^N}F(u,v)dx<0.\end{aligned}\tag{3.2}$$

So, by the Implicit Function Theorem, there exists a C^1 function $t:(-\varepsilon_0,\varepsilon_0)$ $(\subseteq(-\varepsilon,\varepsilon))\to\mathbb{R}$ such that $t(0)=1$ and $\omega(s,t(s))=0$ for all $s\in(-\varepsilon_0,\varepsilon_0)$. This also shows that $t(s)\neq0$, at least for ε_0 small enough. Therefore, $t(s)(u+s\varphi,v+s\psi)\in\mathcal{N}$. Denote $t=t(s)$ and

$$\begin{aligned}\chi(s)&=J(t(u+s\varphi),t(v+s\psi))=\frac{at^p}{p}(\|u+s\varphi\|_Y^p\\&\quad+\|v+s\psi\|_Y^p)+\frac{ct^{p(\tau+1)}}{p(\tau+1)}(\|u+s\varphi\|_Y^{p(\tau+1)}+\|v+s\psi\|_Y^{p(\tau+1)})\\&\quad-\frac{1}{d}\int_{\mathbb{R}^N}F(t(u+s\varphi),t(v+s\psi))dx-\frac{1}{q}(\lambda\|t(u+s\varphi)\|_q^q+\mu\|t(v+s\psi)\|_q^q.\end{aligned}$$

We see that the function $\chi(s)$ is differentiable and has a minimum point at $s=0$. Therefore,

$$0=\chi'(0)=t'(0)\omega(0,1)+J'(u,v)(\varphi,\psi).\tag{3.3}$$

Since $(u,v)\in\mathcal{N}$, it follows from (3.1) that $J'(u,v)(\varphi,\psi)=0$ for every $(\varphi,\psi)\in E$ and thus $J'(u,v)=0$ in E^*. So, (u,v) is a critical point for J and then (u,v) is a pair of weak solutions of the problem (1.1) in E. Since $J(u,v)=J(|u|,|v|)=d>0$, we can assume $u,v\geq0$ a.e. in $\mathbb{R}^N$. Furthermore, the application of maximum principle in [15] yields $u(x),v(x)>0$ in $\mathbb{R}^N$. Thus, the proof of Theorem 1.2 is completed.

4. **Proof of Theorem 1.3**

In this Section, we use the Mountain Pass Theorem in [23] to prove Theorem 1.3. First, we have

Lemma 4.1. *Let $p < q < p(\tau+1) < d$ and (H) hold. Then, the functional J defined by (1.11) satisfies $J(0,0) = 0$ and*
(I). there exist $\rho > 0; \alpha > 0$ such that $J(u,v) \geq \alpha$ for $\|(u,v)\|_E = \rho$;
(II). there exists $(u_0, v_0) \in E$ with $\|(u_0,v_0)\|_E \geq \rho$ such that $J(u_0,v_0) < 0$.

Proof. In fact, it follows from (H) and (1.8) that

$$\begin{aligned} J(u,v) &= \frac{a}{p}A(u,v) + \frac{c}{p(\tau+1)}B(u,v) \\ &\quad - \frac{1}{d}\int_{\mathbb{R}^N} F(u,v)dx - \frac{1}{q}\int_{\mathbb{R}^N}(\lambda|u|^q + \mu|v|^q)dx \\ &\geq \frac{2^{1-p}a}{p}\|(u,v)\|_E^p + \frac{2^{1-p(\tau+1)}c}{p(\tau+1)}\|(u,v)\|_E^p \\ &\quad - C_3\|(u,v)\|_E^d - C_4\|(u,v)\|_E^q \end{aligned} \tag{4.1}$$

where C_3, C_4 are the given constants. Since $p < q \leq p(\tau+1) < d$, The relation (4.1) implies that (I) is true. To prove (II) , we choose $(u_1,v_1) \in E, u_1, v_1 \neq 0$ and $\|(u_1,v_1)\|_E = 1$ such that $F(u_1,v_1) > 0$ and write

$$\begin{aligned} \eta(t) = J(tu_1,tv_1) &= \frac{a}{p}(\|tu_1\|_Y^p + \|tv_1\|_Y^p) \\ &\quad + \frac{c}{p(\tau+1)}(\|tu_1\|_Y^{p(\tau+1)} + \|tv_1\|_Y^{p(\tau+1)}) \\ &\quad - \frac{t^d}{d}\int_{\mathbb{R}^N} F(u_1,v_1)dx - \frac{t^q}{q}\int_{\mathbb{R}^N}(\lambda|u_1|^q + \mu|v_1|^q)dx. \end{aligned} \tag{4.2}$$

Clearly, $\eta(t) > 0$ for small $t > 0$ and $\eta(t) \to -\infty$ as $t \to +\infty$. Then there exists large $t_1 > \rho$ such that $\eta(t_1) < 0$, that is, $J(u_1,v_1) < 0$. Denote $u_0 = t_1u_1, v_0 = t_1v_0$. Then $\|(u_0,v_0)\|_E = t_1 > \rho$ and $J(u_0,v_0) < 0$. This completes the proof of Lemma 4.1. □

Lemma 4.2. *Assume (H) and $p < q < p(\tau+1) < d$. Let $\{(u_n,v_n)\}$ be a $(PS)_c$ sequence of J in E. Then $\{(u_n,v_n)\}$ has a strongly convergent subsequence in E.*

Proof. Let $\{(u_n,v_n)\}$ be a $(PS)_c$ sequence in E, that is,

$$J(u_n,v_n) \to c\,, J'(u_n,v_n) \to 0,\ in\, E^*. \tag{4.3}$$

We first show that $\{(u_n, v_n)\}$ is bounded in E. In fact, for large n, we have from Hölder inequality that

$$
\begin{aligned}
1 + c + \|u_n, v_n\|_E &\geq J(u_n, v_n) - d^{-1} J'(u_n, v_n)(u_n, v_n) \\
&= p_4 A(u_n, v_n) + p_5 B(u_n, v_n) - p_6 \int_{\mathbb{R}^N} (\lambda |u_n|^q + \mu |v_n|^q) dx \\
&\geq 2^{1-p} p_4 \|(u_n, v_n)\|_E^p + p_5 2^{1-p(\tau+1)} \|(u_n, v_n)\|_E^{p(\tau+1)} \\
&\quad - p_6 c_q^q \|(u_n, v_n)\|_E^q (|\lambda|^\theta + |\mu|^\theta)^{1/\theta}
\end{aligned}
\tag{4.4}
$$

with $p_4 = a(\frac{1}{p} - \frac{1}{d}) > 0, p_5 = c(\frac{1}{p(\tau+1)} - \frac{1}{d}) > 0, p_6 = (\frac{1}{q} - \frac{1}{d}) > 0, \theta = \frac{p(\tau+1)}{p(\tau+1)-q}$.

Since $1 < p < q < p(\tau + 1)$, (4.4) implies that $\{(u_n, v_n)\}$ is bounded in E. Then there exists $(u, v) \in E$ and a subsequence of $\{(u_n, v_n)\}$, still denoted by $\{(u_n, v_n)\}$, such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in E. Without loss of generality, we assume $\|(u_n v_n)\|_E \leq M$ for some constant M and all $n \in \mathbb{N}$. By Lemma 2.2, we have $u_n \to u, v_n \to v$ in $L^m(\mathbb{R}^N)$ with $p < m < p^*$ and then $u_n(x) \to u(x), v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$. In particular, we have $u_n(x) \to u(x), v_n(x) \to v(x)$ in $L^d(\mathbb{R}^N)$.

In the following we prove that $(u_n, v_n) \to (u, v)$ in E. Denote

$$
\begin{aligned}
P_n &= J'(u_, v_n)(u_n - u, v_n - v) \\
&= (a + c\|u_n\|_Y^{p\tau}) \int_{\mathbb{R}^N} (|\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - u) + b|u_n|^{p-2} u_n (u_n - u)) dx \\
&\quad + (a + c\|v_n\|_Y^{p\tau}) (\int_{\mathbb{R}^N} (|\nabla v_n|^{p-2} \nabla v_n \nabla (v_n - v) \\
&\quad + b|v_n|^{p-2} v_n (v_n - v)) dx) - R_n - S_n - T_n - K_n,
\end{aligned}
\tag{4.5}
$$

where

$$
\begin{aligned}
R_n &= \textstyle\int_{\mathbb{R}^n} F_u(u_n, v_n)(u_n - u) dx, \\
S_n &= \textstyle\int_{\mathbb{R}^N} F_v(u_n, v_n)(v_n - v) dx, \\
T_n &= \lambda \textstyle\int_{\mathbb{R}^N} |u_n|^{q-2} u_n (u_n - u) dx, \\
K_n &= \mu \textstyle\int_{\mathbb{R}^N} |v_n|^{q-2} v_n (v_n - v) dx.
\end{aligned}
\tag{4.6}
$$

The fact $J'(u_n, v_n) \to 0$ in E^* implies that $P_n \to 0$ as $n \to \infty$. Similarly, the fact $u_n \rightharpoonup u, v_n \rightharpoonup v$ in Y implies that $Q_n \to 0$ as $n \to \infty$, where

$$
\begin{aligned}
Q_n = (a + c\|u_n\|_Y^{p\tau}) &\int_{\mathbb{R}^N} (|\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - u) \\
&+ b|u_n|^{p-2} u_n (u_n - u)) dx \\
&+ (a + c\|v_n\|_Y^{p\tau}) (\int_{\mathbb{R}^N} (|\nabla v_n|^{p-2} \nabla v_n \nabla (v_n - v) \\
&+ b|v_n|^{p-2} v_n (v_n - v)) dx).
\end{aligned}
\tag{4.7}
$$

We now prove $R_n \to 0, S_n \to 0, T_n \to 0$ and $K_n \to 0$ as $n \to \infty$. It follows from the assumption (H_2) and (1.8) that

$$
\begin{aligned}
|R_n| &\le \int_{\mathbb{R}^N} |F_u(u_n, v_n)(u_n - u)|dx \\
&\le k_1 \int_{\mathbb{R}^N} (|u_n|^{d-1} + |v_n|^{d-1})|u_n - u|dx \\
&\le k_1(\|u_n\|_d^{d-1} + \|v_n\|_d^{d-1})\|u_n - u\|_d \\
&\le k_1 c_d(\|u_n\|_Y^{d-1} + \|v_n\|_Y^{d-1})\|u_n - u\|_d \\
&\le 2k_1 c_d M^{d-1}\|u_n - u\|_d \to 0 \ as \ n \to \infty.
\end{aligned} \tag{4.8}
$$

Similarly, we have

$$
\begin{aligned}
&|S_n| \le 2k_1 c_d M^{d-1}\|v_n - v\|_d, \\
&|T_n| \le |\lambda| c_q M^{q-1}\|u_n - u\|_q, \\
&|K_n| \le |\mu| c_q M^{q-1}\|v_n - v\|_q
\end{aligned} \tag{4.9}
$$

and $S_n \to 0, T_n \to 0, K_n \to 0$ as $n \to \infty$, where the constant c_m was in (1.8). Notice that

$$
\begin{aligned}
P_n - Q_n = {} & (a + c\|u_n\|_Y^{p\tau})U_n + (a + c\|v_n\|_Y^{p\tau})V_n \\
& - R_n - S_n - T_n - K_n
\end{aligned} \tag{4.10}
$$

where

$$
\begin{aligned}
U_n \quad = {} & \textstyle\int_{\mathbb{R}^N}((|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)\nabla(u_n - u) \\
& +b(|u|^{p-2}u_n - |u|^{p-2}u)(u_n - u))dx,
\end{aligned} \tag{4.11}
$$

and

$$
\begin{aligned}
V_n \quad = {} & \textstyle\int_{\mathbb{R}^N}((|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v)\nabla(v_n - v) \\
& +b(|v|^{p-2}v_n - |v|^{p-2}v)(v_n - v))dx.
\end{aligned} \tag{4.12}
$$

Then it follows from (4.9) and $P_n - Q_n \to 0$ that $U_n \to 0, V_n \to 0$ as $n \to \infty$, that is, $(u_n, v_n) \to (u, v)$ in E as $n \to \infty$. Thus $J(u, v)$ satisfies (PS) condition on E and we finish the proof of Lemma 4.2. □

Proof of Theorem 1.3. By Lemmas 4.1 and 4.2, $J(u, v)$ satisfies all assumptions in Mountain Pass Theorem in [16]. Then there exists $(u, v) \in E$ such that (u, v) is a pair of solutions of (1.1). Furthermore, $J(u, v) > \alpha$. This completes the proof of Theorem 1.3. □

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(Lihua Liu) College of Sciences, Hohai University, Nanjing 210098, P.R. of China.
E-mail address: `liulihua11111@163.com`

(Caisheng Chen) College of Sciences, Hohai University, Nanjing 210098, P.R. of China.
E-mail address: `cshengchen@hhu.edu.cn`