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WEAK LOG-MAJORIZATION INEQUALITIES OF SINGULAR VALUES BETWEEN NORMAL MATRICES AND THEIR ABSOLUTE VALUES

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Abstract. This paper presents two main results that the singular values of the Hadamard product of normal matrices A_i are weakly log-majorized by the singular values of the Hadamard product of $|A_i|$ and the singular values of the sum of normal matrices A_i are weakly log-majorized by the singular values of the sum of $|A_i|$. Some applications to these inequalities are also given. In addition, several related and new inequalities are obtained.

Keywords: Unitarily invariant norms, singular values, weak log-majorization, normal matrices, Hadamard product.
MSC(2010): Primary: 47A30; Secondary: 15A15, 15A42, 15B57.

1. Introduction

We first recall some notations and definitions. Let M_n denote the vector space of all complex $n \times n$ matrices and let H_n be the set of all Hermitian matrices of order n. We always denote the eigenvalues of $A \in H_n$ in decreasing order by $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$. The *singular values* of $A \in M_n$ are defined to be the nonnegative square roots of the eigenvalues of A^*A, where A^* denotes the conjugate transpose of a matrix A. The absolute value of $A \in M_n$ is defined and denoted by $|A| = (A^*A)^{\frac{1}{2}}$. Thus the singular values of A are the eigenvalues of $|A|$. We always denote the singular values of $A \in M_n$ by $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ and write $s(A) = (s_1(A), s_2(A), \ldots, s_n(A))$. We denote by $\|\cdot\|_\infty$ the spectral norm, and for $A \in M_n$, $\|A\|_\infty = s_1(A)$. We know that the spectral norm $\|\cdot\|_\infty$ is submultiplicative. For $1 \leq k \leq n$, the norm

$$\|A\|_{(k)} := \sum_{j=1}^{k} s_j(A)$$

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is called the *Fan k-norm*. Note that $\|\cdot\|_{(1)} = \|\cdot\|_\infty$ is the spectral norm and $\|\cdot\|_{(n)}$ is called the *trace norm*. For $A, B \in H_n$, we use the notation $A \le B$ or $B \ge A$ to mean that $B - A$ is positive semidefinite. Clearly, " $\le$ " and " $\ge$ " define two partial orders on H_n, each of which is called a *Löwner partial order*. In particular, $B \ge 0$ means that B is positive semidefinite. Given a real vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we rearrange its components as $x_{[1]} \ge x_{[2]} \ge \cdots \ge x_{[n]}$. For $x = (x_1, x_2, \dots, x_n),\ y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, if

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]},\ k = 1, 2, \dots, n,$$

then we say that x is *weakly majorized* by y and write $x \prec_w y$. If $x \prec_w y$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, then we say that x is *majorized* by y and write $x \prec y$. Let the components of $x = (x_1, x_2, \dots, x_n),\ y = (y_1, y_2, \dots, y_n)$ be nonnegative. If

$$\prod_{i=1}^{k} x_{[i]} \le \prod_{i=1}^{k} y_{[i]},\ k = 1, 2, \dots, n,$$

then we say that x is *weakly log-majorized* by y and write $x \prec_{wlog} y$. If $x \prec_{wlog} y$ and $\prod_{i=1}^n x_i = \prod_{i=1}^n y_i$, then we say that x is *log-majorized* by y and write $x \prec_{log} y$.

Log-majorization is a powerful technique for matrix norm inequalities. See [1, 8, 10, 14, 15] for the theory of log-majorziation and its applications. Let us write $\{x_i\}$ for a vector $(x_1, x_2, \dots, x_n)$. The following two well known log-majorization relations are called Weyl's theorem and Horn's theorem respectively [9, 10, 15].

Let $\lambda_1(A), \dots, \lambda_n(A)$ be the eigenvalues of a matrix $A \in M_n$ with $|\lambda_1(A)| \ge \cdots \ge |\lambda_n(A)|$. Then

$$\{|\lambda_i(A)|\} \prec_{log} s(A). \tag{1.1}$$

Let $A, B \in M_n$. Then

$$s(AB) \prec_{log} \{s_i(A)s_i(B)\}. \tag{1.2}$$

In [7], Bourin and Uchiyama proved a triangle inequality for normal matrices. Let A and B be normal matrices. Then for every unitarily invariant norm $\|\cdot\|$,

$$\|A + B\| \le \||A| + |B|\|,$$

which is equivalent to

$$s(A + B) \prec_w s(|A| + |B|).$$

Let A, B be positive semidefinite matrices. Bhatia and Kittaneh [5] showed that

$$s(A + zB) \prec_w s(A + |z|B) \tag{1.3}$$

for any complex number z. The stronger inequality

$$s(A+zB) \prec_{wlog} s(A+|z|B) \tag{1.4}$$

for any complex number z was proved by Zhan [13]. Recently, Zou and Wu [17] proved that if $A, B_1, \ldots, B_n$ are positive semidefinite matrices and if $z_1, z_2, \ldots, z_n$ are complex numbers, then for every unitarily invariant norm $\|\cdot\|$,

$$\|A + z_1 B_1 + \cdots + z_n B_n\| \leq \|A + \mid z_1 \mid B_1 + \cdots + \mid z_n \mid B_n\|. \tag{1.5}$$

The main purpose of this paper is to prove the following weak log-majorization of singular values related to normal matrices and its applications. We prove that if A_i are normal matrices, $i = 1, \ldots, m$, then

$$s\left(\sum_{i=1}^{m} A_i\right) \prec_{wlog} s\left(\sum_{i=1}^{m} |A_i|\right)$$

and

$$s\left(\circ_{i=1}^{m} A_i\right) \prec_{wlog} s\left(\circ_{i=1}^{m} |A_i|\right).$$

An application of the above result is the following. Let $P_i \in M_n$ be positive semidefinite matrices, $i = 1, 2, \ldots, m$. Then for any complex numbers $z_1, \ldots, z_m$,

$$s\left(\sum_{i=1}^{m} z_i P_i\right) \prec_{wlog} s\left(\sum_{i=1}^{m} |z_i| P_i\right),$$

which sharpens the result due to Zou and Wu. In addition, several related and new inequalities are also given.

2. Main results

We first need several lemmas. Most of these are either well known or easy to prove.

Lemma 2.1. *[15, p.67] Let the components of $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ be nonnegative. Then*

$$x \prec_{wlog} y \quad \text{implies} \quad x \prec_w y.$$

This lemma shows that weak log-majorization is stronger than weak majorization.

For $A \in M_n$, the k-th compound matrix of A is denoted by $C_k(A)$. We list some useful properties that we need in our proofs. For the topic of compound matrices see [15, Section 2.4].

Lemma 2.2. *Let $A, B \in M_n$, $1 \leq k \leq n$. Then*

(1) $C_k(A)^T = C_k(A^T)$, $C_k(A)^* = C_k(A^*)$, $C_k(AB) = C_k(A)C_k(B)$;

(2) *if* $s(A) = (s_1(A), \ldots, s_n(A))$, *then*

$$s(C_k(A)) = \{s_{i_1}(A)s_{i_2}(A)\cdots s_{i_k}(A) \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}.$$

Lemma 2.3. *[15, p.13] Let $A, B \in M_n$. If A, B are positive semidefinite, then so is $A \circ B$.*

Denote the block diagonal matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ by $A \oplus B$.

Lemma 2.4. *Let $A, B \in M_n$ be contraction matrices. Then so is $C_k(A \oplus B)$.*

Proof. Recall that C is a contraction if $||C||_\infty \leq 1$. It is clear that $A \oplus B$ is also a contraction, if A and B are contraction matrices. Note that

$$||C_k(A \oplus B)||_\infty = \prod_{i=1}^{k} s_i(A \oplus B) \leq 1.$$

This completes the proof. □

The following lemma is well-known. For the reader's convenience, we provide a short proof.

Lemma 2.5. *If T has the polar decomposition $T = U|T|$ with U unitary, then*

$$T = |T^*|^{\frac{1}{2}} U |T|^{\frac{1}{2}}.$$

Proof. Since $T = U|T|$ with U unitary, we have $|T| = U^*T = T^*U$. Hence

$$|T|^2 = U^*TT^*U = U^*|T^*|^2U.$$

The uniqueness of square roots of positive semidefinite matrices yields $|T| = U^*|T^*|U$ and so $U|T| = |T^*|U$. Then

$$U\ p(|T|) = p(|T^*|)\ U$$

for every polynomial $p(t)$. By Stone-Weierstrass Theorem, we have

$$U|T|^{\frac{1}{2}} = |T^*|^{\frac{1}{2}}U.$$

Therefore $T = |T^*|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$. This completes the proof. □

Now, we prove a very useful lemma.

Lemma 2.6. *Let $P_i,\ Q_i \in M_n$ be positive semidefinite matrices, and let $C_i \in M_n$ be contraction matrices, $i = 1, 2, \ldots, m$. Then*

$$s\left(\sum_{i=1}^{m} P_i C_i Q_i\right) \prec_{wlog} \left\{ s_i\left(\left(\sum_{i=1}^{m} P_i^2\right)^{\frac{1}{2}}\right) s_i\left(\left(\sum_{i=1}^{m} Q_i^2\right)^{\frac{1}{2}}\right)\right\}. \tag{2.1}$$

Proof. Write

$$A = \begin{pmatrix} P_1 & P_2 & \cdots & P_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} Q_1 & 0 & \cdots & 0 \\ Q_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ Q_m & 0 & \cdots & 0 \end{pmatrix},$$

and $X = \mathrm{diag}(C_1, C_2, \ldots, C_m)$. Then X is a contraction and

$$AXB = \left(\sum_{i=1}^{m} P_i C_i Q_i\right) \oplus 0.$$

Then for any positive integer k with $1 \le k \le n$, we have

$$\begin{aligned}\prod_{i=1}^{k} s_i(\sum_{i=1}^{m} P_i C_i Q_i) &= ||C_k(\sum_{i=1}^{m} P_i C_i Q_i)||_\infty \\ &= ||C_k(\left(\sum_{i=1}^{m} P_i C_i Q_i\right) \oplus 0)||_\infty \\ &= ||C_k(AXB)||_\infty.\end{aligned}$$

On the other hand, since the spectral norm $\|\cdot\|_\infty$ is submultiplicative, it follows from Lemma 2.4 that k with $1 \le k \le n$,

$$\begin{aligned}||C_k(AXB)||_\infty &\le ||C_k(A)||_\infty\ ||C_k(X)||_\infty\ ||C_k(B)||_\infty \\ &\le ||C_k(A)||_\infty\ ||C_k(B)||_\infty \\ &= ||C_k(A)C_k(A)^*||_\infty^{\frac{1}{2}}\ ||C_k(B)^*C_k(B)||_\infty^{\frac{1}{2}} \\ &= ||C_k(AA^*)||_\infty^{\frac{1}{2}}\ ||C_k(B^*B)||_\infty^{\frac{1}{2}} \\ &= \left\|C_k\left(\left(\sum_{i=1}^{m} P_i^2\right) \oplus 0\right)\right\|_\infty^{\frac{1}{2}} \left\|C_k\left(\left(\sum_{i=1}^{m} Q_i^2\right) \oplus 0\right)\right\|_\infty^{\frac{1}{2}} \\ &= \left\|C_k\left(\left(\sum_{i=1}^{m} P_i^2\right)^{\frac{1}{2}} \oplus 0\right)\right\|_\infty \left\|C_k\left(\left(\sum_{i=1}^{m} Q_i^2\right)^{\frac{1}{2}} \oplus 0\right)\right\|_\infty \\ &= \prod_{i=1}^{k} s_i\left(\left(\sum_{i=1}^{m} P_i^2\right)^{\frac{1}{2}}\right) s_i\left(\left(\sum_{i=1}^{m} Q_i^2\right)^{\frac{1}{2}}\right).\end{aligned}$$

In the first four equalities above we have used the fact that for any square matrix F, $\|FF^*\|_\infty = \|F^*F\|_\infty = \|F\|_\infty^2 = \|F^*\|_\infty^2$. This completes the proof. □

Applying Lemma 2.6 yields the following weak log-majorization for normal matrices:

Theorem 2.7. *Let $A_i \in M_n$ be normal matrices, $i = 1, 2, \ldots, m$. Then*

$$s\left(\sum_{i=1}^{m} A_i\right) \prec_{wlog} s\left(\sum_{i=1}^{m} |A_i|\right). \tag{2.2}$$

Proof. It is well known that X is a normal matrix if and only if $|X| = |X^*|$. Let $A_i = U_i|A_i|$ be the polar decompositions of A_i with U_i unitary, $i = 1, 2, \ldots, m$.

By Lemma 2.5, we have $A_i = |A_i^*|^{\frac{1}{2}} U |A_i|^{\frac{1}{2}} = |A_i|^{\frac{1}{2}} U |A_i|^{\frac{1}{2}}$. Then

$$\sum_{i=1}^{m} A_i = \sum_{i=1}^{m} |A_i^*|^{\frac{1}{2}} U_i |A_i|^{\frac{1}{2}} = \sum_{i=1}^{m} |A_i|^{\frac{1}{2}} U_i |A_i|^{\frac{1}{2}},$$

where $A_i = |A_i|^{\frac{1}{2}} U_i |A_i|^{\frac{1}{2}}$ with U_i unitary. Note that U_i are also contraction matrices. By Lemma 2.6,

$$s\left(\sum_{i=1}^{m} A_i\right) \prec_{wlog} \left\{ s_i\left(\left(\sum_{i=1}^{m} |A_i|\right)^{\frac{1}{2}}\right) s_i\left(\left(\sum_{i=1}^{m} |A_i|\right)^{\frac{1}{2}}\right)\right\}_{i=1}^{n} = s\left(\sum_{i=1}^{m} |A_i|\right).$$

This completes the proof. □

Remark 2.8. Recall the Fan Dominance Principle [15]: Let $A, B \in M_n$. If $\|A\|_{(k)} \leq \|B\|_{(k)}$ for all Fan k-norms with $1 \leq k \leq n$, then $\|A\| \leq \|B\|$ for every unitarily invariant norm $\|\cdot\|$. It can be equivalently stated as: $\|A\| \leq \|B\|$ holds for all unitarily invariant norms if and only if $s(A) \prec_w s(B)$. In [7], Bourin and Uchiyama proved a triangle inequality for normal matrices. Let A and B be normal matrices. Then for every unitarily invariant norm $\|\cdot\|$,

$$\|A + B\| \leq \||A| + |B|\|, \tag{2.3}$$

which is equivalent to

$$s(A + B) \prec_w s(|A| + |B|).$$

Using Theorem 2.7 and Lemma 2.1, we have the following more general triangle inequality of unitarily invariant norms for normal matrices:

Corollary 2.9. *Let $A_i \in M_n$ be normal matrices, for $i = 1, 2, \ldots, m$. Then*

$$\left\|\sum_{i=1}^{m} A_i\right\| \leq \left\|\sum_{i=1}^{m} |A_i|\right\|, \tag{2.4}$$

for every unitarily invariant norm $\|\cdot\|$.

Let A and B be positive semidefinite matrices. In [5], Bhatia and Kittaneh showed that

$$s(A + zB) \prec_w s(A + |z|B) \tag{2.5}$$

for any complex number z. The stronger inequality

$$s(A + zB) \prec_{wlog} s(A + |z|B) \tag{2.6}$$

for any complex number z was proved by Zhan [13]. An application of Theorem 2.7 can be seen in the following result. This result generalizes the inequality (1.5) and sharpens the result due to Zou and Wu.

Theorem 2.10. *Let $P_i \in M_n$ be positive semidefinite matrices, $i = 1, 2, \ldots, m$. Then for any complex numbers $z_1, \ldots, z_m$,*

$$s\left(\sum_{i=1}^{m} z_i P_i\right) \prec_{wlog} s\left(\sum_{i=1}^{m} |z_i| P_i\right). \tag{2.7}$$

Proof. Note that each $z_i P_i$ is a normal matrix and $|z_i P_i| = |z_i| P_i$. Applying Theorem 2.7 completes the proof. □

Using Lemma 2.6, we have the following interesting inequality.

Corollary 2.11. *Let $A \in M_n$. Then*

$$s(A + A^*) \prec_{wlog} s((|A| + |A^*|). \tag{2.8}$$

Proof. Suppose that $A = U|A|$ is the polar decomposition of A with U unitary. By Lemma 2.5, we have

$$A + A^* = |A^*|^{\frac{1}{2}} U |A|^{\frac{1}{2}} + |A|^{\frac{1}{2}} U^* |A^*|^{\frac{1}{2}}.$$

For each $i = 1, \ldots, n$,

$$s_i((|A^*| + |A|)^{\frac{1}{2}}) s_i((|A| + |A^*|)^{\frac{1}{2}}) = s_i(|A| + |A^*|).$$

Applying Lemma 2.6 completes the proof. □

Corollary 2.12. *Let $A \in M_n$. Then*

$$\|A + A^*\| \leq \| \, |A| + |A^*| \, \|, \tag{2.9}$$

for every unitary invariant norm $\| \cdot \|$.

Proof. Combining Corollary 2.11 and Lemma 2.1 completes the proof. □

An application of Lemma 2.6 gives an inequality involving a partitioned matrix.

Theorem 2.13. *If A, B, C and D are normal matrices of the same order, then*

(2.10)

$$s\begin{pmatrix} A & B \\ C & D \end{pmatrix} \prec_{wlog} \{ s_i\left((|A| + |B|)^{\frac{1}{2}} \oplus (|C| + |D|)^{\frac{1}{2}}\right) s_i\left((|A| + |C|)^{\frac{1}{2}} \oplus (|B| + |D|)^{\frac{1}{2}}\right) \}.$$

Proof. Since A, B, C and D are normal matrices, we have

$$\left|\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\right| = \left|\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}^*\right| = \begin{pmatrix} |A| & 0 \\ 0 & |D| \end{pmatrix},$$

$$\left|\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}^*\right| = \begin{pmatrix} |B| & 0 \\ 0 & |C| \end{pmatrix}$$

and

$$\left|\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right| = \begin{pmatrix} |C| & 0 \\ 0 & |B| \end{pmatrix}.$$

Then

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = (|A|^{\frac{1}{2}} \oplus |D|^{\frac{1}{2}}) U_1 (|A|^{\frac{1}{2}} \oplus |D|^{\frac{1}{2}})$$

and

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} = (|B|^{\frac{1}{2}} \oplus |C|^{\frac{1}{2}})U_2(|C|^{\frac{1}{2}} \oplus |B|^{\frac{1}{2}})$$

with U_1, U_2 unitary. Using Lemma 2.6 completes the proof. □

Applying Theorem 2.13 yields the inequality for the spectral norm due to Bouring and Uchiyama in [7].

Corollary 2.14. *If A, B, C and D are normal matrices of the same order, then*

$$\left\| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\|_\infty \le \max\{\| |A|+|B| \|_\infty,\ \| |C|+|D| \|_\infty,\ \| |A|+|C| \|_\infty,\ \| |B|+|D| \|_\infty\}. \tag{2.11}$$

Proof. Write

$$L := \max\{\| |A| + |B| \|_\infty,\ \| |C| + |D| \|_\infty,\ \| |A| + |C| \|_\infty,\ \| |B| + |D| \|_\infty\}.$$

Since $s_1(X) = \|X\|_\infty$ for any $X \in M_n$, we have

$$s_1\left((|A| + |B|)^{\frac{1}{2}} \oplus (|C| + |D|)^{\frac{1}{2}}\right) = \max\{\| |A| + |B| \|_\infty^{\frac{1}{2}}, \| |C| + |D| \|_\infty^{\frac{1}{2}}\} \le L^{\frac{1}{2}}$$

and

$$s_1\left((|A| + |C|)^{\frac{1}{2}} \oplus (|B| + |D|)^{\frac{1}{2}}\right) = \max\{\| |A| + |C| \|_\infty^{\frac{1}{2}}, \| |B| + |D| \|_\infty^{\frac{1}{2}}\}\} \le L^{\frac{1}{2}}.$$

Applying the case $i = 1$ of Theorem 2.13 completes the proof. □

An inequality in Bhatia's book [4, p.271] is the following:
Let $A, B \in M_n$. Then

$$|\det(A + B)|^2 \ \le\ \det(\ (|A| + |B|\ ) \det(\ (|A^*| + |B^*|\ ). \tag{2.12}$$

The following corollary generalizes this determinant inequality, which is pretty well known, as determinant functional is Liebian [12, p. 70-71].

Corollary 2.15. *Let $A_i \in M_n$, for $i = 1, \dots, m$. Then*

$$\left|\det\left(\sum_{i=1}^m A_i\right)\right|^2 \ \le\ \det\left(\sum_{i=1}^m |A_i|\right)\ \det\left(\sum_{i=1}^m |A_i^*|\right). \tag{2.13}$$

Proof. Let $A_i = U_i|A_i|$ be the polar decompositions of A_i with U_i unitary, $i = 1, 2, \dots, m$. By Lemma 2.5, for any positive integer $i = 1, \dots, m$, $A_i = |A_i^*|^{\frac{1}{2}} U_i |A_i|^{\frac{1}{2}}$ with U_i unitary and so

$$\sum_{i=1}^m A_i = \sum_{i=1}^m |A_i^*|^{\frac{1}{2}} U_i |A_i|^{\frac{1}{2}}.$$

By Lemma 2.6, we have

$$s\left(\sum_{i=1}^m A_i\right) \prec_{wlog} \left\{ s_i\left(\sum_{i=1}^m |A_i^*|\right)^{\frac{1}{2}} s_i\left(\sum_{i=1}^m |A_i|\right)^{\frac{1}{2}} \right\}.$$

Note that $|\det X| = \prod_{i=1}^{n} s_i(X)$ for any $X \in M_n$. Using the case $i = n$ of the above inequality yields

$$\left|\det\left(\sum_{i=1}^{m} A_i\right)\right| \leq \left[\det\left(\sum_{i=1}^{m} |A_i|\right)\right]^{\frac{1}{2}} \left[\det\left(\sum_{i=1}^{m} |A_i^*|\right)\right]^{\frac{1}{2}}.$$

This completes the proof. □

Let $A, B \in M_n$ be positive semidefinite. Bhatia and Kittaneh [6] stated that " The formulation $2s_i(AB) \leq s_i(A^2 + B^2)$ $(1 \leq i \leq n)$ is somewhat delicate. For example, another possible formulation could be

$$s_i(AB + BA) \leq s_i(A^2 + B^2).$$

But this is not always true." However, Zou and He pointed out the following inequality is true:

$$\|AB + BA\| \leq \|A^2 + B^2\|.$$

Using Lemma 2.6, the next corollary sharpens the corresponding result:

Corollary 2.16. *Let $A,\ B \in M_n$ be positive semidefinite matrices. Then*

$$s(AB + BA) \prec_{wlog} s(A^2 + B^2) \tag{2.14}$$

Proof. Using Lemma 2.6 completes the proof. □

At last, we prove some inequalities involving Hadamard products of matrices.

Lemma 2.17. *Let $A_i \in M_n$, $i = 1, 2, \ldots, m$. Then*

$$s\left(\circ_{i=1}^{m} A_i\right) \prec_{wlog} \left\{s_i\left(\left(\circ_{i=1}^{m}|A_i|\right)^{\frac{1}{2}}\right) s_i\left(\left(\circ_{i=1}^{m}|A_i^*|\right)^{\frac{1}{2}}\right)\right\}. \tag{2.15}$$

Proof. Note that for any $X \in M_n$, we have

$$\begin{pmatrix} |X| & X^* \\ X & |X^*| \end{pmatrix} \geq 0.$$

Then for each $i = 1, \ldots, m$,

$$\begin{pmatrix} |A_i| & A_i^* \\ A_i & |A_i^*| \end{pmatrix} \geq 0.$$

By Lemma 2.3, we have

$$0 \leq \circ_{i=1}^{m} \begin{pmatrix} |A_i| & A_i^* \\ A_i & |A_i^*| \end{pmatrix} = \begin{pmatrix} \circ_{i=1}^{m}|A_i| & \circ_{i=1}^{m} A_i^* \\ \circ_{i=1}^{m} A_i & \circ_{i=1}^{m}|A_i^*| \end{pmatrix}.$$

Then there exists a contraction W [15, Th. 3.34] such that

$$\circ_{i=1}^{m} A_i^* = \left(\circ_{i=1}^{m}|A_i|\right)^{\frac{1}{2}} W \left(\circ_{i=1}^{m}|A_i^*|\right)^{\frac{1}{2}}.$$

Note that for any $A \in M_n$, $s(A) = s(A^*)$. Using Lemma 2.6 completes the proof. □

Theorem 2.18. *Let $A_i \in M_n$ be normal matrices, $i = 1, 2, \ldots, m$. Then*

$$s\left(\circ_{i=1}^{m} A_i\right) \prec_{wlog} s\left(\circ_{i=1}^{m} |A_i|\right).$$

Proof. Note that $|A_i| = |A_i^*|$, for each i. Applying Lemma 2.17 completes the proof. □

Remark 2.19. Combining Lemma 2.17 with Lemma 2.1, we can obtain the following inequalities:

Let $A \in M_n$.

$$s(A \circ A^*) \prec_{wlog} s(|A| \circ |A^*|) \tag{2.16}$$

and so for every unitarily invariant norm

$$||A \circ A^*|| \leq |||A| \circ |A^*|||. \tag{2.17}$$

Let $A_i \in M_n$, $i = 1, 2, \ldots, m$.

$$\left|\det\left(\circ_{i=1}^{m} A_i\right)\right|^2 \leq \det\left(\circ_{i=1}^{m} |A_i|\right) \det\left(\circ_{i=1}^{m} |A_i^*|\right). \tag{2.18}$$

Moreover, for all normal matrices A_j,

$$\left|\mathrm{tr}(\circ_{j=1}^{m} A_j)\right| \leq \mathrm{tr}(\circ_{j=1}^{m} |A_j|) \tag{2.19}$$

and

$$|| \circ_{j=1}^{m} A_j|| \leq || \circ_{j=1}^{m} |A_j| \, || \tag{2.20}$$

for every unitarily invariant norm.

Theorem 2.20. *Let $A,\ B \in M_n$ be positive semidefinite matrices. Then*

$$s(AB \circ BA) \prec_{wlog} s(A^2 \circ B^2). \tag{2.21}$$

Proof. Since A and B are positive semidefinite, so are $\begin{pmatrix} A^2 & AB \\ BA & B^2 \end{pmatrix}$ and $\begin{pmatrix} B^2 & BA \\ AB & A^2 \end{pmatrix}$. By Lemma 2.3, $\begin{pmatrix} A^2 \circ B^2 & AB \circ BA \\ BA \circ AB & B^2 \circ A^2 \end{pmatrix}$ is positive semidefinite. Then there exists a contraction W [15, Th. 3.34] such that

$$AB \circ BA = (A^2 \circ B^2)^{\frac{1}{2}} W (B^2 \circ A^2)^{\frac{1}{2}}.$$

Note that $A^2 \circ B^2 = B^2 \circ A^2$. Using Lemma 2.6 completes the proof. □

Using Theorem 2.20 and Lemma 2.1 we have

Corollary 2.21. *Let $A,\ B \in M_n$ be positive semidefinite matrices. Then*

$$\|AB \circ BA\| \leq \|A^2 \circ B^2\| \tag{2.22}$$

for every unitarily invariant norm.

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