ISSN: 1017-060X (Print)

ISSN: 1735-8515 (Online)

Bulletin of the

Iranian Mathematical Society

Vol. 42 (2016), No. 1, pp. 155–173

Title:

Positive solution for Dirichlet $p(t)$-Laplacian BVPs

Author(s):

A. Benmezai and S. Mechrouk

Published by Iranian Mathematical Society
http://bims.ims.ir

Bull. Iranian Math. Soc.
Vol. 42 (2016), No. 1, pp. 155–173
Online ISSN: 1735-8515

POSITIVE SOLUTION FOR DIRICHLET $p(t)$-LAPLACIAN BVPS

A. BENMEZAI* AND S. MECHROUK

(Communicated by Asadollah Aghajani)

Abstract. In this paper we provide existence results for positive solution to Dirichlet p(t)-Laplacian boundary value problems. The sublinear and superlinear cases are considerd.
Keywords: $p(t)$-Laplacian, positive solution, fixed point index theory.
MSC(2010): Primary: 34B15; Secondary: 34B16.

1. Introduction

We investigate in this paper existence of at least one positive solution to the p(t)-Laplacian Dirichlet boundary value problem (bvp for short)

$$\begin{cases} -(\varphi(t,u'(t)))' = f(t,u(t)),\ t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $\varphi(t,x) = |x|^{p(t)-2}\,x,\ p \in C\left([0,1], (1,+\infty)\right), f \in C\left([0,1] \times \mathbb{R}^+, \mathbb{R}^+\right)$ and $\mathbb{R}^+ = [0,+\infty)$.

We mean by a positive solution to bvp (1.1), a function $u \in C^1([0,1],\mathbb{R}^+)$ with $\phi(t,u'(t)) \in C^1([0,1],\mathbb{R})$ and $u(t_0) > 0$ for some $t_0 \in (0,1)$, satisfying both the differential equation and Dirichlet boundary conditions in bvp (1.1).

The differential operator $-(\varphi(t,u'(t)))'$ is known in the literature as the monodimensional $p(t)$-Laplacian and becomes the p-Laplacian when $p(t) \equiv p \in (1,+\infty)$. Because of the physical interests (see for example [1, 5, 11] and references cited therein), differential equations involving the $p(t)$-Laplacian have received a great attention in recent years and many interesting results have been obtained (see for example [2, 6–9, 12–15], and [16]).

Note that if the exponent $p(t)$ is not constant the $p(t)$-Laplacian is not linear and nonhomogeneous. This makes the study of differential equations involving

Article electronically published on February 22, 2016.
Received: 15 April 2014, Accepted: 24 November 2014.
*Corresponding author.

$p(t)$-Laplacian more complicated than the case where $p(t) \equiv p \in (1,+\infty)$. The difficulties encountered when studying various aspects of differential equations involing the $p(t)$-Laplacian are clearly indicated in [9, 12, 14] and [15].

This work is motivated by that in [2], where the authors consider the three point bvp

$$\left\{ \begin{array}{l} -(\varphi(t,u'(t)))' = f(t,u(t)),\ t \in (0,1), \\ u(0) = \alpha u(\eta),\ u'(1) = 0, \end{array} \right. \tag{1.2}$$

with $\alpha, \eta \in (0,1)$ and they provide existence results for at least one positive solution in both the sublinear case and the superlinear case. The main goal of this paper is to see if the results obtained in [2] hold (in form not in details) or not, when we replace boundary conditions in bvp (1.2) by Dirichlet ones. As in [2], we will use the fixed point index theory and obviously, we have encountred the same difficulties as those described in [2].

To overcome these difficulties, we have used a characterization of the positive eigenvalue of the $p-$Laplacian proved in [4] to compute the fixed point index near 0 for the operator associated with the fixed point formulation of bvp (1.1). The fixed point index computation at ∞, is obtained by straighforward calculations for the sublinear case; unlike, for the superlinear case, it is obtained by combining quadrature techniques with the homotopy property of the fixed point index.

The paper is organized as follows. In section 2, we recall first, some lemmas giving fixed point index calculations. Then, we present a fixed point formulation for bvp (1.1) and we close this section by some lemmas making use of homotopical argument for the superlinear case. In Section 3, we present our main results and their proofs and it is ended by illustrative examples.

2. **Preliminaries**

First, let us recall some elements related to fixed point index theory. Let X be a real Banach space. A nonempty closed convex subset K of X is said to be an ordered cone if $K \cap (-K) = \{0\}$ and $(tK) \subset K$ for all $t \geq 0$.

Let K be an ordered cone of X, for $R > 0$, K_R denotes the intersection of the cone K with $B(0,R)$ the ball in X of radius R centered at 0. Let $T : \overline{K_R} \to K$ be a compact mapping. The following lemmas can be found in [10]. They provide fixed point index computations.

Lemma 2.1. *If $Tu \ngeq u$ for all $u \in \partial K_R$, then $i(T, K_R, K) = 1$.*

Lemma 2.2. *If $Tu \nleq u$ for all $u \in \partial K_R$, then $i(T, K_R, K) = 0$.*

Lemma 2.3. *If $T(u) \neq \lambda u$ for all $u \in \partial K_R$ and $\lambda \geq 1$ then $i(f, K_R, K) = 1$.*

Lemma 2.4. *If*

- *$T(x) \neq \lambda x$ for all $x \in \partial K_R = \partial B(0,R) \cap K$ and $\lambda \in (0,1]$ and*

• $\inf\{\|T(x)\| : x \in \partial K_R\} > 0$

then

$$i(f, K_R, K) = 0.$$

Throughout this paper, E will denote the Banach space of all continuous functions defined on $[0,1]$ equipped with the sup-norm denoted by $\|\cdot\|$, K and P are cones of E defined by

$$K = \{u \in E : u \geq 0 \text{ in } [0,1]\},$$

and

$$P = \{u \in K : u(t) \geq \rho(t)\|u\|, \text{ for } t \in [0,1]\}$$

where

$$\rho(t) = \min(t, 1-t).$$

Throughout this paper, $\psi(t,\cdot)$ denotes the inverse function of $\varphi(t,\cdot)$ and we have

$$\psi(t,x) = |x|^{q(t)-2}x \text{ where } q(t) = \tfrac{p(t)}{p(t)-1}.$$

The real numbers, p^-, p^+ are defined by

$$p^- = \min_{t\in[0,1]} p(t), \quad p^+ = \max_{t\in[0,1]} p(t)$$

and we have

$$q^- = \min_{t\in[0,1]} q(t) = \tfrac{p^+}{p^+-1}, \quad q^+ = \max_{t\in[0,1]} q(t) = \tfrac{p^-}{p^--1}.$$

We need also to introduce the following functions:

$$\varphi^-(x) = \begin{cases} x^{p^+-1} & \text{if } x \leq 1, \\ x^{p^--1} & \text{if } x \geq 1, \end{cases} \qquad \varphi^+(x) = \begin{cases} x^{p^--1} & \text{if } x \leq 1, \\ x^{p^+-1} & \text{if } x \geq 1, \end{cases}$$

and

$$\psi^-(x) = \begin{cases} x^{q^+-1} & \text{if } x \leq 1, \\ x^{q^--1} & \text{if } x \geq 1, \end{cases} \qquad \psi^+(x) = \begin{cases} x^{q^--1} & \text{if } x \leq 1, \\ x^{q^+-1} & \text{if } x \geq 1. \end{cases}$$

Note that ψ^+ and ψ^- are respectively the inverse functions of φ^- and φ^+.

Note also that for all $s \in (0,1)$ and $x \geq 0$,

$$\begin{aligned} s^{p^+-1}\phi^-(x) &\leq \phi^-(sx) \leq s^{p^--1}\phi^-(x), \\ s^{p^+-1}\phi^+(x) &\leq \phi^+(sx) \leq s^{p^--1}\phi^+(x), \\ s^{q^+-1}\psi^-(x) &\leq \psi^-(sx) \leq s^{q^--1}\psi^-(x), \\ s^{q^+-1}\psi^+(x) &\leq \psi^+(sx) \leq s^{q^--1}\psi^+(x). \end{aligned} \tag{2.1}$$

Let

$$\phi_i^-(x) = \begin{cases} \phi^-(x) & \text{if } x \geq 0, \\ -\phi^+(-x) & \text{if } x \leq 0, \end{cases} \qquad \phi_i^+(x) = \begin{cases} \phi^+(x) & \text{if } x \geq 0, \\ -\phi^-(-x) & \text{if } x \leq 0, \end{cases}$$

$$\psi_i^-(x) = \begin{cases} \psi^-(x) & \text{if } x \geq 0, \\ -\psi^+(-x) & \text{if } x \leq 0, \end{cases} \qquad \psi_i^+(x) = \begin{cases} \psi^+(x) & \text{if } x \geq 0, \\ -\psi^-(-x) & \text{if } x \leq 0. \end{cases}$$

and observe that for all $t \in [0,1]$ and $x \in \mathbb{R}$,

$$\phi_i^-(x) \leq \phi(t,x) \leq \phi_i^+(x)$$

and

$$\psi_i^-(x) \leq \psi(t,x) \leq \psi_i^+(x).$$

In the particular case where the exponent p is constant, the function φ is denoted by ϕ_p and in this case we have $\psi = \phi_q$.

The main goal of the three following lemmas is to provide a fixed point formulation to bvp (1.1).

Lemma 2.5. *For all $h \in L^1[0,1]$ there exists a unique $c(h)$ solution of*

$$\int_0^1 \psi\left(t, \phi(0,c) - \int_0^t h(s)\,ds\right) dt = 0.$$

Moreover the map $H : L^1[0,1] \to \mathbb{R}$ defined by $H(h) = c(h)$ is continuous.

Proof. Fix $h \in L^1[0,1]$ and consider the mapping $\hat{h} : \mathbb{R} \to \mathbb{R}$ with

$$\hat{h}(c) = \int_0^1 \psi\left(t, \varphi(0,c) - \int_0^t h(s)\,ds\right) dt.$$

Since for all $t \in \mathbb{R}$, $\psi(t,\cdot)$ is an increasing function, the mapping $\hat{h}$ is increasing, moreover $\hat{h}$ satisfies for all $c \in \mathbb{R}$.

$$\psi_i^-\left(\varphi(0,c) - |h|_1\right) \leq \hat{h}(c) \leq \psi_i^+\left(\varphi(0,c) + |h|_1\right).$$

These inequalities lead to $\lim_{c\to-\infty} \hat{h}(c) = -\infty$ and $\lim_{c\to+\infty} \hat{h}(c) = +\infty$ and so equation $\hat{h}(c) = 0$ admits a unique solution.

Now, suppose that the mapping H is not continuous and there exists $\epsilon_0 > 0$ and a sequence $(h_n) \subset L^1[0,1]$ converging to $h \in L^1[0,1]$ such that $|c(h_n) - c(h)| > \epsilon_0$.

Observe that for all $n \in \mathbb{N}$, there exists $t_n \in (0,1)$ such that

$$\psi\left(t_n, \varphi(0, c(h_n)) - \int_0^{t_n} h(s)\,ds\right) = 0$$

leading to

$$|c(h_n)| \leq \psi(0, |h_n|_1).$$

This inequality together with the convergence of (h_n) to h in $L^1[0,1]$ means that the sequence $(c(h_n))$ is bounded and up to a subsequence, $(c(h_n))$ converges to some c_*.

At this stage, we obtain from Lebesgue dominated convergence theorem, that

$$\int_0^1 \psi\left(t, \varphi(0, c_*) - \int_0^t h(s)\, ds\right) dt = 0$$

then uniqueness of $c(h)$ leads to $c(h) = c_*$.

Combining all the above, yields the contradiction

$$0 < \epsilon_0 \leq \lim |c(h_n) - c(h)| = |c_* - c(h)| = 0,$$

ending the proof □

Now, let $N_\varphi : L^1[0,1] \to C^1[0,1]$ be the operator defined for $h \in L^1[0,1]$ by

$$N_\varphi h(x) = \int_0^x \psi\left(t, \varphi(0, c(h)) - \int_0^t h(s)\, ds\right) dt$$

where the real number $c(h)$ is that in Lemma 2.5.

It is easy to prove the following lemma.

Lemma 2.6. *Consider for $h \in L^1[0,1]$, the bvp*

$$\left\{\begin{array}{l} -(\varphi(t, u'(t)))' = h(t), \quad a.\ e.\ t \in (0,1), \\ u(0) = u(1) = 0. \end{array}\right. \tag{2.2}$$

Then u is a solution to bvp (2.2) if and only if $u = N_\varphi h$.

Let $F : K \to K$ be the mapping given for $u \in K$ by $Fu(t) = f(t, u(t))$ and let i, j be respectively the compact embeding of $C^1[0,1]$ in E and the continuous embeding of E in $L^1[0,1]$ and consider the operator $T_\varphi = i \circ N_\varphi \circ j \circ F$. The proof of the following lemma is easy, so we omit it.

Lemma 2.7. *We have that*

- $T_\varphi(K) \subset K$
- *T_φ is completely continuous*
- *u is a positive solution of bvp (1.1) if and only if u is a fixed point of T_φ.*

In the remainder of this section, we will recall and prove some results for the case of p-Laplacian operator (the case where the weight p is constant). These results are use singular form for proofs of the main results of this paper. Especially, we need to establish a fixed point index calculation for the case of p-Laplacian operator in order to obtain by a homotopical argument, fixed point index computation at ∞ for the superlinear case.

Let $\lambda(p_0)$ be the positive eigenvalue of

$$\begin{cases} -(\phi_{p_0}(u'(t)))' = \lambda\phi_{p_0}(u(t)), \ t \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

It is well known that

$$\lambda(p_0) = (p_0 - 1)\left(2\int_0^1 \frac{ds}{\sqrt[p_0]{1-s^{p_0}}}\right)^{p_0}.$$

Consider the operator $N_{p_0} : E \to E$ defined by

$$N_{p_0}u(t) = \begin{cases} \int_0^t \phi_{q_0}\left(\int_s^{1/2} \phi_{p_0}(u(s))\,ds\right)dt & \text{if } t \in [0,1/2] \\ \int_t^1 \phi_{q_0}\left(\int_{1/2}^s \phi_{p_0}(u(s))\,ds\right)dt & \text{if } t \in [1/2,1] \end{cases}$$

where $q_0 = \frac{p_0}{p_0-1}$. Clearly, we have that $(\lambda(p_0))^{-1}$ is the unique positive eigenvalue of N_{p_0} and Theorem 3.15 in [4] states that

$$(\lambda(p_0))^{-1} = \sup \Lambda_K^{N_{p_0}} = \inf \Theta_K^{N_{p_0}}$$

where

$$\Lambda_K^{N_{p_0}} = \{\lambda \geq 0 : \text{ there exists } u \in K \setminus \{0\} \text{ such that } N_{p_0}u \leq \lambda u\},$$

$$\Theta_K^{N_{p_0}} = \{\theta \geq 0 : \text{ there exists } u \in K \setminus \{0\} \text{ such that } N_{p_0}u \geq \theta u\}.$$

Observe also that $\lambda(p_0)$ is the positive eigenvalue to each of the bvps,

$$\begin{cases} -(\phi_{p_0}(u'(t)))' = \lambda\phi_{p_0}(u(t)), \ t \in (0,1/2), \\ u(0) = u'(1/2) = 0 \end{cases}$$

and

$$\begin{cases} -(\phi_{p_0}(u'(t)))' = \lambda\phi_{p_0}(u(t)), \ t \in (1/2,1), \\ u'(1/2) = u(1) = 0. \end{cases}$$

Similarly, we have that

$$(\lambda(p_0))^{-1} = \sup \Lambda_K^{N_{p_0}^r} = \inf \Theta_K^{N_{p_0}^r} = \sup \Lambda_K^{N_{p_0}^l} = \inf \Theta_K^{N_{p_0}^l}$$

where

$$\begin{aligned} N_{p_0}^r u(t) &= \int_t^1 \phi_{q_0}\left(\int_{1/2}^s \phi_{p_0}(u(s))\,ds\right)dt, \ t \in [1/2,1] \\ N_{p_0}^l u(t) &= \int_0^t \phi_{q_0}\left(\int_s^{1/2} \phi_{p_0}(u(s))\,ds\right)dt, \ t \in [0,1/2] \end{aligned}$$

and for $\nu = r$ or l,

$$\Lambda_K^{N_{p_0}^\nu} = \left\{\lambda \geq 0 : \text{ there exists } u \in K \setminus \{0\} \ \text{ such that } N_{p_0}^\nu u \leq \lambda u\right\},$$

$$\Theta_K^{N_{p_0}^\nu} = \left\{\theta \geq 0 : \text{ there exists } u \in K \setminus \{0\} \text{ such that } N_{p_0}^\nu u \geq \theta u\right\}.$$

Lemma 2.8. *(Lemma 2.4 in [3])Assume that* $p \equiv p_0$ *is constant, then* $T_{\phi_{p_0}}(K) \subset P$.

Lemma 2.9. *Assume that* $p \equiv p_0$ *is constant,*

$$f(t,u) > 0 \quad \text{for all } (t,u) \in [0,1] \times (0,+\infty). \tag{2.3}$$

and $f_\infty^0 > \lambda(p_0)$ *where*

$$f_\infty^0 = \lim_{u \to \nu} \inf \left(\min_{t \in [0,1]} \frac{f(t,u)}{u^{p_0-1}} \right).$$

Then there exists $R_\infty > 0$ *such that* $i\left(T_{\phi_{p_0}}, K_R, K\right) = 0$ *for all* $R > R_\infty$.

Proof. Because of Lemma 2.8 and the permanance property of the fixed point index, we have that for all $R > 0,\ i\left(T_{\phi_{p_0}}, K_R, K\right) = i\left(T_{\phi_{p_0}}, P_R, P\right)$.

In order to use Lemma 2.4, we claim that there exists $R_\infty^1 > 0$ such that for all $R > R_\infty^1$, $T_{\phi_{p_0}} u \neq \lambda u$ for all $u \in \partial P_R$ and $\lambda \in (0,1]$. To the contrary, assume that there exists sequences $(\lambda_n) \subset (0,1]$ and $(u_n) \subset P$ such that $\lim_{n\to\infty} \|u_n\| = +\infty$ and $T_{\phi_{p_0}} u_n = \lambda_n u_n$. We have that

$$\begin{cases} -(\phi_{p_0}(u_n'(t)))' = \phi_{p_0}\left(\lambda^{-1}\right) f(t, u_n(t)),\ t \in (0,1), \\ u_n(0) = u_n(1) = 0. \end{cases} \tag{2.4}$$

Let $t_n \in (0,1)$ be such that $u_n'(t_n) = 0$ and $u_n(t_n) = \|u_n\|$. We deduce from the hypothesis (2.3) that t_n is the unique critical point of u_n. Indeed, if there exists $t_n^* \in (0,t_n) \cup (t_n,1)$ such that $u_n'(t_n^*) = 0$ then one gets the contradiction,

$$0 = \int_{t_n^*}^{t_n} -\left(\phi_{m_0}\left(u_n'(s)\right)\right)' ds = \int_{t_n^*}^{t_n} \phi_{m_0}\left(\lambda^{-1}\right) f\left(s, u_n(s)\right) ds \neq 0.$$

Let $\epsilon > 0$ be such that $\left(f_\infty^0 - \epsilon\right) > \lambda(p_0)$, there exists R_* large such that $f(t,x) \geq \left(f_\infty^0 - \epsilon\right) x^{m_0-1}$ for all $x \geq R_*$ and $t \in [0,1]$. Multiplying the differential equation in (2.4) by u_n' and integrating between t and t_n, we obtain:

$$\tfrac{1}{p_0}\left|u_n'(t)\right|^{p_0} = \int_t^{t_n} \phi_{p_0}\left(\lambda^{-1}\right) f(s, u_n(s)) u'(s) ds$$

then,

$$\begin{aligned} u_n'(t_n) &= \left(p_0 \int_t^{t_u} \phi_{p_0}\left(\lambda^{-1}\right) f(s,u_n(s))u'(s)ds\right)^{1/m_0}, \text{ for } t \in (0,t_u) \\ -u_n'(t_n) &= \left(p_0 \int_t^{t_u} \phi_{p_0}\left(\lambda^{-1}\right) f(s,u_n(s))u'(s)ds\right)^{1/m_0}, \text{ for } t \in (t_u,1). \end{aligned}$$

Consequently, we have

$$\begin{aligned} t_n &= \int_{u_n(0)}^{u_n(t_u)} \frac{du_n(t)}{u_n'(t)} = \int_0^{\|u_n\|} \frac{du_n(t)}{\left(p_0 \int_t^{t_n} \phi_{p_0}(\lambda^{-1}) f(s,u_n(s)) u_n'(s) ds\right)^{1/p_0}} \\ &\leq \int_0^{\|u_n\|} \frac{du_n(t)}{\left(p_0 \int_t^{t_n} f(s,u_n(s)) u_n'(s) ds\right)^{1/p_0}}. \end{aligned}$$

Let t_*^n be the unique point in $(0, t_n)$ such that $u_n(t_*^n) = R_*$. We have

$$\begin{aligned} t_n &\leq \int_0^{R_*} \frac{du_n(t)}{\left(p_0 \int_t^{t_n} f(s,u_n(s)) u_n'(s) ds\right)^{1/p_0}} + \int_{R_*}^{\|u_n\|} \frac{du_n(t)}{\left(p_0 \int_t^{t_n} f(s,u_n(s)) u_n'(s) ds\right)^{1/p_0}} \\ &\leq \int_0^{R_*} \frac{du_n(t)}{\left(p_0 \int_{t_*^n}^{t_n} f(s,u_n(s)) u_n'(s) ds\right)^{1/p_0}} + \int_{R_*}^{\|u_n\|} \frac{du_n(t)}{\left(p_0 \int_t^{t_n} f(s,u_n(s)) u_n'(s) ds\right)^{1/p_0}} \\ &\leq \frac{R_*}{\left((f_\infty^0 - \epsilon)\left(\|u_n\|^{p_0} - R_*^{p_0}\right)\right)^{1/p_0}} + \int_0^{\|u_n\|} \frac{du_n(t)}{\left((f_\infty^0 - \epsilon)\left(\|u_n\|^{p_0} - u_n^{p_0}(t)\right)\right)^{1/p_0}} \end{aligned}$$

which leads to

$$t_n \leq \frac{R_*}{\left((f_\infty^0 - \epsilon)\left(\|u_n\|^{p_0} - R_*^{p_0}\right)\right)^{1/p_0}} + \left(\frac{\lambda(p_0)}{2^{p_0}(f_\infty^0 - \epsilon)}\right)^{1/p_0}. \tag{2.5}$$

Similarly, we have

$$1 - t_n = \frac{R_*}{\left((f_\infty^0 - \epsilon)\left(\|u_n\|^{p_0} - R_*^{p_0}\right)\right)^{1/p_0}} + \left(\frac{\lambda(p_0)}{2^{p_0}(f_\infty^0 - \epsilon)}\right)^{1/p_0}. \tag{2.6}$$

Adding (2.5) and (2.6), we get

$$1 \leq \frac{2R_*}{\left((f_\infty^0 - \epsilon)\left(\|u_n\|^{p_0} - R_*^{p_0}\right)\right)^{1/p_0}} + \left(\frac{\lambda(p_0)}{(f_\infty^0 - \epsilon)}\right)^{1/p_0}. \tag{2.7}$$

Letting $n \to \infty$ in (2.7), yields the contradiction

$$1 \leq \left(\frac{\lambda(p_0)}{(f_\infty^0 - \epsilon)}\right)^{1/p_0} < 1.$$

Thus, the claim is proved.

Now, set $R_\infty = 4R_\infty^1$ and let $u \in \partial P_R$ with $R \geq R_\infty$ and $t_u \in (0,1)$ such that $T_{\phi_{p_0}} u(t_u) = \left\|T_{\phi_{p_0}} u\right\|$ and $\left(T_{\phi_{p_0}} u\right)'(t_u) = 0$. We distinguish two cases

1) $t_u \geq 1/2$, in this case we have

$$u(s) \geq \rho(s)\|u\| \geq \tfrac{1}{4}\|u\| \geq R_\infty^1 \quad \text{for all } s \in [1/4, 1/2]$$

which yields the estimate,

$$\begin{aligned} \left\|T_{\phi_{p_0}}u\right\| &= T_{\phi_{p_0}}u\left(t_u\right) = \int_0^{t_u} \phi_{q_0}\left(\int_t^{t_u} f\left(s,u\left(s\right)\right)ds\right)dt \\ &\geq \int_{1/4}^{1/2} \phi_{q_0}\left(\int_t^{1/2} f\left(s,u\left(s\right)\right)ds\right)dt \\ &\geq \int_{1/4}^{1/2} \phi_{q_0}\left(\int_t^{1/2} \left(f_\infty^0-\epsilon\right)\phi_{p_0}\left(u\left(s\right)\right)ds\right)dt \\ &\geq \phi_{q_0}\left(f_\infty^0-\epsilon\right)\int_0^{1/2} \phi_{q_0}\left(\int_t^{1/2} \phi_{p_0}\left(\rho\left(s\right)\left\|u\right\|\right)ds\right)dt \\ &\geq \phi_{q_0}\left(f_\infty^0-\epsilon\right)R_\infty^1\int_{1/4}^{1/2} \phi_{q_0}\left(\int_t^{1/2} \phi_{p_0}\left(\rho\left(s\right)\right)ds\right)dt > 0 \end{aligned}$$

2) $t_u \geq 1/2$, in this case we have,

$$u\left(s\right) \geq \rho\left(s\right)\left\|u\right\| \geq \tfrac{1}{4}\left\|u\right\| \geq R_\infty^1 \text{ for all } s \in [1/2, 3/4]$$

which yields the estimate

$$\begin{aligned} \left\|T_{\phi_{p_0}}u\right\| &= T_{\phi_{p_0}}u\left(t_u\right) = \int_{t_u}^{1} \phi_{q_0}\left(\int_{t_u}^{t} f\left(s,u\left(s\right)\right)ds\right)dt \\ &\geq \int_{1/2}^{3/4} \phi_{q_0}\left(\int_{1/2}^{t} f\left(s,u\left(s\right)\right)ds\right)dt \\ &\geq \int_{1/2}^{3/4} \phi_{q_0}\left(\int_{1/2}^{t} \left(f_\infty-\epsilon\right)\phi_{p_0}\left(u\left(s\right)\right)ds\right)dt \\ &\geq \phi_{n_0}\left(f_\infty^0-\epsilon\right)\int_{1/2}^{3/4} \phi_{q_0}\left(\int_{1/2}^{t} \phi_{p_0}\left(\rho\left(s\right)\left\|u\right\|\right)ds\right)dt \\ &\geq \phi_{q_0}\left(f_\infty^0-\epsilon\right)R_\infty^1\int_{1/2}^{3/4} \phi_{q_0}\left(\int_t^{1/2} \phi_{p_0}\left(\rho\left(s\right)\right)ds\right)dt > 0. \end{aligned}$$

Therefore, we have that

$$\inf\left\{\left\|T_{\phi_{m_0}}u\right\|, u \in \partial P_R\right\} \geq \phi_{q_0}\left(f_\infty^0-\epsilon\right)R_\infty^1 I_0 > 0$$

with

$$I_0 = \min\left(\int_{1/4}^{1/2} \phi_{q_0}\left(\int_t^{1/2} \phi_{p_0}\left(\rho\left(s\right)\right)ds\right)dt, \int_{1/2}^{3/4} \phi_{q_0}\left(\int_t^{1/2} \phi_{p_0}\left(\rho\left(s\right)\right)ds\right)dt\right)$$

and so, from Lemma 2.4 we deduce that

$$i\left(T_{\phi_{m_0}}, K_R, K\right) = i\left(T_{\phi_{m_0}}, P_R, P\right) = 0 \text{ for all } R > R_\infty.$$

This ends the proof $\square$

3. Main results

We begin this section by introducing some notations. Set

$$\gamma^{-}(p) = q^{-}\left(p^{-}-1\right) = \frac{p^{+}\left(p^{-}-1\right)}{\left(p^{+}-1\right)},$$

$$\gamma^{+}(p) = q^{+}\left(p^{+}-1\right) = \frac{p^{-}\left(p^{+}-1\right)}{\left(p^{-}-1\right)} \text{ if } p^{+}\left(p^{-}-2\right)+1>0,$$

$$I(p) = \int_0^1 \frac{ds}{\left(1-s^{\gamma^{+}(p)}\right)^{1/\gamma^{-}(p)}},$$

$$\Lambda_\infty(p) = \left(2I(p)\right)^{\gamma^{+}(p)}\left(p^{+}-1\right) \quad \text{if } p^{+}\left(p^{-}-2\right)+1>0,$$

$$W_{+} = \Gamma^{+}\circ\phi^{+} \quad \text{where for } x\geq 0,\ \Gamma^{+}(x) = \int_0^x \psi^{+}(t)\,dt,\ \Pi = W_{+}^{-1}.$$

Note that $\gamma^{+}(p)\geq p^{+}$, $I(p)<\infty$ if and only if $p^{+}(p^{-}-2)+1>0$ and for all $s\in(0,1)$, $x\geq 0$

$$s^{\gamma^{+}(p)}W_{+}(x) \leq W_{+}(sx) \leq s^{\gamma^{-}(p)}W_{+}(x),$$
$$s^{1/\gamma^{-}(p)}\Pi(x) \leq \Pi(sx) \leq s^{1/\gamma^{+}(p)}\Pi(x).$$

We have also,

$$\lim_{x\to+\infty}\frac{W_{+}(x)}{x^{\gamma^{+}(p)}} = \frac{1}{q^{+}},\ \lim_{x\to+\infty}\frac{\Gamma^{-}(x)}{x^2} = \frac{1}{2}.$$

It easy to see from the above limits that for all $\eta>0$

$$\lim_{y\to+\infty}\frac{y}{\Pi\left(\eta y^{\gamma^{+}(p)}\right)} = \left(\frac{1}{\eta q^{+}}\right)^{1/\gamma^{+}(p)}. \tag{3.1}$$

Let

$\Lambda_1 = \varphi^{+}(\lambda(p^{-}))$ $\Lambda_2 = \varphi^{-}(\lambda(p^{+}))$ $\Lambda_3 = q^{-}2^{q^{-}}$ $\Lambda_4 = \max(\Lambda_\infty(p), \lambda(p^{+}))$

and

$$f_0 = \liminf_{u\to 0}\left(\min_{t\in[\eta,1]}\frac{f(t,u)}{u^{p^{-}-1}}\right), \qquad f^0 = \limsup_{u\to 0}\left(\max_{t\in[0,1]}\frac{f(t,u)}{u^{p^{+}-1}}\right),$$
$$f^\infty = \limsup_{u\to+\infty}\left(\max_{t\in[0,1]}\frac{f(t,u)}{u^{p^{-}-1}}\right), \quad f_\infty = \liminf_{u\to+\infty}\left(\min_{t\in[\eta,1]}\frac{f(t,u)}{u^{\gamma^{+}(p)-1}}\right).$$

Theorem 3.1 (sublinear case)**.** *Assume that*

$$f^\infty/\Lambda_3 < 1 < f_0/\Lambda_1. \tag{3.2}$$

Then bvp (1.1) admits at least one positive solution.

Proof. Note that Lemma 2.7 implies that fixed points of the operator T_φ are nonnegative solutions to bvp (1.1). Consequently, we have to prove that T_φ has a fixed point in $K\setminus\{0\}$. To this aim, let $\epsilon>0$ be such that $(f_0-\epsilon)>\Lambda_1$; there exists $\delta>0$ such that for all $t\in[0,1]$ and $u\in[0,\delta]$, $f(t,u)\geq(f_0-\epsilon)\,u^{p^- -1}$.

Set $r_0=\min(1,\delta)$ and let $r\in(0,r_0)$ and $u\in\partial K_r$ such that $T_\varphi u\leq u$.

Thus, we have

$$u(t)\geq T_\varphi u(t)=\int_0^t\psi\left(s,\int_s^{t_u}f(\tau,u(\tau))d\tau\right)ds \text{ for all } t\in[0,t_u]$$

and

$$u(t)\geq T_\varphi u(t)=\int_t^1\psi\left(s,\int_{t_u}^{s}f(\tau,u(\tau))d\tau\right)ds \text{ for all } t\in[t_u,1]$$

where t_u is such that $T_\varphi u(t_u)=\|T_\varphi u\|$ and $(T_\varphi u)'(t_u)=0$.

Without loss of generality, assume that $t_u\geq 1/2$, then we have from (2.1)

$$\begin{aligned} u(t) &\geq T_\varphi u(t)\geq\int_0^t\psi^-\left(\int_s^{1/2}f(\tau,u(\tau))d\tau\right)ds\\ &\geq \int_0^t\psi^-\left(\int_s^{1/2}(f_0-\epsilon)\,u^{p^- -1}(\tau)d\tau\right)ds\\ &\geq \psi^-(f_0-\epsilon)\int_0^t\phi_{p^-}\left(\int_s^{1/2}\phi_{p^-}(u(\tau))\,d\tau\right)ds\\ &= \psi^-(f_0-\epsilon)\,N^l_{\phi_{p^-}}u(t). \end{aligned}$$

This implies that $(\psi^-(f_0-\epsilon))^{-1}\in\Lambda_K^{N^l_{\phi_{p^-}}}$ and

$$\left(\psi^-(f_0-\epsilon)\right)^{-1}\geq\left(\lambda\left(p^-\right)\right)^{-1}=\inf\Lambda_K^{N^l_{\phi_{p^-}}}.$$

So, we obtain the contradiction,

$$\Lambda_1=\varphi^+\left(\lambda\left(p^-\right)\right)\geq(f_0-\epsilon)>\Lambda_1.$$

At this stage, we have proved that for all $r\in(0,r_0)$, $T_\varphi u\not\leq u$ for all $u\in\partial K_r$. Thus, Lemma 2.2 guarantees that

$$i(T_\varphi,K_r,K)=0 \text{ for all } r\in(0,r_0).$$

Let $\varepsilon>0$ be such that $(f^\infty+\varepsilon)<\Lambda_3$. There exists a positive constant c_∞ such that

$$f(t,x)\leq(f^\infty+\varepsilon)\,x^{p^- -1}+c_\infty \text{ for all } t\in[0,1] \text{ and } x\geq 0.$$

We claim that there exists R_∞ large such that for all $R\in(R_\infty,+\infty)$,

$$T_\varphi u\neq\lambda u \text{ for all } u\in\partial K_R \text{ and } \lambda\geq 1.$$

Suppose that this is not the case and there are sequences $(\lambda_n) \subset [1, +\infty)$, $(u_n) \subset K$ with $\|u_n\| \to \infty$, such that $T_\varphi u_n = \lambda_n u_n$. Let for all $n \in \mathbb{N}$, $t_n \in (0,1)$ be such that $u_n(t_n) = \|u_n\|$ and $u_n'(t_n) = 0$.

We have

$$\begin{aligned}
\|u_n\| &= u_n(t_n) \le \lambda_n u_n(t_n) = \int_0^{t_n} \psi\left(s, \int_s^{t_n} f(\tau, u(\tau))d\tau\right) ds \\
&\le \int_0^{t_n} \psi^+\left(\int_s^{t_n} f(\tau, u(\tau))d\tau\right) ds \\
&\le \int_0^{t_n} \psi^+\left(\int_s^{t_n} \left((f^\infty + \varepsilon) u_n^{p^- -1}(\tau) + c_\infty\right) d\tau\right) ds \\
&\le \int_0^{t_n} \psi^+\left((t_n - s)\left((f^\infty + \varepsilon)\|u_n\|^{p^- -1} + c_\infty\right)\right) ds.
\end{aligned}$$

This together with (2.1) leads to

$$\begin{aligned}
1 &\le \frac{1}{\|u_n\|}\int_0^{t_n} \psi^+\left((t_n - s)\left((f^\infty + \varepsilon)\|u_n\|^{p^- -1} + c_\infty\right)\right) ds \\
&\le \int_0^{t_n} \psi^+\left((t_n - s)\left((f^\infty + \varepsilon) + \tfrac{c_\infty}{\|u_n\|^{p^- -1}}\right)\right) ds \\
&\le \psi^+\left((f^\infty + \varepsilon) + \tfrac{c_\infty}{\|u_n\|^{p^- -1}}\right)\int_0^{t_n} (t_n - s)^{q^- -1} ds \\
&= \psi^+\left((f^\infty + \varepsilon) + \tfrac{c_\infty}{\|u_n\|^{p^- -1}}\right)\tfrac{t_n^{q^-}}{q^-}.
\end{aligned}$$

Similarly, we obtain from

$$\|u_n\| = u_n(t_n) \le \lambda_n u_n(t_n) = \int_{t_n}^1 \psi\left(s, \int_{t_n}^s f(\tau, u(\tau))d\tau\right) ds$$

that

$$\begin{aligned}
1 &\le \psi^+\left((f^\infty + \varepsilon) + \tfrac{c_\infty}{\|u_n\|^{p^- -1}}\right)\int_{t_n}^1 (s - t_n)^{q^- -1} ds \\
&= \psi^+\left((f^\infty + \varepsilon) + \tfrac{c_\infty}{\|u_n\|^{p^- -1}}\right)\tfrac{(1-t_n)^{q^-}}{q^-}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
1 &\le \psi^+\left((f^\infty + \varepsilon) + \tfrac{c_\infty}{\|u_n\|^{p^- -1}}\right)\frac{\min\left((1-t_n)^{q^-}, t_n^{q^-}\right)}{q^-} \\
&\le \psi^+\left((f^\infty + \varepsilon) + \tfrac{c_\infty}{\|u_n\|^{p^- -1}}\right)\tfrac{1}{q^- 2^{q^-}}.
\end{aligned}$$

Letting $n \to \infty$, yields the contradiction

$$\Lambda_3 = q^- 2^{q^-} \le \psi^+(f^\infty + \varepsilon) < \Lambda_3.$$

Our claim is then proved and there exists R_∞ large such that for all $R \in (R_\infty, +\infty)$,

$$T_\varphi u \neq \lambda u \text{ for all } u \in \partial K_R \text{ and } \lambda \geq 1$$

and so, we have from Lemma 2.3 that

$$i\left(T_\varphi, K_R, K\right) = 1 \text{ for all } R \in \left(R_\infty, +\infty\right).$$

At the end, if r_1 and R_1 are such that $0 < r_1 < r_0 < R_\infty < R_1$ then we have

$$i\left(T_\varphi, K_{R_1} \setminus \overline{K_{r_1}}, K\right) = i\left(T_\varphi, K_{R_1}, K\right) - i\left(T_\varphi, K_{r_1}, K\right) = 1$$

and T_φ has a fixed point $u \in K_{R_1} \setminus \overline{K_{r_1}}$ which by iii) of Lemma 2.7, is a positive solution to bvp (1.1). □

Theorem 3.2 (superlinear case)**.** *Assume that*

$$f(t,u) > 0 \text{ for all } t \in [0,1] \text{ and } u > 0,$$
$$p^+\left(p^- - 2\right) + 1 > 0 \text{ and}$$
$$f^0/\Lambda_2 < 1 < f_\infty/\Lambda_4.$$

Then bvp (1.1) admits at least one positive solution.

Proof. As in the proof of Theorem 3.1 we have to prove that the operator T_φ has a fixed point in $K \setminus \{0\}$.

Let $\epsilon > 0$ be such that $\psi^+\left(f^0 + \epsilon\right) < \lambda\left(p^+\right)$. There exists $\delta > 0$ such that for all $t \in [0,1]$ and $x \in [0,\delta]$, $f(t,x) \leq \left(f_0 - \epsilon\right) x^{p^+ - 1}$.

Set $r_0 = \min(1,\delta)$ and let $r \in (0, r_0)$, $u \in \partial K_r$ and $\lambda \geq 1$ be such that $T_\varphi u = \lambda u$. Let $t_u \in (0,1)$ be such that $u\left(t_u\right) = \|u\|$, $u'\left(t_u\right) = 0$ and without loss of generality assume that $t_u \leq 1/2$, then we have for all $t \in [0, t_u]$,

$$\begin{aligned}
u(t) &\leq \lambda u(t) = T_\varphi u(t) = \int_0^t \psi\left(s, \int_s^{t_u} f(\tau, u(\tau)) d\tau\right) ds \\
&\leq \int_0^t \psi^+\left(\int_s^{t_u} f(\tau, u(\tau)) d\tau\right) ds \\
&\leq \int_0^t \psi^+\left(\int_s^{t_u} \left(f^0 - \epsilon\right) u^{p^+ - 1}(\tau) d\tau\right) ds \\
&\leq \int_0^t \psi^+\left(\int_s^{1/2} \left(f^0 - \epsilon\right) u^{p^+ - 1}(\tau) d\tau\right) ds \\
&\leq \psi^+\left(f^0 - \epsilon\right) \int_0^t \psi_{p^+}\left(\int_s^{1/2} \phi_{p^+}\left(u(\tau)\right) d\tau\right) ds \\
&= \psi^+\left(f^0 - \epsilon\right) N^l_{\phi_{p^+}} u\left(t\right)
\end{aligned}$$

and for all $t \in [t_u, 1/2]$

$$\begin{aligned} u(t) &\leq \|u\| = u(t_u) \leq \lambda u(t_u) = Tu(t_u) \\ &\leq \int_0^{t_u} \psi^+ \left(\int_s^{t_u} f(\tau, u(\tau)) d\tau \right) ds \\ &\leq \int_0^{t} \psi \left(s, \int_s^{t_u} f(\tau, u(\tau)) d\tau \right) ds \\ &\leq \int_0^{t} \psi \left(s, \int_s^{1/2} f(\tau, u(\tau)) d\tau \right) ds \\ &\leq \int_0^{t} \psi^+ \left(\int_s^{1/2} \left(f^0 - \epsilon \right) u^{p^+ -1}(\tau) d\tau \right) ds \\ &\leq \psi^+ \left(f^0 - \epsilon \right) \int_0^{t} \phi_{p^+} \left(\int_s^{1/2} \phi_{p^+} \left(u(\tau) \right) d\tau \right) ds \\ &= \psi^+ \left(f^0 - \epsilon \right) N^l_{\phi_{p^+}} u(t). \end{aligned}$$

Thus, the above estimates show that for all $t \in [0, 1/2]$

$$u(t) \leq \psi^+ \left(f^0 - \epsilon \right) N^l_{\phi_{p^+}} u(t),$$

leading to

$$\left(\psi^+ \left(f^0 - \epsilon \right) \right)^{-1} \in \Theta_K^{N^l_{\phi_{p^+}}} \text{ and } \left(\psi^+ \left(f^0 - \epsilon \right) \right)^{-1} \leq \left(\lambda \left(p^+ \right) \right)^{-1}$$

and so, this yields the contradiction

$$\lambda \left(p^+ \right) \leq \psi^+ \left(f^0 - \epsilon \right) < \lambda \left(p^+ \right).$$

Therefore, we conclude that for all $r \in (0, r_0)$ and $u \in \partial K_r$, $T_\varphi u \neq \lambda u$ for all $\lambda \geq 1$ and by Lemma 2.3, we have

$$i \left(T_\phi, K_r, K \right) = 1 \text{ for all } r \in (0, r_0).$$

Now consider the homotopical equation

(3.3) $$u = T_{\varphi_\theta} u$$

where for $\theta \in [0, 1]$, $\varphi_\theta(x) = |x|^{p_\theta(t)-2} x$ and $p_\theta(t) = (1-\theta) p(t) + \theta p^+$.

Clearly, Equation (3.3) is equivalent to bvp

(3.4) $$\begin{cases} -(\varphi_\theta(t, u'(t)))' = f(t, u(t)), \ t \in (0, 1), \\ u(0) = u(1) = 0. \end{cases}$$

In order to use the homotopy property of the fixed point index, let us prove existence of R_∞ large, such that for all $R > R_\infty$ bvp (3.4) (respectively, Equation (3.3)) has no solution in ∂K_R.

To the contrary, suppose that there exist sequences $(\theta_n) \subset [0,1]$ and $(u_n) \subset K$ with $\|u_n\| \to \infty$ such that

$$\left\{ \begin{array}{l} -(\varphi_{\theta_n}(t, u_n'(t)))' = f(t, u_n(t)),\ t \in (0,1), \\ u_n(0) = u_n(1) = 0. \end{array} \right. \tag{3.5a}$$

Arguing as in the proof of Lemma 2.9, we see that u_n admits a unique critical point $t_n \in (0,1)$ at which it reaches its supremum.

Let $\epsilon > 0$ be such that $(f_\infty - \epsilon) > \Lambda_4 \geq (2I(p))^{\gamma^+(p)}(p^+ - 1)$. There exists $M_\infty > 0$ such that $f(t,x) > (f_\infty - \epsilon)\, x^{\gamma^+(p)-1}$ for all $t \in [0,1]$ and $x > M_\infty$.

It is easy to see that $p^+ \leq \gamma^+(p)$ and $\gamma^+(p_\theta) \leq \gamma^+(p)$. Therefore, we have

$$\lim \inf_{x \to +\infty} \left(\min_{t \in [0,1]} \frac{f(t,x)}{x^{\gamma^+(p_\theta)-1}} \right) \geq f_\infty > \Lambda_4 \geq (2I(p))^{\gamma^+(p)} \left(p^+ - 1\right) \tag{3.6}$$

and

$$\lim \inf_{x \to +\infty} \left(\min_{t \in [0,1]} \frac{f(t,x)}{x^{p^+-1}} \right) \geq f_\infty > \Lambda_4 \geq \lambda\left(p^+\right). \tag{3.7}$$

We have also that for all $(t,x) \in [0,1] \times \mathbb{R}^+$ and $\theta \in [0,1]$

$$\varphi_\theta(t,x) \leq \varphi^+(x) \text{ and } \psi_\theta(t,x) \leq \psi^+(x)$$

where $\psi_\theta(t,\cdot)$ is the inverse function of $\varphi_\theta(t,\cdot)$.

Therefore, multiplying the differential equation in (3.5a) by u_n' and inegrating between t and t_n we get,

$$\int_t^{t_n} -(\varphi_{\theta_n}(s, u_n'(s)))' u_n'(s) ds = \int_t^{t_n} f(s, u_n(s)) u_n'(s) ds.$$

Therefore, we have for each of the cases $t < t_n$ and $t_n < t$,

$$\begin{aligned} \int_t^{t_n} -(\varphi_{\theta_n}(s, u_n'(s)))' u_n'(s) ds &= \int_t^{t_n} -(\varphi_{\theta_n}(s, u_n'(s)))' \psi_{\theta_n}(s, \varphi_{\theta_n}(s, u_n'(s))\, ds \\ &\leq \int_t^{t_n} -(\varphi_{\theta_n}(s, u_n'(s)))' \psi^+(\varphi_{\theta_n}(s, u_n'(s))\, ds \\ &= \Gamma^+(\varphi_{\theta_n}(t, |u_n'(t)|) \leq \Gamma^+\left(\varphi^+(|u_n'(t)|\right) \\ &= W_+(|u_n'(t)|). \end{aligned}$$

That is

$$W_+(|u_n'(t)|) \geq \int_t^{t_n} f(s, u_n(s)) u_n'(s) ds, \quad \text{for all } t \in [0,1]$$

and so,

$$u_n'(t) \geq \Pi\left(\int_t^{t_n} f(s, u_n(s)) u_n'(s) ds \right), \quad \text{for all } t \in (0, t_n)$$

and

$$-u_n'(t) \geq \Pi\left(\int_t^{t_n} f(s,u_n(s))u_n'(s)ds\right), \quad \text{for all } t \in (0,t_n).$$

The above estimates lead to

$$t_n = \int_{u_n(0)}^{u_n(t_n)} \frac{du_n(t)}{u_n'(t)} \leq \int_0^{\|u_n\|} \frac{du_n(t)}{\Pi\left(\int_t^{t_n} f(s,u_n(s))u_n'(s)ds\right)} \tag{3.8}$$

and

$$1-t_n = \int_{u_n(1)}^{u_n(t_n)} \frac{du_n(t)}{-u_n'(t)} \leq \int_0^{\|u_n\|} \frac{du_n(t)}{\Pi\left(\int_t^{t_n} f(s,u_n(s))u_n'(s)ds\right)}. \tag{3.9}$$

Let ξ_n and η_n be such that $0 < \xi_n < t_n < \eta_n < 1$ and $u_n(\xi_n) = u_n(\eta_n) = M_\infty$. We have from (3.8) and (3.9),

$$\begin{aligned} t_n &\leq \int_0^{M_\infty} \frac{du_n(t)}{\Pi\left(\int_t^{t_n} f(s,u_n(s))u_n'(s)ds\right)} + \int_{M_\infty}^{\|u_n\|} \frac{du_n(t)}{\Pi\left(\int_t^{t_n} f(s,u_n(s))u_n'(s)ds\right)} \\ &\leq \int_0^{M_\infty} \frac{du_n(t)}{\Pi\left(\int_{\xi_n}^{t_n} f(s,u_n(s))u_n'(s)ds\right)} + \int_{M_\infty}^{\|u_n\|} \frac{du_n(t)}{\Pi\left(\int_t^{t_n} f(s,u_n(s))u_n'(s)ds\right)} \\ &\leq \int_0^{M_\infty} \frac{du_n(t)}{\Pi\left(\int_{\xi_n}^{t_n} (f_\infty-\epsilon)u_n^{\gamma^+(p)-1}(s))u_n'(s)ds\right)} + \int_{M_\infty}^{\|u_n\|} \frac{du_n(t)}{\Pi\left(\int_t^{t_n} (f_\infty-\epsilon)u_n^{\gamma^+(p)-1}(s))u_n'(s)ds\right)} \\ &\leq \frac{M_\infty}{\Pi\left(\frac{f_\infty-\epsilon}{\gamma^+}\left(\|u_n\|^{\gamma^+(p)} - M_\infty^{\gamma^+(p)}\right)\right)} + \int_{M_\infty}^{\|u_n\|} \frac{du_n(t)}{\Pi\left(\frac{f_\infty-\epsilon}{\gamma^+}\left(\|u_n\|^{\gamma^+(p)} - u_n^{\gamma^+(p)}(t)\right)\right)} \end{aligned}$$

and

$$1-t_n \leq \frac{M_\infty}{\Pi\left((f_\infty-\epsilon)\left(\|u_n\|^{\gamma^+(p)-1} - M_\infty^{\gamma^+(p)-1}\right)\right)} + \int_{M_\infty}^{\|u_n\|} \frac{du_n(t)}{\Pi\left((f_\infty-\epsilon)\left(\|u_n\|^{\gamma^+(p)-1} - u_n^{\gamma^+(p)-1}(t)\right)\right)}.$$

Adding, we get

$$\begin{aligned} 1 &\leq \frac{2M_\infty}{\Pi\left(\frac{f_\infty-\epsilon}{\gamma^+(p)}\left(\|u_n\|^{\gamma^+(p)} - M_\infty^{\gamma^+(p)}\right)\right)} + 2\int_{M_\infty}^{\|u_n\|} \frac{du_n(t)}{\Pi\left(\frac{f_\infty-\epsilon}{\gamma^+}\left(\|u_n\|^{\gamma^+(p)} - u_n^{\gamma^+(p)}(t)\right)\right)} \\ &\leq \frac{2M_\infty}{\Pi\left(\frac{f_\infty-\epsilon}{\gamma^+(p)}\left(\|u_n\|^{\gamma^+(p)} - M_\infty^{\gamma^+(p)}\right)\right)} + 2\int_0^{\|u_n\|} \frac{du_n(t)}{\Pi\left(\frac{f_\infty-\epsilon}{\gamma^+}\left(\|u_n\|^{\gamma^+(p)} - u_n^{\gamma^+(p)}(t)\right)\right)} \\ &\leq \frac{2M_\infty}{\Pi\left(\frac{f_\infty-\epsilon}{\gamma^+(p)}\left(\|u_n\|^{\gamma^+(p)} - M_\infty^{\gamma^+(p)}\right)\right)} + \frac{2\|u_n\|I(p)}{\Pi\left(\frac{f_\infty-\epsilon}{\gamma^+(p)}\|u_n\|^{\gamma^+(p)}\right)}. \end{aligned}$$

Letting $n \to \infty$, we get from (3.1) the contradiction

$$\begin{aligned} 1 &\leq \lim \frac{2\|u_n\|I(p)}{\Pi\left(\frac{f_\infty-\epsilon}{\gamma^+(p)}\|u_n\|^{\gamma^+(p)}\right)} = \left((2I(p))^{\gamma^+(p)} \frac{\gamma^+(p)}{(f_\infty-\epsilon)q^+}\right)^{1/\gamma^+(p)} \\ &= \left(\frac{(2I(p))^{\gamma^+(p)}\left(p^+-1\right)}{(f_\infty-\epsilon)}\right)^{1/\gamma^+(p)} < 1. \end{aligned}$$

Thus, we have proved existence of such a real number R_∞ and we have from the homotopy property of the fixed point index, (3.7) and Lemma 2.9 that for all $R \geq R_\infty$

$$i\left(T_\varphi, K_R, K\right) = i\left(T_{\varphi_\theta}, K_R, K\right) = i\left(T_{\phi_{p^+}}, K_R, K\right) = 0.$$

At the end, if r_1 and R_1 are such that $0 < r_1 < r_0 < R_\infty < R_1$ then we have

$$i\left(T_\varphi, K_{R_1} \setminus \overline{K_{r_1}}, K\right) = i\left(T_\varphi, K_{R_1}, K\right) - i\left(T_\varphi, K_{r_1}, K\right) = -1$$

and T_φ has a fixed point $u \in K_{R_1} \setminus \overline{K_{r_1}}$ which by iii) of Lemma 2.7, is a positive solution to bvp (1.1) □

Remark 3.3. Observe that if $p^- \geq 2$ then $p^+\left(p^- - 2\right) + 1 > 0$.

Observe also that Theorem 3.2 does not hold if $p^- < 3/2 < 2 < p^+$. Indeed, in this case we have $1/\left(2 - p^-\right) < 2$.

It is easy to check that in the case where the weight $p \equiv p_0 \in (1, +\infty)$, we have $\gamma^+(p) = \gamma^-(p) = p_0$ and then $\Lambda_\infty(p) = \lambda(p_0)$.

Example 3.4. Consider bvp (1.1) and $f(t,u) = u^\sigma$ with $\sigma \geq 0$. We conclude from Theorem 3.1 and Theorem 3.2 that in this case bvp (1.1) admits a positive solution for all $\sigma \in [0, p^- - 1)$ and for all $\sigma \in \left(\gamma^+(p), +\infty\right)$ if $p^+\left(p^- - 2\right) + 1 > 0$.

Example 3.5. Consider bvp (1.1) with

$$p(t) = \tfrac{3+t}{1+t} \text{ and } f(t,u) = \tfrac{Au + Bu^2}{1+u}$$

where A, B are positive real numbers.

By simple computations we get that $p^- = 2,\ p^+ = 3$, and

$$f_0 = A,\ f^\infty = B,\ \Lambda_1 = \pi^4,\ \Lambda_3 = 3\sqrt{2}.$$

We deduce from Theorem 3.1 that bvp (1.1) admits a positive solution if $B < 3\sqrt{2} < \pi^4 < A$.

Example 3.6. Consider bvp (1.1) with

$$p(t) = 2 + 2t \text{ and } f(t,u) = \tfrac{Au + Bu^5}{1+u}$$

where A, B are positive real numbers.

By simple computations we get that

$$p^- = 2 \quad p^+ = 4 \quad \gamma^-(p) = 4/3 \quad \gamma^+(p) = 6$$

and

$$f^0 = A,\ f_\infty = B,\ \Lambda_2 = \lambda(4),\ \Lambda_4 = \max\left(\lambda(4), 3\left(2\int_0^1 \tfrac{ds}{(1-s^6)^{3/4}}\right)^6\right).$$

We deduce from Theorem 3.2 that bvp (1.1) admits a positive solution if $A < \lambda(4) < \Lambda_4 < B$.

Concluding remarks. In the end of this work, we want to compare results obtained in this paper with those in [2]. Since the boundary conditions in bvp (1.1) considerd in [2] are of diffrent nature than Dirichlet ones, the comparison will be in form not in details.

As in [2], there is no restriction here on the weight p in the sublinear case. Contrary to [2], there is a restriction here on the weight p in the superlinear case. For example, the case where $p(t) = t + 3/2$ (we have $p^+(p^- - 2) + 1 < 0$) is not covered by Theorem 3.2.

In fact the condition $p^+(p^- - 2) + 1 > 0$ in Theorem 3.2, was imposed by quadrature techniques used in the proof of this result and note that it is satisfied in the case $p \equiv p_0 \in (1, +\infty)$. So, Theorems 3.1 and 3.2 cover the case where the exponent p is constant.

The fact that there was no restriction on the weight p in [2] is mainly due to the fact that the boundary conditions in [2], set in advance the point $x_0 = 1$ as the point where a possible positive solution reaches its maximum. In fact, this is what allowed us technically to convert the problem of seeking positive solution to bvp (1.1) in [2] to the problem of finding those of a first order initial value problem, on which homotopical argument on the nonlinearity permits us to compute the fixed point index at ∞ . Clearly, here it is not the case, the point where a possible positive solution to bvp (1.1) reaches it maximum is unknown.

Acknowledgments

The authors would like to thank their laboratory Fixed Point Theory and Applications for supporting this work. They are grateful to anonymous referees, for their careful reading of the first version of this manuscript and their constructive suggestions and comments.

References

[1] B. Amaziane, L. Pankratov and V. Prytula, Nonlocal effects in homogenization of $p_\epsilon(x)$-Laplacian in perforated domains, *Nonlinear Anal.* **71** (2009), no. 9, 4078–4097.

[2] N. Benkaci-Ali, A. Benmezai and J. Henderson, Positive solutions for a $p(t)$-Laplacian three point boundary value problem *Dynam. System Appl.* **22** (2013), no. 4, 493–502.

[3] Benmezai, S. Djebali and T. Moussaoui, Positive solution for $\phi-$Laplacian Dirichlet bvps, *Fixed Point Theory* **8** (2007), no. 2, 167–186.

[4] A. Benmezai and S. Mechrouk, Positive solution to the nonlinear abstract Hammerstein equation and application to $\phi-$Laplacian BVPs, *Nonlinear Differential Equations Appl.* **20** (2013), no. 3, 489–510.

[5] G. Dai, Infinitely many non-negative solutions for a Dirichlet problem involving $p(x)$-Laplacian, *Nonlinear Anal.* **71** (2009), no. 11, 5840–5849.

[6] X. Fan, On the sub-supersolution method for $p(x)$-Laplacian equations, *J. Math. Anal. Appl.* **330** (2007), no. 1, 665–682.

[7] X. Fan and Q. Zhang, Existence of solutions for p(x)-Laplacian Dirichlet problems, *Nonlinear Anal.* **52** (2003), no. 8, 1843–1852.

[8] X. Fan, H. Wu and F. Wang, Hartman-type results for p(t)-Laplacian systems, *Nonlinear Anal.* **52** (2003), no. 2, 585–594.

[9] X. Fan, Remarks on eigenvalue problems involving the $p(x)$-Laplacian, *J. Math. Anal. Appl.* **352** (2009), no. 1, 85–98.

[10] D. Guo and Lakshmikantham, Nonlinear problems in abstract cones, Notes and Reports in Mathematics in Science and Engineering, 5, Academic Press, Inc., Boston,

[11] M. Mihăilescu and D. Repovš, On a PDE involving the $A_p(\cdot)$-Laplace operator, *Nonlinear Anal.* **75** (2012), no. 2, 975–981.

[12] Q. Zhang, X. Liu and Z. Qiu, Existence of solutions and multiple solutions for a class of weighted p(r)-Laplacian system, *J. Math. Anal. Appl.* **355** (2009), no. 2, 620–633.

[13] Q. Zhang, X. Liu and Z. Qiu, Existence of solutions for weighted $p(t)$-Laplacian system multi-point boundary value problems, *Nonlinear Anal.* **71** (2009), no. 9, 3715–3727.

[14] Q. Zhang, Y. Wang and Z. Qiu, Existence of solutions and boundary asymptotic behavior of $p(r)$-Laplacian equation multi-point boundary value problems, *Nonlinear Anal.* **72** (2010), no. 6, 2950–2973.

[15] Q. Zhang, Oscillatory property of solutions for $p(t)$-Laplacian equations, *J. Inequal. Appl.* **2007** Article ID 58548, 8 pages.

[16] Q. Zhang, Z. Qiu and X. Liu, Existence of solutions and nonnegative solutions for weighted $p(r)$-Laplacian impulsive system multi-point boundary value problems, *Nonlinear Anal.* **71** (2009), no. 9, 3814–3825.

(Abdelhamid Benmezai) Faculty of Mathematics, USTHB, Algiers, Algeria.
E-mail address: `aehbenmezai@gmail.com`

(Salima Mechrouk) Faculty of Sciences, UMB, Boumerdes, Algeria.
E-mail address: `mechrouksalima@yahoo.fr`