Title:
A note on lacunary series in $\mathcal{Q}_K$ spaces

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A NOTE ON LACUNARY SERIES IN $Q_K$ SPACES

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Abstract. In this paper, under the condition that $K$ is concave, we characterize lacunary series in $Q_K$ spaces. We improve a result due to H. Wulan and K. Zhu.

Keywords: $Q_K$ spaces; lacunary series; concave.

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1. Introduction

Let $D$ be the unit disk in the complex plane $\mathbb{C}$ and denote by $\partial D$ the boundary of $D$. As usual, $H(D)$ is the class of functions analytic in $D$. The Green function in the unit disk with singularity at $a \in D$ is given by

$$g(z,a) = \log \frac{1}{|\sigma_a(z)|}, \ z \in D.$$  

Here

$$\sigma_a(z) = \frac{a - z}{1 - \overline{a}z},$$

is the Möbius transformation of $D$.

Throughout this paper, we assume that $K : [0, \infty) \to [0, \infty)$ is an increasing function. A function $f \in H(D)$ belongs to the space $Q_K$ if

$$\left\|f\right\|_{Q_K}^2 = \sup_{a \in D} \int_D |f'(z)|^2 K(g(z,a)) \ dA(z) < \infty,$$

where $dA$ is the element of Euclidean area on $D$ normalized so that $dA(z) = \pi^{-1} dx dy$. $Q_K$ spaces are Möbius invariant in the sense that $\left\|f \circ \sigma_a\right\|_{Q_K} = \left\|f\right\|_{Q_K}$ for every $f \in Q_K$ and $a \in D$. See [3–5] for more results of $Q_K$ spaces. If $K(t) = t^p$, $0 \leq p < \infty$, then the space $Q_K$ reduces to the space $Q_p$ (cf. [7,8]). In particular, $Q_0$ is the Dirichlet space; $Q_1 = BMOA$, the space of bounded mean oscillation; $Q_p$ is the Bloch space for all $p > 1$. 


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Recall that a function \( f(z) = \sum_{k=1}^{\infty} a_k z^{n_k} \in H(\mathbb{D}) \) is called a lacunary series if
\[
\lambda = \inf_k \frac{n_{k+1}}{n_k} > 1.
\]
Such series are often used to give examples of functions in various analytic function spaces. It is well known that a lacunary series belongs to BMOA if and only if it is in the Hardy space \( H^2 \) (see [2]). By [1], if \( 0 < p < 1 \), then the lacunary series \( f \in Q_p \) if and only if \( \sum_{k=1}^{\infty} n_k^{1-p} |a_k|^2 < \infty \).


**Theorem A** ([6]). Let \( K \) satisfy
\[
\int_1^{\infty} \frac{\varphi_K(s)}{s^2} ds < \infty,
\]
where
\[
\varphi_K(s) = \sup_{0 \leq t \leq 1} K(st)/K(t), \quad 0 < s < \infty.
\]
Then a lacunary series
\[
f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}
\]
belongs to \( Q_K \) if and only if
\[
\sum_{k=1}^{\infty} n_k |a_k|^2 K \left( \frac{1}{n_k} \right) < \infty.
\]

By [3], the space \( Q_K \) only depends on the weight function \( K \) in a neighbourhood of the origin. If \( K_0(t) = t \log \frac{e}{t}, \ 0 < t < 1 \), then \( Q_{K_0} \) is the analytic version of \( Q_1(\partial \mathbb{D}) \) space (see [7] and [9]). An elementary calculation shows that \( \varphi_{K_0}(s) = s \) when \( s \geq 1 \). Thus, \( K_0 \) does not satisfy the condition (1.1). In other words, Theorem A misses the case of the analytic version of \( Q_1(\partial \mathbb{D}) \) space. It was pointed out in [6] that Theorem A also misses the classical case of BMOA. The goal of this article is to characterize lacunary series in \( Q_K \) spaces under a weaker condition of \( K \). Our result covers the cases of BMOA and the analytic version of \( Q_1(\partial \mathbb{D}) \) space.

We write \( A \lesssim B \) if there exists a constant \( C \) such that \( A \leq CB \), in addition, the symbol \( A \approx B \) means that \( A \lesssim B \lesssim A \).

### 2. Main result

In [6], we can find many nice estimates of the weight function \( K \) under the condition (1.1). In recent years, the study of \( Q_K \) spaces benefits from these estimates. In particular, if \( K \) satisfies (1.1), then there exists an increasing function \( K^* \) on \( (0, \infty) \) such that \( K^*(t) \approx K(t) \) for all \( t \in (0, \infty) \). Moreover,
$K^*$ is twice differentiable on $(0, \infty)$ and concave. Namely, $K''(t) \leq 0$ for $t \in (0, \infty)$. Next we state the main result of this paper.

**Theorem 2.1.** Let $K$ be a concave function. Then a lacunary series

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$$

belongs to $Q_K$ if and only if

$$\sum_{k=1}^{\infty} n_k |a_k|^2 K \left( \frac{1}{n_k} \right) < \infty. \quad (2.1)$$

**Proof.** We prove the result by following the proof of Theorem A in [6]. First suppose that $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k} \in Q_K$. Then

$$\int |f'(z)|^2 K \left( \log \frac{1}{|z|} \right) dA(z) < \infty.$$

Bearing in mind that $K$ is increasing, we get

$$\infty > \sum_{k=1}^{\infty} n_k^2 |a_k|^2 \int_0^1 r^{2n_k-1} K \left( \frac{1}{r} \right) dr$$

$$\geq \sum_{k=1}^{\infty} n_k^2 |a_k|^2 \int_0^\infty e^{-2n_k t} K(t) dt$$

$$\geq \sum_{k=1}^{\infty} n_k |a_k|^2 K \left( \frac{1}{n_k} \right).$$

On the other hand, suppose that the condition (2.1) holds. Since $K$ is concave, the estimates in [6, page 226] show that

$$\sup_{a \in D} \int \int |f'(z)|^2 K(g(z,a)) dA(z) \leq 2 \int_0^1 r \left( \sum_{k=1}^{\infty} n_k |a_k| r^{n_k-1} \right)^2 K \left( \log \frac{1}{r} \right) dr.$$

Write $I_n = \{ k : 2^n \leq k < 2^{n+1}, k \in \mathbb{N} \}$. The Hölder inequality gives

$$\left( \sum_{k=1}^{\infty} n_k |a_k|^2 r^{n_k} \right)^2 \leq \left( \sum_{n=0}^{\infty} \sum_{n_k \in I_n} n_k |a_k|^2 r^{n_k} \right)^2$$

$$\leq \sum_{n=0}^{\infty} 2^{n/2} r^{2n} \sum_{n=0}^{\infty} 2^{-n/2} r^{-2n} \left( \sum_{n_k \in I_n} n_k |a_k| \right)^2$$

$$\leq \left( \log \frac{1}{r} \right)^{-1/2} \sum_{n=0}^{\infty} 2^{-n/2} r^{2n} \left( \sum_{n_k \in I_n} n_k |a_k| \right)^2.$$
Thus,
\[
\sup_{a \in \mathcal{D}} \int_D |f'(z)|^2 K(g(z, a)) dA(z) \lesssim \sum_{n=0}^{\infty} 2^{-n/2} \left( \sum_{n_k \in I_n} n_k|a_k| \right)^2 \int_0^1 r^{2^n-1} \left( \log \frac{1}{r} \right)^{-1/2} K \left( \log \frac{1}{r} \right) dr.
\]

Since \( K \) is increasing, we see that
\[
\int_{e^{-2^n}}^1 r^{2^n-1} \left( \log \frac{1}{r} \right)^{-1/2} K \left( \log \frac{1}{r} \right) dr \leq K \left( \frac{1}{2^n} \right) \int_{1/2^n}^{1/2^n} e^{-2^n t} t^{-1/2} dt = 2^{-\frac{3}{4}} K \left( \frac{1}{2^n} \right) \int_0^1 e^{-s} s^{-1/2} ds.
\]
If \( K(0) \neq 0 \), by [3], we can assume that \( K \) is constant. Of course, \( K(t)/t \) is decreasing on \((0, \infty)\). If \( K(0) = 0 \), we claim that \( K(t)/t \) is decreasing on \((0, \infty)\). Choose \( s, t \in (0, \infty) \) such that \( s < t \). By the basic property of a concave function, we have
\[
\frac{K(s)}{s} = \frac{K(s) - K(0)}{s - 0} \geq \frac{K(t) - K(s)}{t - s}.
\]
Thus
\[
\frac{K(s)}{s} - \frac{K(t)}{t} = t \frac{K(s) - sK(t)}{st} = \frac{1}{st} [s(K(s) - K(t)) + (t - s)K(s)] = \frac{t - s}{t} \left[ \frac{K(s)}{s} - \frac{K(t) - K(s)}{t - s} \right] \geq 0.
\]
Hence,
\[
\int_{e^{-2^n}}^1 r^{2^n-1} \left( \log \frac{1}{r} \right)^{-1/2} K \left( \log \frac{1}{r} \right) dr \leq 2^n K \left( \frac{1}{2^n} \right) \int_{1/2^n}^{\in\infty} e^{-2^n t} t^{1/2} dt = 2^{-\frac{3}{2}} K \left( \frac{1}{2^n} \right) \int_1^{\infty} e^{-s} s^{1/2} ds.
\]
Therefore,
\[
\sup_{a \in \mathcal{D}} \int_D |f'(z)|^2 K(g(z, a)) dA(z) \lesssim \sum_{n=0}^{\infty} 2^{-n} K \left( \frac{1}{2^n} \right) \left( \sum_{n_k \in I_n} n_k \right)^2.
\]
If \( n_k \in I_n \), then \( n_k \geq 2^n \). Using the monotonicity of \( K(t)/t \), one gets
\[
n_k K \left( \frac{1}{n_k} \right) \geq 2^n K \left( \frac{1}{2^n} \right).
\]
This gives
\[
\sup_{a \in \mathbb{D}} \int_{D} |f'(z)|^2 K(g(z,a)) dA(z)
\leq \sum_{n=0}^{\infty} 2^{-2n} \left( \sum_{n_k \in I_n} n_k |a_k| \sqrt{n_k K \left( \frac{1}{n_k} \right)} \right)^2
\approx \sum_{n=0}^{\infty} \left( \sum_{n_k \in I_n} n_k |a_k| \sqrt{n_k K \left( \frac{1}{n_k} \right)} \right)^2.
\]
Since \( f \) is a lacunary series, then there exists a positive constant \( \lambda \) such that \( \frac{n_{k+1}}{n_k} \geq \lambda > 1 \) for all \( k \). It is well known that the Taylor series of \( f(z) \) has at most \( \lfloor \log_2 2 \rfloor + 1 \) terms \( a_k z^{n_k} \) such that \( n_k \in I_n \) for all \( n \in \mathbb{N} \). By Hölder inequality, we obtain
\[
\sup_{a \in \mathbb{D}} \int_{D} |f'(z)|^2 K(g(z,a)) dA(z)
\leq (\lfloor \log_2 2 \rfloor + 1) \sum_{n=0}^{\infty} \sum_{n_k \in I_n} n_k |a_k| K \left( \frac{1}{n_k} \right)
\approx \sum_{k=1}^{\infty} n_k |a_k| K \left( \frac{1}{n_k} \right) < \infty.
\]
The proof is complete. \( \square \)

Note that \( K(t) = t^p \), \( 0 \leq p \leq 1 \) is concave. The following result follows easily from Theorem 2.1.

**Corollary 2.2** ([7]). Let \( p \in [0, 1] \). Then a lacunary series
\[
f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}
\]
belongs to \( Q_p \) if and only if
\[
\sum_{k=1}^{\infty} n_k^{1-p} |a_k|^2 < \infty.
\]
Since \( K_0(t) = t \log \frac{t}{t} \) is also concave, we have the following statement.

**Corollary 2.3** ([9]). A lacunary series
\[
f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}
\]
belongs to $Q_1(\partial \mathbb{D})$ if and only if
\[
\sum_{k=1}^{\infty} \log(1 + n_k) |a_k|^2 < \infty.
\]

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