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A NOTE ON LACUNARY SERIES IN $\mathcal{Q}_K$ SPACES

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Abstract. In this paper, under the condition that K is concave, we characterize lacunary series in $\mathcal{Q}_K$ spaces. We improve a result due to H. Wulan and K. Zhu.

Keywords: $\mathcal{Q}_K$ spaces; lacunary series; concave.
MSC(2010): Primary: 30H25; Secondary: 30B10, 30H05.

1. **Introduction**

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$ and denote by $\partial\mathbb{D}$ the boundary of $\mathbb{D}$. As usual, $H(\mathbb{D})$ is the class of functions analytic in $\mathbb{D}$. The Green function in the unit disk with singularity at $a \in \mathbb{D}$ is given by

$$g(z,a) = \log \frac{1}{|\sigma_a(z)|},\ z \in \mathbb{D}.$$

Here

$$\sigma_a(z) = \frac{a-z}{1-\overline{a}z},$$

is the Möbius transformation of $\mathbb{D}$.

Throughout this paper, we assume that $K : [0,\infty) \to [0,\infty)$ is an increasing function. A function $f \in H(\mathbb{D})$ belongs to the space $\mathcal{Q}_K$ if

$$\|f\|^2_{\mathcal{Q}_K} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K\left(g(z,a)\right) dA(z) < \infty,$$

where dA is the element of Euclidean area on $\mathbb{D}$ normalized so that $dA(z) = \pi^{-1}dxdy$. $\mathcal{Q}_K$ spaces are Möbius invariant in the sense that $\|f \circ \sigma_a\|_{\mathcal{Q}_K} = \|f\|_{\mathcal{Q}_K}$ for every $f \in \mathcal{Q}_K$ and $a \in \mathbb{D}$. See [3–5] for more results of $\mathcal{Q}_K$ spaces. If $K(t) = t^p$, $0 \le p < \infty$, then the space $\mathcal{Q}_K$ reduces to the space $\mathcal{Q}_p$ (cf. [7,8]). In particular, $\mathcal{Q}_0$ is the Dirichlet space; $\mathcal{Q}_1 = BMOA$, the space of bounded mean oscillation; $\mathcal{Q}_p$ is the Bloch space for all $p > 1$.

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Recall that a function $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k} \in H(\mathbb{D})$ is called a lacunary series if

$$\lambda = \inf_k \frac{n_{k+1}}{n_k} > 1.$$

Such series are often used to give examples of functions in various analytic function spaces. It is well known that a lacunary series belongs to $BMOA$ if and only if it is in the Hardy space H^2 (see [2]). By [1], if $0 < p < 1$, then the lacunary series $f \in \mathcal{Q}_p$ if and only if $\sum_{k=1}^{\infty} n_k^{1-p}|a_k|^2 < \infty$.

H. Wulan and K. Zhu [6] characterized lacunary series in $\mathcal{Q}_K$ spaces as follows.

Theorem A ([6]). Let K satisfy

$$\int_1^{\infty} \frac{\varphi_K(s)}{s^2} ds < \infty, \tag{1.1}$$

where

$$\varphi_K(s) = \sup_{0\le t\le 1} K(st)/K(t), \quad 0 < s < \infty.$$

Then a lacunary series

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$$

belongs to $\mathcal{Q}_K$ if and only if

$$\sum_{k=1}^{\infty} n_k |a_k|^2 K\left(\frac{1}{n_k}\right) < \infty.$$

By [3], the space $\mathcal{Q}_K$ only depends on the weight function K in a neighbourhood of the origin. If $K_0(t) = t \log \frac{e}{t}$, $0 < t < 1$, then $\mathcal{Q}_{K_0}$ is the analytic version of $\mathcal{Q}_1(\partial\mathbb{D})$ space (see [7] and [9]). An elementary calculation shows that $\varphi_{K_0}(s) = s$ when $s \ge 1$. Thus, K_0 does not satisfy the condition (1.1). In other words, Theorem A misses the case of the analytic version of $\mathcal{Q}_1(\partial\mathbb{D})$ space. It was pointed out in [6] that Theorem A also misses the classical case of $BMOA$. The goal of this article is to characterize lacunary series in $\mathcal{Q}_K$ spaces under a weaker condition of K. Our result covers the cases of $BMOA$ and the analytic version of $\mathcal{Q}_1(\partial\mathbb{D})$ space.

We write $A \lesssim B$ if there exists a constant C such that $A \le CB$, in addition, the symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Main result

In [6], we can find many nice estimates of the weight function K under the condition (1.1). In recent years, the study of $\mathcal{Q}_K$ spaces benefits from these estimates. In particular, if K satisfies (1.1), then there exists an increasing function K^* on $(0, \infty)$ such that $K^*(t) \approx K(t)$ for all $t \in (0, \infty)$. Moreover,

K^* is twice differentiable on $(0,\infty)$ and concave. Namely, $K''(t) \le 0$ for $t \in (0,\infty)$. Next we state the main result of this paper.

Theorem 2.1. *Let K be a concave function. Then a lacunary series*

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$$

belongs to $\mathcal{Q}_K$ if and only if

$$\sum_{k=1}^{\infty} n_k |a_k|^2 K\left(\frac{1}{n_k}\right) < \infty. \tag{2.1}$$

Proof. We prove the result by following the proof of Theorem A in [6]. First suppose that $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k} \in \mathcal{Q}_K$. Then

$$\int_{\mathbb{D}} |f'(z)|^2 K\left(\log \frac{1}{|z|}\right) dA(z) < \infty.$$

Bearing in mind that K is increasing, we get

$$\begin{aligned} \infty &> \sum_{k=1}^{\infty} n_k^2 |a_k|^2 \int_0^1 r^{2n_k-1} K\left(\log\frac{1}{r}\right) dr \\ &\ge \sum_{k=1}^{\infty} n_k^2 |a_k|^2 \int_{\frac{1}{n_k}}^{\infty} e^{-2n_k t} K(t) dt \\ &\gtrsim \sum_{k=1}^{\infty} n_k |a_k|^2 K\left(\frac{1}{n_k}\right). \end{aligned}$$

On the other hand, suppose that the condition (2.1) holds. Since K is concave, the estimates in [6, page 226] show that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(g(z,a)) dA(z) \le 2\int_0^1 r \left(\sum_{k=1}^{\infty} n_k |a_k| r^{n_k-1}\right)^2 K\left(\log\frac{1}{r}\right) dr.$$

Write $I_n = \{k : 2^n \le k < 2^{n+1}, k \in \mathbb{N}\}$. The Hölder inequality gives

$$\begin{aligned} \left(\sum_{k=1}^{\infty} n_k |a_k| r^{n_k}\right)^2 &\le \left(\sum_{n=0}^{\infty} \sum_{n_k \in I_n} n_k |a_k| r^{2^n}\right)^2 \\ &\le \sum_{n=0}^{\infty} 2^{n/2} r^{2^n} \sum_{n=0}^{\infty} 2^{-n/2} r^{2^n} \left(\sum_{n_k\in I_n} n_k |a_k|\right)^2 \\ &\lesssim \left(\log\frac{1}{r}\right)^{-1/2} \sum_{n=0}^{\infty} 2^{-n/2} r^{2^n} \left(\sum_{n_k\in I_n} n_k |a_k|\right)^2. \end{aligned}$$

Thus,

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^2K(g(z,a))dA(z)\\ \lesssim\ &\sum_{n=0}^{\infty}2^{-n/2}\left(\sum_{n_k\in I_n}n_k|a_k|\right)^2\int_0^1 r^{2^n-1}\left(\log\frac{1}{r}\right)^{-1/2}K\left(\log\frac{1}{r}\right)dr.\end{aligned}$$

Since K is increasing, we see that

$$\begin{aligned}&\int_{e^{-\frac{1}{2^n}}}^1 r^{2^n-1}\left(\log\frac{1}{r}\right)^{-1/2}K\left(\log\frac{1}{r}\right)dr\\ \leq\ &K\left(\frac{1}{2^n}\right)\int_0^{1/2^n}e^{-2^nt}t^{-\frac{1}{2}}dt\\ =\ &2^{-\frac{n}{2}}K\left(\frac{1}{2^n}\right)\int_0^1 e^{-s}s^{-\frac{1}{2}}ds.\end{aligned}$$

If $K(0)\neq 0$, by [3], we can assume that K is constant. Of course, $K(t)/t$ is decreasing on $(0,\infty)$. If $K(0)=0$, we claim that $K(t)/t$ is decreasing on $(0,\infty)$. Choose $s,t\in(0,\infty)$ such that $s<t$. By the basic property of a concave function, we have

$$\frac{K(s)}{s}=\frac{K(s)-K(0)}{s-0}\geq\frac{K(t)-K(s)}{t-s}.$$

Thus

$$\begin{aligned}\frac{K(s)}{s}-\frac{K(t)}{t}&=\frac{tK(s)-sK(t)}{st}\\ &=\frac{1}{st}\left[s(K(s)-K(t))+(t-s)K(s)\right]\\ &=\frac{t-s}{t}\left[\frac{K(s)}{s}-\frac{K(t)-K(s)}{t-s}\right]\geq 0.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_0^{e^{-\frac{1}{2^n}}}r^{2^n-1}\left(\log\frac{1}{r}\right)^{-1/2}K\left(\log\frac{1}{r}\right)dr\\ \leq\ &2^nK\left(\frac{1}{2^n}\right)\int_{1/2^n}^{\infty}e^{-2^nt}t^{\frac{1}{2}}dt\\ =\ &2^{-\frac{n}{2}}K\left(\frac{1}{2^n}\right)\int_1^{\infty}e^{-s}s^{\frac{1}{2}}ds.\end{aligned}$$

Therefore,

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^2K(g(z,a))dA(z)\lesssim\sum_{n=0}^{\infty}2^{-n}K\left(\frac{1}{2^n}\right)\left(\sum_{n_k\in I_n}n_k|a_k|\right)^2.$$

If $n_k \in I_n$, then $n_k \geq 2^n$. Using the monotonicity of $K(t)/t$, one gets

$$n_k K\left(\frac{1}{n_k}\right) \geq 2^n K\left(\frac{1}{2^n}\right).$$

This gives

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(g(z,a))dA(z) \\
\lesssim\ & \sum_{n=0}^{\infty} 2^{-2n} \left(\sum_{n_k\in I_n} n_k |a_k| \sqrt{n_k K\left(\frac{1}{n_k}\right)} \right)^2 \\
\approx\ & \sum_{n=0}^{\infty} \left(\sum_{n_k\in I_n} n_k |a_k| \sqrt{\frac{1}{n_k} K\left(\frac{1}{n_k}\right)} \right)^2 .
\end{aligned}$$

Since f is a lacunary series, then there exists a positive constant λ such that $\frac{n_{k+1}}{n_k} \geq \lambda > 1$ for all k. It is well known that the Taylor series of $f(z)$ has at most $[\log_\lambda 2]+1$ terms $a_k z^{n_k}$ such that $n_k \in I_n$ for all $n \in \mathbb{N}$. By Hölder inequality, we obtain

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(g(z,a))dA(z) \\
\lesssim\ & ([\log_\lambda 2]+1) \sum_{n=0}^{\infty} \sum_{n_k\in I_n} n_k |a_k|^2 K\left(\frac{1}{n_k}\right) \\
\approx\ & \sum_{k=1}^{\infty} n_k |a_k|^2 K\left(\frac{1}{n_k}\right) < \infty.
\end{aligned}$$

The proof is complete. □

Note that $K(t) = t^p,\ 0 \leq p \leq 1$ is concave. The following result follows easily from Theorem 2.1.

Corollary 2.2 ([7]). Let $p \in [0,1]$. Then a lacunary series

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$$

belongs to Q_p if and only if

$$\sum_{k=1}^{\infty} n_k^{1-p} |a_k|^2 < \infty.$$

Since $K_0(t) = t\log\frac{e}{t}$ is also concave, we have the following statement.

Corollary 2.3 ([9]). A lacunary series

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$$

belongs to $Q_1(\partial\mathbb{D})$ if and only if

$$\sum_{k=1}^{\infty}\log(1+n_k)|a_k|^2<\infty.$$

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