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FLAG-TRANSITIVE POINT-PRIMITIVE SYMMETRIC DESIGNS AND THREE DIMENSIONAL PROJECTIVE SPECIAL LINEAR GROUPS

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Abstract. The main aim of this article is to study (v, k, λ)-symmetric designs admitting a flag-transitive and point-primitive automorphism group G whose socle is $\mathrm{PSL}(3, q)$. We indeed show that the only possible design satisfying these conditions is a Desarguesian projective plane $\mathrm{PG}(2, q)$ and $G \geqslant \mathrm{PSL}(3, q)$.
Keywords: Automorphism group, point-primitive, flag-transitive, symmetric design.
MSC(2010): Primary: 20B25; Secondary: 05B05.

1. Introduction

A t-(v, k, λ) *design* $\mathcal{D} = (\mathcal{V}, \mathcal{B})$ is an incidence structure consisting of a set $\mathcal{V}$ of v *points*, and a set $\mathcal{B}$ of k-element subsets of $\mathcal{V}$, called *blocks*, such that every t-element subset of points lies in exactly λ blocks. The design is *nontrivial* if $t < k < v - t$, and is *symmetric* if $|\mathcal{B}| = v$. Indeed, if $\mathcal{D}$ is symmetric and nontrivial, then $t \leqslant 2$ (see [5, Theorem 1.1] or [13, Theorem 1.27]). This motivates the study of nontrivial symmetric 2-(v, k, λ) designs which we simply call *symmetric* (v, k, λ) *designs*. A *flag* of $\mathcal{D}$ is an incident pair (α, B) where α and B are a point and a block of $\mathcal{D}$, respectively. An *automorphism* of a symmetric design $\mathcal{D}$ is a permutation of the points permuting the blocks and preserving the incidence relation. An automorphism group G of $\mathcal{D}$ is called *flag-transitive* if it is transitive on the set of flags of $\mathcal{D}$. If G is primitive on the point set $\mathcal{V}$, then G is said to be *point-primitive*. A group G is said to be *almost simple* with socle X if $X \trianglelefteq G \leqslant \mathrm{Aut}(X)$ where X is a (nonabelian) simple group. Further notation and definitions in both design theory and group theory are standard and can be found, for example, in [7, 13, 17].

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Symmetric designs with λ small have been of most interest. Kantor [15] classified flag-transitive symmetric $(v, k, 1)$ designs (projective planes) of order n and showed that either $\mathcal{D}$ is a *Desarguesian projective plane* and $\mathrm{PSL}(3, n) \trianglelefteq G$, or G is a sharply flag-transitive Frobenius group of odd order $(n^2 + n + 1)(n + 1)$, where n is even and $n^2 + n + 1$ is prime. Regueiro [21] gave a complete classification of biplanes ($\lambda = 2$) with flag-transitive automorphism groups apart from those admitting a 1-dimensional affine group (see also [22–25]). Zhou and Dong studied nontrivial symmetric $(v, k, 3)$ designs (triplanes) and proved that if $\mathcal{D}$ is a nontrivial symmetric $(v, k, 3)$ design with a flag-transitive and point-primitive automorphism group G, then $\mathcal{D}$ has parameters $(11, 6, 3)$, $(15, 7, 3)$, $(45, 12, 3)$ or G is a subgroup of $\mathrm{A\Gamma L}(1, q)$ where $q = p^m$ with $p \geqslant 5$ prime [9, 30–33]. Nontrivial symmetric $(v, k, 4)$ designs admitting flag-transitive and point-primitive almost simple automorphism group whose socle is an alternating group or $\mathrm{PSL}(2, q)$ have also been investigated [8, 34]. It is known [28] that if a nontrivial (v, k, λ)-symmetric design $\mathcal{D}$ with $\lambda \leqslant 100$ admitting a flag-transitive, point-primitive automorphism group G, then G must be an affine or almost simple type. Therefore, it is interesting to study such designs whose socle is of almost simple type or affine type.

In this paper, however, we are interested in large λ. In this direction, it is recently shown in [1] that there are only four possible symmetric (v, k, λ) designs admitting a flag-transitive and point-primitive automorphism group G satisfying $X \trianglelefteq G \leqslant \mathrm{Aut}(X)$ where $X = \mathrm{PSL}(2, q)$. In the case where X is a sporadic simple group, there also exist four possible parameters (see [29]). This paper is devoted to studying symmetric designs admitting a flag-transitive and point-primitive almost simple automorphism group G whose socle is $X := \mathrm{PSL}(3, q)$. We prove Theorem 1.1 below in Section 3.1.

Theorem 1.1. *Let $\mathcal{D}$ be a (v, k, λ)-symmetric design and G be an automorphism group of $\mathcal{D}$ with the socle $X = \mathrm{PSL}(3, q)$. If G is flag-transitive and point-primitive, then $\lambda = 1$ and $\mathcal{D}$ is a Desarguesian projective plane* $\mathrm{PG}(2, q)$ *and* $\mathrm{PSL}(3, q) \leqslant G$.

In order to prove Theorem 1.1, we need to know the complete list [3, Table 8.3] of maximal subgroups of almost simple groups with socle $\mathrm{PSL}(3, q)$ (see Lemma 2.4 below). We frequently apply Lemma 2.1 below as a key tool and use GAP [10] for computations.

In the case where G is imprimitive, Praeger and Zhou [26] studied point-imprimitive symmetric (v, k, λ) designs, and determined all such possible designs for $\lambda \leqslant 10$. This motivates Praeger and Reichard [18] to classify flag-transitive symmetric $(96, 20, 4)$ designs. As a result of their work, the only examples for flag-transitive, point-imprimitive symmetric $(v, k, 4)$ designs are $(15, 8, 4)$ and $(96, 20, 4)$ designs. In a recent study of imprimitive flag-transitive designs [4], Cameron and Praeger gave a construction of a family of designs with a specified point-partition, and determined the subgroup of automorphisms leaving invariant the point-partition. They gave necessary and sufficient conditions for a design in the family to possess a flag-transitive group of automorphisms preserving the specified point-partition. Consequently, they gave examples of flag-transitive designs in the family, including a new symmetric 2-$(1408, 336, 80)$ design with automorphism group $2^{12} : ((3 \cdot \mathrm{M}_{22}) : 2)$, and a construction of one of the families of the symplectic designs exhibiting a flag-transitive, point-imprimitive automorphism group.

2. **Preliminaries**

In this section, we state some useful facts in both design theory and group theory. Our notation and terminology are standard and can be found in [6, 12, 17] for design theory and in [7] for group theory. The following Lemma 2.1 is a key result in our approach to prove Theorem 1.1:

Lemma 2.1. *Let $\mathcal{D}$ be a symmetric (v, k, λ) design, and let G be a flag-transitive automorphism group of $\mathcal{D}$. If α is a point in $\mathcal{V}$ and $M := G_\alpha$, then*

(*a*) $k(k-1) = \lambda(v-1)$*;*
(*b*) $k \mid |M|$ *and* $\lambda v < k^2$*;*
(*c*) $k \mid \gcd(\lambda(v-1), |M|)$*;*
(*d*) $k \mid \lambda d$*, for all subdegrees d of G.*

Proof. (*a*) This part follows from [17, Proposition 1.1].

(*b*) The equality $k(k-1) = \lambda(v-1)$ implies that $k^2 = \lambda v - \lambda + k$. Since G is flag-transitive, M is transitive on the set of blocks containing α, and so k divides $|M|$. Moreover, as $\lambda < k$, we have that $\lambda v = k^2 - k + \lambda < k^2$.

(*c*) This part follows from (a) and (b).

(*d*) To prove this part, we use the same treatment as in [34, Lemma 2.2]. Suppose that Γ is a nontrivial suborbit of G of size d and let Δ be an orbital of the G-action on $\mathcal{V} \times \mathcal{V}$. Define $S = \{(\alpha, \beta, B) | \alpha \neq \beta, \beta \in B, (\alpha, \beta) \in \Delta\}$. Then we can count S in two ways, and so $\lambda|\Delta| = vkt$, where vk is the number of flags (α, B) and t is the number of triples containing the flag (α, B). Note that t is independent of the choice of the flag (α, B). Since $|\Delta| = vd$, it follows that $\lambda vd = vkt$. Thus $\lambda d = kt$, and hence $k \mid \lambda d$. □

Recall that a group G is called almost simple if $X \trianglelefteq G \leqslant \mathrm{Aut}(X)$ where X is a (nonabelian) simple group. If M is a maximal subgroup of an almost simple group G with socle X, then $G = MX$, and since we may identify X with $\mathrm{Inn}(X)$, the group of inner automorphisms of X, we also conclude that $|M|$ divides $|\mathrm{Out}(X)| \cdot |X \cap M|$. This implies the following elementary and useful fact:

Lemma 2.2. *Let G be an almost simple group with socle X, and let M be maximal in G not containing X. Then*

(*a*) $G = MX$;
(*b*) $|M|$ *divides* $|\mathrm{Out}(X)| \cdot |X \cap M|$.

Lemma 2.3. *Suppose that $\mathcal{D}$ is a symmetric (v, k, λ) design admitting a flag-transitive and point-primitive almost simple automorphism group G with socle X of Lie type in odd characteristic p. Suppose also that the point-stabiliser G_α, not containing X, is not a parabolic subgroup of G. Then* $\gcd(p, v-1) = 1$.

Proof. Note that G_α is maximal in G, then by Tits' Lemma [27, (1.6)], p divides $|G : G_\alpha| = v$, and so $\gcd(p, v-1) = 1$. □

If a group G acts primitively on a set $\mathcal{V}$ and $\alpha \in \mathcal{V}$ (with $|\mathcal{V}| \geqslant 2$), then the point-stabiliser G_α is maximal in G [7, Corollary 1.5A]. Therefore, in our study, we need a list of all maximal subgroups of almost simple group G with socle $X := \mathrm{PSL}(3, q)$. Note that if M is a maximal subgroup of G, then $M_0 := M \cap X$ is not necessarily maximal in X in which case M is called a *novelty*. By [3, Table 8.3], the complete list of maximal subgroups of an almost simple group G with socle $\mathrm{PSL}(3, q)$ are known, and in this case, there arose only three novelties (see also [2, 11, 16, 20]).

Lemma 2.4. *Let G be a group such that $X = \mathrm{PSL}(3, q) \lhd G \leqslant \mathrm{Aut}(X)$, and let M be a maximal subgroup of G not containing X. Then $M_0 = X \cap M$, is (isomorphic to) one of the following subgroups:*

(*a*) $\hat{}[q^2] : \mathrm{GL}(2, q)$ (the stabiliser of a point of the projective space);
(*b*) $\hat{}[q]^{1+2} : (q-1)^2$ (novelty);
(*c*) $\hat{}\mathrm{GL}(2, q)$ (novelty);
(*d*) $\hat{}(q-1)^2 : \mathrm{S}_3$;
(*e*) $\hat{}(q^2+q+1) : 3$ (novelty if $q = 4$),
(*f*) $\hat{}\mathrm{SL}(3, q_0) \cdot \gcd(3, \dfrac{q-1}{q_0-1})$, where $q = q_0^r$;
(*g*) $\mathrm{PSU}(3, q_0)$ for $q = q_0^2$;
(*h*) $3^2 : \mathrm{Q}_8$ with q odd;
(*i*) $\mathrm{SO}_3(q)$ with q odd;
(*j*) $\mathrm{PSL}(2, 7)$ with q odd;
(*k*) $3^2.\mathrm{SL}(2, 3)$ with q odd;
(*l*) A_6 with q odd.

Proof. It follows from [2, 11, 16, 20] and [3, Table 8.3]. □

3. Proof of the main result

In this section, suppose that $\mathcal{D}$ is a nontrivial (v, k, λ)-symmetric design and G is an almost simple automorphism group G with simple socle $X := \mathrm{PSL}(3, q)$, where $q = p^f$ (p prime), that is to say, $X \lhd G \leqslant \mathrm{Aut}(X)$. Suppose also that $V = \mathrm{GF}(q)^3$ is the underlying vector space of X over the finite field $\mathrm{GF}(q)$.

Let now G be a flag-transitive and point-primitive automorphism group of $\mathcal{D}$. Then the point-stabiliser $M := G_\alpha$ is maximal in G [7, Corollary 1.5A]. Set $M_0 := X \cap M$. So M_0 is (isomorphic to) one of the subgroups in Lemma 2.4(a)-(l). Moreover, by Lemma 2.2,

$$v = \frac{|X|}{|M_0|} = \frac{q^3(q^2-1)(q^3-1)}{\gcd(3, q-1).|M_0|}. \tag{3.1}$$

Note that $|\mathrm{Out}(X)| = 2f \cdot \gcd(3, q-1)$. Therefore, by Lemma 2.1(c) and Lemma 2.2(b),

$$k \mid 2f \cdot \gcd(3, q-1) \cdot |M_0|. \tag{3.2}$$

In what follows, considering possible structure for the subgroup M_0 as in Lemma 2.4(a)-(l), we prove that the only case might occur is Lemma 2.4(a). Indeed, we show that other cases lead to a contradiction.

Remark 3.1. Note that we may exclude the case where $X = \mathrm{PSL}(3,2)$ in our arguments. This is as $\mathrm{PSL}(3,2) \cong \mathrm{PSL}(2,7)$ and there exists the unique symmetric $(7,3,1)$ design known as *Fano Plane* where $\mathrm{PSL}(2,7)$ is its full automorphism group and S_4 is its point-stabiliser. Moreover, $\mathrm{PSL}(2,7)$ is flag-transitive and point-primitive. The complement of Fano Plane is the unique symmetric $(7,4,2)$ design which is also flag-transitive and point-primitive (see [23, Section 1.2.1]).

Lemma 3.2. *The subgroup M_0 cannot be* $\hat{}[q]^{1+2} : (q-1)^2$.

Proof. Let V be the underlying vector space of $X = \mathrm{PSL}(3,q)$ over the finite field $\mathrm{GF}(q)$. Then, in this case, M stabilises a pair $\{U, W\}$ of subspaces of dimension 1 and 2, respectively, with $U \subseteq W$.

By (3.1), we have that $v = q^3 + 2q^2 + 2q + 1$. It follows from [19, Lemma 3.9] and Lemma 2.1(e) that k divides $2\lambda q$. Let now m be a positive integer such that $mk = 2\lambda q$. Since $\lambda < k$, we have that

$$m < 2q. \tag{3.3}$$

By Lemma 2.1(a), $k(k-1) = \lambda(v-1)$, and so

$$\frac{2\lambda q}{m}(k-1) = \lambda(q^3 + 2q^2 + 2q).$$

Thus,

$$2k = m(q^2 + 2q + 2) + 2 \tag{3.4}$$

$$2\lambda = m^2(q+2) + \frac{2m^2 + m}{q} \tag{3.5}$$

Since λ is integer, (3.5) implies that

$$q \mid 2m^2 + m. \tag{3.6}$$

Therefore, q divides either m, or $2m+1$. We now consider these two cases:

Case 1. Let q divide m. By (3.3), we must have $m = q$, and so by (3.4) and (3.5), k and λ must satisfy

$$2k = q(q^2 + 2q + 2) + 2; \tag{3.7}$$

$$2\lambda = q^2(q + 2) + 2q + 1. \tag{3.8}$$

The equation (3.8) implies that q is odd, and by (3.7), q must divide $2k-2$, and hence q divides $k-1$. Consequently, $\gcd(k, q) = 1$, and since k divides $2fq^3(q-1)^2$ by (3.2), the parameter k must divide $2f(q-1)^2$. Thus by (3.7), $q(q^2 + 2q + 2) + 2$ divides $4f(q-1)^2$. Therefore,

$$\frac{q(q^2 + 2q + 2) + 2}{(q-1)^2} < 4f.$$

This leads to a contradiction as this inequality does not hold for any $q = p^f$.

Case 2: Let q divide $2m+1$. Then p is odd and is coprime to m. Since q is odd, (3.3) implies that $2m+1 = q$ or $2m+1 = 3q$.

If $2m+1 = q$, then $m = (q-1)/2$, and so (3.4) implies that $4k = q^3+q^2+2$. It follows from (3.2) that k divides $2fq^3(q-1)^2$, then $4k = q^3 + q^2 + 2$ divides $8fq^3(q-1)^2$. Since q is odd, $\gcd(q^3+q^2+2, q) = \gcd(2, q) = 1$, and so q^3+q^2+2 must divide $8f(q-1)^2$. Therefore, $q^3 + q^2 + 2 < 8f(q-1)^2$. This holds only for $q = 9$. Thus $k = 203$ and $\lambda = mk/2q = 406/9$ which is impossible.

If $2m + 1 = 3q$, then $m = (3q-1)/2$, and so $4k = 3q^3 + 5q^2 + 4q + 2$ by (3.4). By (3.2), $4k = 3q^3 + 5q^2 + 4q + 2$ divides $8fq^3(q-1)^2$. Note that $\gcd(q, 3q^3 + 5q^2 + 4q + 2) = 1$. Then $3q^3 + 5q^2 + 4q + 2$ must divide $8f(q-1)^2$, and so $3q^3 + 5q^2 + 4q + 2 < 8f(q-1)^2$. This does not hold for each value of $q = p^f$, which is a contradiction. □

Lemma 3.3. *The subgroup M_0 cannot be* $\hat{}\mathrm{GL}(2, q)$.

Proof. Suppose that V is the underlying vector space of $X = \mathrm{PSL}(3, q)$ over the finite field $\mathrm{GF}(q)$. Then, M is the stabiliser of a pair $\{U, W\}$ of subspaces of dimension 1 and 2, respectively, where $V = U \oplus W$.

By (3.1), we have that $v = q^2(q^2 + q + 1)$. Let $x = \{\langle v_1\rangle, \langle v_2, v_3\rangle\}$ and $y = \{\langle v_1, v_2\rangle, \langle v_3\rangle\}$. Then $|G_x : G_{xy}| = q^2(q + 1)$ is a subdegree of G. Thus by Lemma 2.1(e), we conclude that k divides $\lambda q^2(q + 1)$. On the other hand, k divides $\lambda(v-1)$, where $v = q^2(q^2 + q + 1)$. As $v-1$ and q are coprime, k must divide $\lambda(q + 1)$, and hence there exists a positive integer m such that $mk = \lambda(q + 1)$. Since $k(k-1) = \lambda(v-1)$, it follows that

$$\frac{\lambda(q+1)}{m}(k-1) = \lambda(q^4 + q^3 + q^2 - 1).$$

Thus,

$$k = m(q^3 + q - 1) + 1 \tag{3.9}$$

Note by (3.2) that $k \mid 2fq(q-1)(q^2-1)$. Then, by (3.9), we must have

$$m(q^3 + q - 1) + 1 \mid 2mfq(q-1)(q^2-1). \tag{3.10}$$

Note also that

$$\begin{aligned} 2f(q-1)[m(q^3+q-1)+1] &- 2mfq(q-1)(q^2-1) \\ &= 4mf(q^2-q) - 2(m-1)f(q-1). \end{aligned}$$

Then by (3.10), we must have that $m(q^3+q-1)+1$ divides $4mf(q^2-q) - 2(m-1)f(q-1)$, and so

$$m(q^3+q-1)+1 < 4mf(q^2-q) - 2(m-1)f(q-1).$$

Therefore, $m(q^3+q-1) < 4mf(q^2-q)$, and hence

$$\frac{q^3+q-1}{q^2-q} < 4f.$$

This holds only for $q = 2^f$ with $f \leqslant 3$. Recall that $mk = \lambda(q+1)$, and since $\lambda < k$, we have that

$$m < q + 1. \tag{3.11}$$

If $q = 2$, then $v = q^2(q^2+q+1) = 28$ and $m = 1, 2, 3$ by (3.11), and so (3.9) implies that $k = 8, 17, 26$, respectively. This is impossible as for each values of k and v the fraction $\lambda = k(k-1)/(v-1)$ is not integer. If $q = 4$ or 8, then m is at most 5 or 9, respectively. In both cases, by the same argument, we observe that the fraction $k(k-1)/(v-1)$ is not an integer number, which is a contradiction. □

Lemma 3.4. *The subgroup M_0 cannot be* $\hat{}(q^2+q+1):3$.

Proof. Here, by (3.1), we have $v = q^3(q^2-1)(q-1)/3$. Note that $|\mathrm{Out}(X)| = 2 \cdot \gcd(3, q-1) \cdot f$. Then by (3.2), we conclude that k divides $6f(q^2+q+1)$. By [24,33], we may assume that $\lambda \geqslant 4$, and so Lemma 2.1(c) yields

$$\frac{4q^3(q^2-1)(q-1)}{3} \leqslant \lambda v < k^2 \leqslant 36f^2(q^2+q+1)^2.$$

Then $q^6 - q^5 - q^4 + q^3 < 27f^2(q^2+q+1)^2$, and so

$$\frac{q^6 - q^5 - q^4 + q^3}{(q^2+q+1)^2} < 27f^2.$$

This inequality holds when

$$\begin{array}{ll} p = 2, & f \leqslant 4; \\ p = 3, & f \leqslant 2; \\ p = 5 & f = 1. \end{array} \tag{3.12}$$

Recall that k is a divisor of $6f(q^2+q+1)$. Then, for each $q = p^f$ with p and f as in (3.12), the possible values of k and v are listed in Table 1 below:

Table 1. Possible value for k and v when $q = p^f$ with p and f as in (3.12).

q	2	3	4	5	8	9	16
v	8	144	960	4000	75264	155520	5222400
k divides	42	78	252	186	1314	1092	6552

This is a contradiction as for each k and v as in Table 1, the fraction $k(k-1)/(v-1)$ is not integer. □

Lemma 3.5. *The subgroup M_0 cannot be* $\hat{}(q-1)^2 : \mathrm{S}_3$.

Proof. The argument here is the same as that of Lemma 3.4. By (3.1), we have $v = q^3(q+1)(q^2+q+1)/6$, and since $|\mathrm{Out}(X)| = 2f \cdot \gcd(3, q-1)$, it follows from (3.2) that k divides $12f(q-1)^2$. By [24,33] and Lemma 2.1(c), we may assume that λ is at least 4, and so

$$\frac{4q^3(q+1)(q^2+q+1)}{6} \leqslant \lambda v < k^2 \leqslant 144f^2(q-1)^4.$$

This implies that $q^3(q+1)(q^2+q+1) < 216f^2(q-1)^4$, and so

$$\frac{q^3(q+1)(q^2+q+1)}{(q-1)^4} < 216f^2.$$

This is true only when

$$\begin{array}{ll} p=2, & f \leqslant 6; \\ p=3, & f \leqslant 3; \\ p=5, & f \leqslant 2; \\ p=7,11 & f=1. \end{array} \tag{3.13}$$

Recall that k is a divisor of $12f(q-1)^2$. Then for each $q = p^f$ with p and f as in (3.13), the possible values of k and v are listed in Table 2 below:
This leads us to a contradiction as, for each parameter k and v as in Table 2, the fraction $k(k-1)/(v-1)$ is not integer. □

Lemma 3.6. *The subgroup M_0 cannot be* A_6*, with q odd.*

Proof. By (3.1), we have that

$$v = \frac{q^3(q^2-1)(q^3-1)}{360 \cdot \gcd(3, q-1)}. \tag{3.14}$$

TABLE 2. Possible value for k and v when $q = p^f$ is as in (3.13).

q	v	k divides	q	v	k divides
2	28	12	11	354046	1200
3	234	48	16	3168256	10800
4	1120	216	25	44078125	13824
5	3875	192	27	69533478	24336
7	26068	432	32	190496768	57660
8	56064	1764	64	11816796160	285768
9	110565	1536			

Note by (3.2) that k divides $2160f$. By [24, 33] ,we may only focus on $\lambda \geqslant 4$, and so Lemma 2.1(c) yields

$$\frac{4q^3(q^2-1)(q^3-1)}{1080} \leqslant \lambda v < k^2 \leqslant 2160^2 f^2.$$

This implies that

$$q^8 - q^6 - q^5 + q^3 < 1259712000f^2. \tag{3.15}$$

Since $q = p^f$ is odd, (3.15) implies that $q \in \{3, 5, 7, 9, 11, 13\}$. Since also the fraction (3.14) must be integer, the only acceptable value of q is $q = 9$, and so $v = 117936$. It follows from (3.2) that k divides 1440. We then easily observe that, for each divisor k of 1440, the fraction $k(k-1)/(v-1)$ is not integer, which is a contradiction. □

Our arguments to prove Lemmas 3.7–3.9 below are the same as those of Lemma 3.6.

Lemma 3.7. *The subgroup M_0 cannot be $3^2 : \mathrm{Q}_8$ with q odd.*

Proof. By (3.1), we have that

$$v = \frac{q^3(q^2-1)(q^3-1)}{72 \cdot \gcd(3, q-1)}. \tag{3.16}$$

Note that $|\mathrm{Out}(X)| = 2f \cdot \gcd(3, q-1)$. Then by (3.2), we conclude that k divides $432f$. By [24, 33], we may assume that $\lambda \geqslant 4$, and so Lemma 2.1(c) implies that

$$\frac{4q^3(q^2-1)(q^3-1)}{216} \leqslant \lambda v < k^2 \leqslant 432^2 f^2.$$

Therefore

$$q^8 - q^6 - q^5 + q^3 < 10077696f^2. \tag{3.17}$$

As $q = p^f$ is odd, it follows from (3.17) that $q \in \{3, 5, 7\}$. Since the fraction in (3.16) must be integer, the only possible value for q is 3 or 7. Recall that k is a divisor of $144f \cdot \gcd(3, q-1)$, and so, for each $q \in \{3, 7\}$, the possible values of k and v are listed in Table 3 below:

TABLE 3. Possible values for k and v when $q = 3$ and 7.

q	3	7
v	78	26068
k divides	144	432

For each parameter k and v as in Table 3, by straightforward calculation, we observe that the fraction $k(k-1)/(v-1)$ is not integer, which is a contradiction. □

Lemma 3.8. *The subgroup M_0 cannot be $3^2 \cdot \mathrm{SL}(2, 3)$ with q odd.*

Proof. By (3.1), we have that

$$v = \frac{q^3(q^2-1)(q^3-1)}{216 \cdot \gcd(3, q-1)}. \tag{3.18}$$

Since by (3.2), k divides $1296f$, and since $\lambda \geqslant 4$ by [24, 33], it follows from Lemma 2.1(c) that

$$\frac{4q^3(q^2-1)(q^3-1)}{648} \leqslant \lambda v < k^2 \leqslant 1296^2 f^2.$$

This implies that

$$q^3(q^2-1)(q^3-1) < 272097792 f^2. \tag{3.19}$$

Since $q = p^f$ is odd, it follows from (3.19) that $q \in \{3, 5, 7, 9, 11\}$. Note that the fraction in (3.18) must be integer. Then $q = 3$ or 9. Since k is a divisor of $432f \cdot \gcd(3, q-1)$, for each value of $q \in \{3, 9\}$, the possible values of k and v are given in Table 4 below. Again for each parameter k and v as in Table 4,

TABLE 4. Possible values for k and v when $q = 3$ and 9.

q	3	9
v	26	196560
k divides	432	864

the fraction $k(k-1)/(v-1)$ is not integer, which is a contradiction. □

Lemma 3.9. *The subgroup M_0 cannot be $\mathrm{PSL}(2, 7)$ with q odd.*

Proof. By (3.1), we have that

$$v = \frac{q^3(q^2-1)(q^3-1)}{168 \cdot \gcd(3, q-1)}. \tag{3.20}$$

Note by (3.2) that k divides $1008f$. Moreover, we may assume that $\lambda \geqslant 4$ by [24,33]. Then by Lemma 2.1(c),

$$\frac{4q^3(q^2-1)(q^3-1)}{504} \leqslant \lambda v < k^2 \leqslant 1008^2 f^2.$$

Then

$$q^3(q^2-1)(q^3-1) < 128024064 f^2. \tag{3.21}$$

Since $q = p^f$ is odd and the fraction in (3.20) must be integer, the inequality (3.21) implies that $q \in \{7, 9\}$. Again using the fact that k is a divisor of $336f \cdot \gcd(3, q-1)$, possible values of k and v are obtained in Table 5 below:

TABLE 5. Possible valued for k and v when $q = 7$ and 9.

q	7	9
v	33516	252720
k divides	1008	672

None of the values of k and v is acceptable as for each of those, the fraction $k(k-1)/(v-1)$ is not integer in each case, which is a contradiction. □

Lemma 3.10. *The subgroup M_0 cannot be $\mathrm{SO}_3(q)$ with q odd.*

Proof. By (3.1), we have that $v = q^2(q^3-1)/d$ with $d = \gcd(3, q-1)$. It follows from (3.2) that k divides $2dfq(q^2-1)$, and so k is a divisor of $6fq(q^2-1)$. Moreover, Lemma 2.1(a) implies that k divides $\lambda(v-1)$. Note by Lemma 2.3 that $v-1$ is coprime to q. Thus k divides $6\lambda f \gcd(q^2-1, v-1)$. Since every divisor of q^2-1 which also divides $(q^5-q^2-d)/d$ is a divisor of 15, we conclude that k divides $90\lambda f$. Then there exists a positive integer m such that $mk = 90\lambda f$. Since $k(k-1) = \lambda(v-1)$, it follows that

$$\frac{90\lambda f}{m}(k-1) = \frac{\lambda(q^5-q^2-d)}{d},$$

where $d = \gcd(3, q-1)$. Thus

$$k = \frac{m(q^5-q^2-d)}{90 \cdot d \cdot f} + 1. \tag{3.22}$$

Since $d = 1, 3$, we have by (3.2) that $k \mid 6fq(q^2-1)$. Then (3.22) yields

$$m(q^5-q^2-d) \leqslant 540df^2q(q^2-1).$$

Since also $m \geqslant 1$ and $d \leqslant 3$, we have that

$$\frac{q^5-q^2-3}{q(q^2-1)} \leqslant 1620f^2.$$

This inequality only holds for

$$q \in \{3, 5, 7, 9, 11, 13, 17, 19, 23, 25, 27, 29, 31, 37\}. \tag{3.23}$$

For these values of q, since k divides $2dfq(q^2-1)$, the possible values of k can be found as in Table 6.

TABLE 6. Possible values for k and v when q is as in (3.23).

q	v	k divides	q	v	k divides
3	234	48	19	825246	41040
5	3100	240	23	6435814	24288
7	5586	2016	25	3255000	187200
9	58968	2880	27	14348178	117936
11	160930	2640	29	20510308	48720
13	123708	13104	31	9542730	178560
17	1419568	9792	37	23114196	303696

This leads us to a contradiction as for each value of v and k as in Table 6, the fraction $k(k-1)/(v-1)$ is not integer. □

Lemma 3.11. *The subgroup* M_0 *cannot be* $\mathrm{PSU}(3, q_0)$*, where* $q = q_0^2$*.*

Proof. By (3.1), we have that

$$v = q_0^3(q_0^2+1)(q_0^3-1)\cdot b, \tag{3.24}$$

where $b = \gcd(3, q_0+1)/\gcd(3, q_0^2-1)$. We now consider the following two cases:

Case 1: Let $b = 1$. Then $v = q_0^3(q_0^2+1)(q_0^3-1)$ and (3.2) implies that k divides $2fq_0^3(q_0^2-1)(q_0^3+1)$. It also follows from Lemma 2.1(a) that k divides $\lambda(v-1)$. By Lemma 2.3, $v-1$ is coprime to q_0. Thus k divides $2\lambda f\gcd(v-1, (q_0^2-1)(q_0^3+1))$. Note that

$$v - 1 = (q_0^2-1)(q_0^3+1)\cdot(q_0^3+2q_0-2) + h(q_0),$$

where $h(q_0) = 2q_0^4 - 4q_0^3 + 2q_0^2 + 2q_0 - 3$. Note also that $h(q_0)$ is odd and

$$2(q_0^2-1)(q_0^3+1) = h(q_0)\cdot(q_0+2) + r(q_0),$$

where $r(q_0) = 4q_0^3 - 4q_0^2 - q_0 + 4$. Therefore,

$$\gcd(v-1, (q_0^2-1)(q_0^3+1)) = \gcd(h(q_0), r(q_0)).$$

Set $R := R(f, q_0) = 2f\cdot r(q_0)$. Then k divides λR, and so there exists a positive integer m such that $mk = \lambda R$. Since $k(k-1) = \lambda(v-1)$, it follows that

$$\frac{\lambda R}{m}(k-1) = \lambda(v-1),$$

where $v = q_0^3(q_0^2+1)(q_0^3-1)$, and so

$$k = \frac{m(v-1)}{R} + 1. \tag{3.25}$$

Since $k \mid 2fq_0^3(q_0^2-1)(q_0^3+1)$, it follows from (3.25) that

$$m(v-1) + R \mid 2mfq_0^3(q_0^2-1)(q_0^3+1) \cdot R. \tag{3.26}$$

Let now

$$\begin{aligned} T &:= T(q_0) = 4q_0^3 - 4q_0^2 - 9q_0 + 20; \\ G &:= G(q_0) = 2q_0^7 - 34q_0^6 + 24q_0^5 - 8q_0^4 + 20q_0^3 - 4q_0^2 - 9q_0 + 20. \end{aligned}$$

Then

$$\begin{aligned} 4mf^2G - 4f^2T \cdot R =& 2mfq_0^3(q_0^2-1)(q_0^3+1) \cdot R \\ & - 4f^2T \cdot [m(v-1)+R]. \end{aligned}$$

Therefore (3.26) implies that

$$\begin{aligned} m(v-1) + R &\leqslant 4f^2|mG - T \cdot R| \\ &\leqslant 4f^2(mG + T \cdot R). \end{aligned}$$

So $m[(v-1) - 4f^2G] < 4f^2T \cdot R$, and since $m \geqslant 1$, it follows that

$$q_0^3(q_0^2+1)(q_0^3-1) \leqslant 4f^2(G + T \cdot R) + 1. \tag{3.27}$$

Since also $G + T \cdot R < 2fq_0^7$ for all $q_0 \geqslant 2$, the inequality (3.27) implies that $p^{f/2} = q_0 < 8f^3$, and this holds when $p \leqslant 61$ and $f \leqslant 36$. Since $q_0 = p^{f/2}$, for these values of q_0, considering the fact that $b = \gcd(3, q_0+1)/\gcd(3, q_0^2-1) = 1$, it follows from (3.27) that

$$q_0 \in \{2, 3, 5, 8, 9, 11, 17, 23, 27, 29, 32, 41, 81, 125, 128, 243, 512, 729, 2048\}. \tag{3.28}$$

Recall that k is a divisor of $2fq_0^3(q_0^2-1)(q_0^3+1)$, and so for each value of q_0 as in (3.28), the possible values of k and v are listed in Table 7 below. This leads us a contradiction as for each value of v and k as in Table 7, the fraction $k(k-1)/(v-1)$ is not integer.

Case 2: Let $b = 1/3$. Then $\gcd(3, q_0-1) = 3$. By (3.24), we have that $v = q_0^3(q_0^2+1)(q_0^3-1)/3$ and (3.2) implies that k divides $6fq_0^3(q_0^2-1)(q_0^3+1)$. Moreover, Lemma 2.1(a) implies that k divides $3\lambda(v-1)$. By Lemma 2.3 and the fact that $\gcd(3, q_0) = 1$, we conclude that $3(v-1)$ and q_0 are coprime. Since also

$$\gcd(3(v-1), q_0+1) = 1,$$

it follows that k divides $6\lambda f \gcd(3(v-1), (q_0-1)(q_0^2-q_0+1))$. Set

$$R := R(f, q_0) = 6f(q_0-1)(q_0^2-q_0+1).$$

Table 7. Possible values for k and v.

q_0	v	k divides
2	280	864
3	7020	24192
5	403000	1512000
8	17006080	198567936
9	43518384	340588800
11	215968060	850988160
17	6998470240	27812139264
23	78452572660	312677494272
27	282802588380	3384677342592
29	500820700744	1998688305600
32	1100551782400	21969428152320
41	7989559408240	31921163648640
81	1853299131072480	29643859929093120
125	59608428953125000	715210327125000000
128	72061957722603520	2017490449773625344
243	12157870502886065100	243149208303981893760
512	472238446208364838 9120	1700045461315934497 01376
729	7976659296561628734 7344	1914391036515070980921600
2048	30948508357229255795 4088960	136173371870974927418865 74592

Then there exists a positive integer m such that $mk = \lambda R$. Since $k(k-1) = \lambda(v-1)$, it follows that

$$\frac{\lambda R}{m}(k-1) = \lambda(v-1).$$

where $v = q_0^3(q_0^2+1)(q_0^3-1)/3$. Thus

$$k = \frac{m(v-1)}{R} + 1. \tag{3.29}$$

Note by (3.2) that $k \mid 6fq_0^3(q_0^2-1)(q_0^3+1)$. Then by (3.35), we must have that

$$m(v-1) + R \mid 6mf \cdot q_0^3(q_0^2-1)(q_0^3+1) \cdot R. \tag{3.30}$$

Let

$$\begin{aligned} T &:= T(q_0) = 3q_0^3 - 6q_0^2 + 15; \\ G &:= G(q_0) = 6q_0^7 + 2q_0^6 - 4q_0^5 + 2q_0^4 - 9q_0^3 + 6q_0^2 - 15. \end{aligned}$$

Then

$$\begin{aligned} 36mf^2 \cdot G + 36f^2 \cdot T \cdot R =& 36f^2 \cdot T \cdot (m(v-1) + R) \\ & - 6mfq_0^3(q_0^2-1)(q_0^3+1) \cdot R, \end{aligned} \tag{3.31}$$

and so it follows from (3.30) that $m(v-1)+R$ divides $36mf^2\cdot G+36f^2\cdot T\cdot R$. Thus

$$m(v-1)+R\leqslant 36mf^2\cdot G+36f^2\cdot T\cdot R.$$

So $m(v-1-36f^2\cdot G)\leqslant 36f^2\cdot T\cdot R$, and since $m\geqslant 1$, we conclude that

$$q_0^3(q_0^2+1)(q_0^3-1)<108f^2(G+T\cdot R)+3. \tag{3.32}$$

Note that $G+T\cdot R+3<8fq_0^7$, for all $q_0\geqslant 2$. Then (3.32) implies that $p^{f/2}=q_0<108\cdot 8f^3$, and this holds when $p\leqslant 6911$ and $f\leqslant 54$. Note also that $\gcd(3,q_0-1)=3$ and k is a divisor of $6fq_0^3(q_0^2-1)(q_0^3+1)$. As in Case 1, considering these two facts, we obtain possible values of v and k, and hence for such v and k, the fraction $k(k-1)/(v-1)$ is not integer, which is a contradiction. □

Lemma 3.12. *The subgroup M_0 cannot be $\hat{}\mathrm{SL}(3,q_0)\cdot c$, where $q=q_0^r$ and $c:=\gcd\left(3,\dfrac{q-1}{q_0-1}\right)$.*

Proof. In this case, $|M_0|=c\cdot q_0^3(q_0^2-1)(q_0^3-1)/\gcd(3,q-1)$. It follows from (3.1) that

$$v=\frac{1}{c}\cdot\frac{q_0^{3r}(q_0^{2r}-1)(q_0^{3r}-1)}{q_0^3(q_0^2-1)(q_0^3-1)}. \tag{3.33}$$

Note by (3.2) that k divides $6fq_0^3(q_0^2-1)(q_0^3-1)$. By [24, 33], we may assume that $\lambda\geqslant 4$. Moreover, $c\leqslant 3$ and $f^2\leqslant q_0^r$ as $q=q_0^r$. Since $\lambda v<k^2$ by Lemma 2.1(c), we must have

$$\begin{aligned}\frac{4q_0^{3r}(q_0^{2r}-1)(q_0^{3r}-1)}{3q_0^3(q_0^2-1)(q_0^3-1)}\leqslant\lambda v<k^2&\leqslant 36f^2\cdot q_0^6(q_0^2-1)^2(q_0^3-1)^2\\&\leqslant 36\cdot q_0^{6+r}(q_0^2-1)^2(q_0^3-1)^2\end{aligned}$$

and hence

$$\lambda\cdot q_0^{3r}(q_0^{2r}-1)(q_0^{3r}-1)<27\cdot q_0^{9+r}(q_0^2-1)^3(q_0^3-1)^3.$$

Note that $q_0^{8r-1}\leqslant q_0^{3r}(q_0^{2r}-1)(q_0^{3r}-1)$ and $q_0^{9+r}(q_0^2-1)^3(q_0^3-1)^3\leqslant q_0^{24+r}$. Then $q_0^{8r-1}<27\cdot q_0^{24+r}$, and so $q_0^{7r-25}<27$. As $q_0\geqslant 2$, this implies that $r=2$ or 3.

Case 1. Suppose first $r=2$. By (3.33), we have that

$$v=\frac{q_0^3(q_0^2+1)(q_0^3+1)}{c}, \tag{3.34}$$

where $c=\gcd(3,q_0+1)$. We now consider the following two cases:

Subcase 1. If $c=1$, then $v=q_0^3(q_0^2+1)(q_0^3+1)$. It follows from (3.2) that k divides $2fq_0^3(q_0^2-1)(q_0^3-1)$. Moreover, by Lemma 2.1(a), k divides $\lambda(v-1)$, and since $v-1$ is coprime to q_0 by Lemma 2.3, k must divide $2\lambda f\gcd((q_0^2-$

$1)(q_0^3-1), v-1)$. Since also $\gcd(v-1, q_0+1) = 1$, k must divide $2\lambda f \gcd(v-1, (q_0-1)(q_0^3-1))$. Note that

$$v-1 = (q_0-1)(q_0^3-1)\cdot(q_0^4+q_0^3+2q_0^2+4q_0+4)+r(q_0),$$

where $r(q_0) = 6q_0^3+2q_0^2-5$. Therefore,

$$\gcd(v-1, (q_0^2-1)(q_0^3-1)) = \gcd(r(q_0), (q_0^2-1)(q_0^3+1)).$$

Set $R := R(f, q_0) = 2f\cdot r(q_0)$. Then k divides λR, and so there exists a positive integer m such that $mk = \lambda R$. Since $k(k-1) = \lambda(v-1)$, it follows that

$$\frac{\lambda R}{m}(k-1) = \lambda(v-1),$$

where $v = q_0^3(q_0^2+1)(q_0^3+1)$. Thus

$$k = \frac{m(v-1)}{R}+1. \tag{3.35}$$

Since $k \mid 2fq_0^3(q_0^2-1)(q_0^3-1)$, it follows from (3.35) that

$$m(v-1)+R \mid 2mfq_0^3(q_0^2-1)(q_0^3-1)\cdot R. \tag{3.36}$$

Set

$$T := T(q_0) := 24q_0^3+8q_0^2-48q_0-84;$$
$$G := G(q_0) := 32q_0^7+152q_0^6+104q_0^5+48q_0^4+88q_0^3+8q_0^2-48q_0-84.$$

Then

$$2mfq_0^3(q_0^2-1)(q_0^3+1)\cdot R = f^2T\cdot[m(v-1)+R]-f^2T\cdot R+mf^2G.$$

This together with (3.36) implies that

$$\begin{aligned} m(v-1)+R &\leqslant f^2|mG-T\cdot R| \\ &\leqslant f^2(mG+T\cdot R). \end{aligned} \tag{3.37}$$

Then $m[(v-1)-f^2G] < f^2T\cdot R$, and since $m \geqslant 1$, it follows that

$$q_0^3(q_0^2+1)(q_0^3+1) \leqslant f^2(G+T\cdot R)+1. \tag{3.38}$$

Since also $G+T\cdot R < 190fq_0^7$ for all $q_0 \geqslant 2$, the inequality (3.27) implies that $q_0 < 190f^3$. This holds when $p \leqslant 1511$ and $f \leqslant 48$ (recall that $q_0 = p^{f/2}$).

By the same manner as in the proof of Lemma 3.11, for such possible values of q_0, considering the fact that $c = \gcd(3, q_0+1) = 1$, the inequality (3.37) implies that

$$\begin{aligned} q_0 \in \{&3, 4, 7, 9, 13, 16, 19, 25, 27, 31, 37, 43, 49, 61, 64, 67, 73, 79, 81, 97, \\ &103, 109, 121, 127, 139, 169, 243, 256, 289, 343, 361, 529, 625, 729, \\ &1024, 2187, 4096, 6561, 16384\}. \end{aligned}$$

For each $q_0 = p^{f/2}$ as above, k is a divisor of $2fq_0^3(q_0^2-1)(q_0^3-1)$, but in each case, the fraction $k(k-1)/(v-1)$ is not integer, which is a contradiction.

Subcase 2: If $c = 3$, then $v = q_0^3(q_0^2+1)(q_0^3+1)/3$. By (3.2), k divides $6fq_0^3(q_0^2-1)(q_0^3-1)$. On the other hand, Lemma 2.1(a) implies that k divides $3\lambda(v-1)$, and so k divides $6\lambda f \gcd(q_0^3(q_0^2-1)(q_0^3-1), 3(v-1))$. By Lemma 2.3, $v-1$ and q_0 are coprime. Moreover, as $c = 3$, q_0 is not a power of 3. Then $3(v-1)$ and q_0 are coprime. Since also $q_0 - 1$ and $3(v-1)$ are coprime, k must divide $\gcd((q_0+1)(q_0^2+q_0+1), 3(v-1))$. Therefore, k divides λR, where $R := R(f, q_0) = 6f(q_0+1)(q_0^2+q_0+1)$. Then there is a positive integer m such that $mk = \lambda R$. Since $k(k-1) = \lambda(v-1)$, it follows that

$$\frac{\lambda R}{m}(k-1) = \lambda(v-1),$$

where $v = q_0^3(q_0^2+1)(q_0^3+1)/3$. Thus

$$k = \frac{m(v-1)}{R} + 1. \tag{3.39}$$

Note by (3.2) that $k \mid 6fq_0^3(q_0^2-1)(q_0^3-1)$. Then by (3.39), we must have

$$m(v-1) + R \mid 6mfq_0^3(q_0^2-1)(q_0^3-1) \cdot R. \tag{3.40}$$

Set

$$\begin{aligned} T &:= T(q_0) := 108q_0^3 + 216q_0^2 - 540; \\ G &:= G(q_0) := 216q_0^7 - 72q_0^6 - 144q_0^5 - 72q_0^4 - 324q_0^3 - 216q_0^2 + 540. \end{aligned}$$

Then

$$6mfq_0^3(q_0^2-1)(q_0^3-1) \cdot R = f^2T \cdot [m(v-1) + R] - f^2T \cdot R - f^2mG.$$

This together with (3.40) implies that

$$m(v-1) + R \leqslant f^2(mG + T \cdot R).$$

So $m(v - 1 - f^2G) < f^2T \cdot R$. Since now $m \geqslant 1$, we conclude that

$$q_0^3(q_0^2+1)(q_0^3+1) - 3 \leqslant 3f^2(G + T \cdot R). \tag{3.41}$$

Since also $G + T \cdot R < 1282fq_0^7$ for all $q_0 \geqslant 2$, the inequality (3.27) implies that $q_0 < 1282f^3$. This holds when $p \leqslant 10253$ and $f \leqslant 54$ (recall that $q_0 = p^{f/2}$).

By the same manner as in the proof of Lemma 3.11, for such possible values of q_0, considering the fact that $c = \gcd(3, q_0+1) = 3$, the inequality (3.37) holds for q_0 as one of the following:

$2, 5, 8, 11, 17, 23, 29, 32, 41, 47, 53, 59, 71, 83, 89, 101, 107, 113, 125, 128, 131, 137, 149, 167, 173, 179, 191, 197, 227, 233, 239, 251, 257, 263, 269, 281, 293, 311, 317, 347, 353, 359, 383, 389, 401, 419, 431, 443, 449, 461, 467, 479, 491, 503, 509, 512, 521, 557, 563, 569, 587, 593, 599, 617, 641, 647, 653, 659, 677, 683, 701, 719, 743, 761, 773, 797, 809, 821, 827, 839, 857, 863, 881, 887, 911, 929, 941, 947, 953, 971, 977, 983, 1013, 1019, 1031, 1049, 1061, 1091, 1097, 1103, 1109, 1151, 1163, 1181, 1187, 1193, 1217, 1223, 1229, 1259, 1277, 1283, 1289, 1301, 1307, 1319, 1331, 1361, 1367, 1373, 1409, 1427, 1433, 1439, 1451, 1481, 1487, 1493, 1499, 1511, 1523, 1553, 1559, 1571, 1583, 1601, 1607, 1613, 1619, 1637, 1667, 1697, 1709, 1721, 1733, 1787, 1811, 1823, 1847, 1871, 1877, 1889, 1901, 1907, 1913, 1931, 1949, 1973, 1979, 1997, 2003, 2027, 2039, 2048, 2063, 2069, 2081, 2087, 2099, 2111, 2129, 2141, 2153, 2207, 2213, 2237, 2243, 2267, 2273, 2297, 2309, 2333, 2339, 2351, 2357, 2381, 2393, 2399, 2411, 2417, 2423, 2441, 2447, 2459, 2477, 2531, 2543, 2549, 2579, 2591, 3125, 4913, 8192, 12167, 32768, 78125, 131072, 524288.$

For each $q_0 = q^{f/2}$ as above, k is a divisor of $6fq_0^3(q_0^2-1)(q_0^3-1)$ and we can obtain $v = q_0^3(q_0^2+1)(q_0^3+1)/3$. But in each case, the fraction $k(k-1)/(v-1)$ is not integer, which is a contradiction.

Case 2. Suppose now $r = 3$. By (3.33), we have that

$$v = (\frac{1}{c}) \cdot \frac{q_0^6(q_0^3+1)(q_0^9-1)}{q_0^2-1} \tag{3.42}$$

where $c = \gcd(3, q_0^2+q_0+1)$. By (3.2), k divides $6fq_0^3(q_0^2-1)(q_0^3-1)$. Then by Lemma 2.1(c), we have that

$$\lambda \cdot \frac{q_0^6(q_0^3+1)(q_0^9-1)}{c\cdot(q_0^2-1)} < k^2 \leqslant 36f^2q_0^6(q_0^2-1)^2(q_0^3-1)^2.$$

Therefore

$$\lambda < 36cf^2 \cdot \frac{(q_0^2-1)^3(q_0^3-1)^2}{(q_0^3+1)(q_0^9-1)} \leqslant 108f^2. \tag{3.43}$$

It follows from (3.2) that k divides $6fq_0^3(q_0^2-1)(q_0^3-1)$. Moreover, Lemma 2.1(a) implies that k divides $\lambda(v-1)$, and since $v-1$ is coprime to q_0 by Lemma 2.3, k must divide $6\lambda f(q_0^2-1)(q_0^3-1)$. Using this fact and Lemma 2.1(c), we have that

$$\lambda \cdot \frac{q_0^6(q_0^3+1)(q_0^9-1)}{c\cdot(q_0^2-1)} < k^2 \leqslant 36\lambda^2f^2(q_0^2-1)^2(q_0^3-1)^2,$$

and so

$$q_0^6(q_0^3+1)(q_0^9-1) \over (q_0^2-1)^3(q_0^3-1)^2 < 36\lambda f^2 c. \tag{3.44}$$

Since $c \leqslant 3$ and $\lambda \leqslant 108f^2$ by (3.43), it follows that

$$q_0^6 < 11664f^4.$$

Since also q_0 is at least 2, $2^{3f} < 11664 \cdot f^4$, and this implies that $f \leqslant 8$. Then $p^{3f} = q_0^6 < 1166f^4$, and so $p \leqslant 7$. Considering (3.44), the only possibilities for q_0 is 2, 3, 4 5, 8 or 9. Since now k divides $6fq_0^3(q_0^2-1)(q_0^3-1)$, for each such value of q_0, the possible values of k and v are listed in Table 8 below:

TABLE 8. Possible values for k and v when $q_0 \in \{2,3,4,5,7,8,9\}$.

q_0	v	k divides
2	32704	2016
3	50218623	67392
4	4652863488	1451520
5	53405734375	4464000
8	95500437815296	593381376
9	1878756575514273	1018967040

For each such parameter k and v as in Table 8, by straightforward calculation, we observe that the fraction $k(k-1)/(v-1)$ is not integer, which is a contradiction. □

3.1. **Proof of Theorem 1.1.** Suppose that $\mathcal{D}$ is a nontrivial (v,k,λ)-symmetric design and G is an almost simple automorphism group G with simple socle $X = \mathrm{PSL}(3,q)$. Suppose also that $V = \mathrm{GF}(q)^3$ is the underlying vector space of X over the finite field $\mathrm{GF}(q)$. If G is a flag-transitive and point-primitive automorphism group of $\mathcal{D}$, then the point-stabiliser $M := G_\alpha$ is maximal in G, and so $M_0 := X \cap M$ is isomorphic to one of the subgroups in Lemma 2.4(a)-(l). It follows from Lemmas 3.2–3.12 that $M_0 = \hat{}[q^2] : \mathrm{GL}(2,q)$. In this case, M_0 is transitive on the set of projective points of V (the set of one dimensional subspaces of V), and so G is 2-transitive. It follows from [14, 15] that $\mathcal{D}$ is a Desarguesian plane and $G \geqslant \mathrm{PSL}(3,q)$. □

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