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COHEN-MACAULAY-NESS IN CODIMENSION FOR SIMPLICIAL COMPLEXES AND EXPANSION FUNCTOR

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Abstract. In this paper we show that expansion of a Buchsbaum simplicial complex is CM_t, for an optimal integer $t \geq 1$. Also, by imposing extra assumptions on a CM_t simplicial complex, we prove that it can be obtained from a Buchsbaum complex.
Keywords: CM_t simplicial complex, expansion functor, simple graph
MSC(2010): Primary: 13H10; Secondary: 05C75.

1. Introduction

Set $[n] := \{x_1, \dots, x_n\}$. Let K be a field and $S = K[x_1, \dots, x_n]$, a polynomial ring over K. Let Δ be a simplicial complex over $[n]$. For an integer $t \geq 0$, Haghighi, Yassemi and Zaare-Nahandi introduced the concept of CM_t-ness which is the pure version of simplicial complexes *Cohen-Macaulay in codimension t* studied in [7]. A reason for the importance of CM_t simplicial complexes is that they generalize two notions for simplicial complexes: being Cohen-Macaulay and Buchsbaum. In particular, by the results from [9,11], CM_0 is the same as Cohen-Macaulayness and CM_1 is identical with Buchsbaum property.

In [3], the authors described some combinatorial properties of CM_t simplicial complexes and gave some characterizations of them and generalized some results of [6, 8]. Then, in [4], they generalized a characterization of Cohen-Macaulay bipartite graphs from [5] and [2] on unmixed Buchsbaum graphs.

Bayati and Herzog defined the expansion functor in the category of finitely generated multigraded S-modules and studied some homological behaviors of this functor (see [1]). The expansion functor helps us to present other multigraded S-modules from a given finitely generated multigraded S-module which may have some of algebraic properties of the primary module. This allows to introduce new structures of a given multigraded S-module with the same

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properties and especially to extend some homological or algebraic results for larger classes (see for example [1, Theorem 4.2]. There are some combinatorial versions of expansion functor which we will recall in this paper.

The purpose of this paper is the study of behaviors of expansion functor on CM_t complexes. We first recall some notations and definitions of CM_t simplicial complexes in Section 1. In the next section we describe the expansion functor in three contexts, the expansion of a simplicial complex, the expansion of a simple graph and the expansion of a monomial ideal. We show that there is a close relationship between these three contexts. In Section 3 we prove that the expansion of a CM_t complex Δ with respect to α is $\mathrm{CM}_{t+e-k+1}$ but it is not CM_{t+e-k} where $e = \dim(\Delta^\alpha)+1$ and k is the minimum of the components of α (see Theorem 4.3). In Section 4, we introduce a new functor, called contraction, which acts in contrast to expansion functor. As a main result of this section we show that if the contraction of a CM_t complex is pure and all components of the vector obtained from contraction are greater than or equal to t then it is Buchsbaum (see Theorem 5.6). The section is finished with a view towards the contraction of simple graphs.

2. Preliminaries

Let t be a non-negative integer. We recall from [3] that a simplicial complex Δ is called CM_t or *Cohen-Macaulay in codimension* t if it is pure and for every face $F \in \Delta$ with $\#(F) \geq t$, $\mathrm{link}_\Delta(F)$ is Cohen-Macaulay. Every CM_t complex is also CM_r for all $r \geq t$. For $t < 0$, CM_t means CM_0. The properties CM_0 and CM_1 are the same as Cohen-Macaulay-ness and Buchsbaum-ness, respectively.

The link of a face F in a simplicial complex Δ is denoted by $\mathrm{link}_\Delta(F)$ and is

$$\mathrm{link}_\Delta(F) = \{G \in \Delta : G \cap F = \emptyset, G \cup F \in \Delta\}.$$

The following lemma is useful for checking the CM_t property of simplicial complexes:

Lemma 2.1. *([3, Lemma 2.3]) Let $t \geq 1$ and let Δ be a nonempty complex. Then Δ is CM_t if and only if Δ is pure and $\mathrm{link}_\Delta(v)$ is CM_{t-1} for every vertex $v \in \Delta$.*

Let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a simple graph with vertex set V and edge set E. The *independence complex* of $\mathcal{G}$ is the complex $\Delta_\mathcal{G}$ with vertex set V and with faces consisting of independent sets of vertices of $\mathcal{G}$. Thus F is a face of $\Delta_\mathcal{G}$ if and only if there is no edge of $\mathcal{G}$ joining any two vertices of F.

The *edge ideal* of a simple graph $\mathcal{G}$, denoted by $I(\mathcal{G})$, is an ideal of S generated by all squarefree monomials x_ix_j with $x_ix_j \in E(\mathcal{G})$.

A simple graph $\mathcal{G}$ is called CM_t if $\Delta_\mathcal{G}$ is CM_t and it is called *unmixed* if $\Delta_\mathcal{G}$ is pure.

For a monomial ideal $I \subset S$, We denote by $G(I)$ the unique minimal set of monomial generators of I.

3. The expansion functor in combinatorial and algebraic concepts

In this section we define the expansion of a simplicial complex and recall the expansion of a simple graph from [10] and the expansion of a monomial ideal from [1]. We show that these concepts are intimately related to each other.

(1) Let $\alpha = (k_1, \ldots, k_n) \in \mathbb{N}^n$. For $F = \{x_{i_1}, \ldots, x_{i_r}\} \subseteq \{x_1, \ldots, x_n\}$ define

$$F^\alpha = \{x_{i_1 1}, \ldots, x_{i_1 k_{i_1}}, \ldots, x_{i_r 1}, \ldots, x_{i_r k_{i_r}}\}$$

as a subset of $[n]^\alpha := \{x_{11}, \ldots, x_{1k_1}, \ldots, x_{n1}, \ldots, x_{nk_n}\}$. F^α is called *the expansion of F with respect to α.*

For a simplicial complex $\Delta = \langle F_1, \ldots, F_r \rangle$ on $[n]$, we define *the expansion of Δ with respect to α* as the simplicial complex

$$\Delta^\alpha = \langle F_1^\alpha, \ldots, F_r^\alpha \rangle.$$

(2) The *duplication* of a vertex x_i of a simple graph $\mathcal{G}$ was first introduced by Schrijver [10] and it means extending its vertex set $V(\mathcal{G})$ by a new vertex x'_i and replacing $E(\mathcal{G})$ by

$$E(\mathcal{G}) \cup \{(e \backslash \{x_i\}) \cup \{x'_i\} : x_i \in e \in E(\mathcal{G})\}.$$

For the n-tuple $\alpha = (k_1, \ldots, k_n) \in \mathbb{N}^n$, with positive integer entries, the *expansion* of the simple graph $\mathcal{G}$ is denoted by $\mathcal{G}^\alpha$ and it is obtained from $\mathcal{G}$ by successively duplicating $k_i - 1$ times every vertex x_i.

(3) In [1] Bayati and Herzog defined the expansion functor in the category of finitely generated multigraded S-modules and studied some homological behaviors of this functor. We recall the expansion functor defined by them only in the category of monomial ideals and refer the reader to [1] for more general case in the category of finitely generated multigraded S-modules.

Let S^α be a polynomial ring over K in the variables

$$x_{11}, \ldots, x_{1k_1}, \ldots, x_{n1}, \ldots, x_{nk_n}.$$

Whenever $I \subset S$ is a monomial ideal minimally generated by $u_1, \ldots, u_r$, the expansion of I with respect to α is defined by

$$I^\alpha = \sum_{i=1}^{r} P_1^{\nu_1(u_i)} \ldots P_n^{\nu_n(u_i)} \subset S^\alpha$$

where $P_j = (x_{j1}, \ldots, x_{jk_j})$ is a prime ideal of S^α and $\nu_j(u_i)$ is the exponent of x_j in u_i.

It was shown in [1] that the expansion functor is exact and so $(S/I)^\alpha = S^\alpha / I^\alpha$. In the following lemmas we describe the relations between the above three concepts of expansion functor.

Lemma 3.1. *For a simplicial complex Δ we have $I_\Delta^\alpha = I_{\Delta^\alpha}$. In particular, $K[\Delta]^\alpha = K[\Delta^\alpha]$.*

Proof. Let $\Delta = \langle F_1, \ldots, F_r \rangle$. Since $I_\Delta = \bigcap_{i=1}^r P_{F_i^c}$, it follows from Lemma 1.1 in [1] that $I_\Delta^\alpha = \bigcap_{i=1}^r P_{F_i^c}^\alpha$. The result is obtained by the fact that $P_{F_i^c}^\alpha = P_{(F_i^\alpha)^c}$. □

Let $u = x_{i_1} \ldots x_{i_t} \in S$ be a monomial and $\alpha = (k_1, \ldots, k_n) \in \mathbb{N}^n$. We set $u^\alpha = G((u)^\alpha)$ and for a set A of monomials in S, A^α is defined by

$$A^\alpha = \bigcup_{u \in A} u^\alpha.$$

One can easily obtain the following lemma.

Lemma 3.2. *Let $I \subset S$ be a monomial ideal and $\alpha \in \mathbb{N}^n$. Then $G(I^\alpha) = G(I)^\alpha$.*

Lemma 3.3. *For a simple graph $\mathcal{G}$ on the vertex set $[n]$ and $\alpha \in \mathbb{N}^n$ we have $I(\mathcal{G}^\alpha) = I(\mathcal{G})^\alpha$.*

Proof. Let $\alpha = (k_1, \ldots, k_n)$ and $P_j = (x_{j1}, \ldots, x_{jk_j})$. Then it follows from Lemma 11(ii,iii) of [1] that

$$I(\mathcal{G}^\alpha) = (x_{ir}x_{js} : x_ix_j \in E(\mathcal{G}), 1 \leq r \leq k_i, 1 \leq s \leq k_j) = \sum_{x_ix_j \in E(\mathcal{G})} P_iP_j$$

$$= \sum_{x_ix_j \in E(\mathcal{G})} (x_i)^\alpha (x_j)^\alpha = (\sum_{x_ix_j \in E(\mathcal{G})} (x_i)(x_j))^\alpha = I(\mathcal{G})^\alpha.$$

□

4. The expansion of a CM_t complex

The following proposition gives us some information about the expansion of a simplicial complex which are useful in the proof of the next results.

Proposition 4.1. *Let Δ be a simplicial complex and let $\alpha \in \mathbb{N}^n$.*

(i) *For all $i \leq \dim(\Delta)$, there exists an epimorphism $\theta : \tilde{H}_i(\Delta^\alpha; K) \to \tilde{H}_i(\Delta; K)$.*
In particular in this case

$$\tilde{H}_i(\Delta^\alpha; K)/\ker(\theta) \cong \tilde{H}_i(\Delta; K);$$

(ii) *For $F \in \Delta^\alpha$ such that $F = G^\alpha$ for some $G \in \Delta$, we have*

$$\mathrm{link}_{\Delta^\alpha}(F) = (\mathrm{link}_\Delta(G))^\alpha;$$

(iii) For $F \in \Delta^\alpha$ such that $F \neq G^\alpha$ for every $G \in \Delta$, we have

$$\mathrm{link}_{\Delta^\alpha} F = \langle U^\alpha \backslash F \rangle * \mathrm{link}_{\Delta^\alpha} U^\alpha$$

for some $U \in \Delta$ with $F \subseteq U^\alpha$. Here $*$ means the join of two simplicial complexes.

In the third case, $\mathrm{link}_{\Delta^\alpha} F$ is a cone and so acyclic, i.e., $\tilde{H}_i(\mathrm{link}_{\Delta^\alpha} F; K) = 0$ for all $i > 0$.

Proof. (i) Consider the map $\pi : [n]^\alpha \to [n]$ by $\pi(x_{ij}) = x_i$ for all i, j. Let the simplicial map $\varphi : \Delta^\alpha \to \Delta$ be defined by $\varphi(\{x_{i_1 j_1}, \dots, x_{i_q j_q}\}) = \{\pi(x_{i_1 j_1}), \dots, \pi(x_{i_q j_q})\} = \{x_{i_1}, \dots, x_{i_q}\}$. Actually, φ is an extension of π to Δ^α by linearity. Define $\varphi_\# : \tilde{\mathcal{C}}_q(\Delta^\alpha; K) \to \tilde{\mathcal{C}}_q(\Delta; K)$, for each q, by

$$\varphi_\#([x_{i_0 j_0}, \dots, x_{i_q j_q}]) = \begin{cases} 0 & \text{if for some indices } i_r = i_t \\ [\varphi(\{x_{i_0 j_0}\}), \dots, \varphi(\{x_{i_q j_q}\})] & \text{otherwise.} \end{cases}$$

It is clear from the definitions of $\tilde{\mathcal{C}}_q(\Delta^\alpha; K)$ and $\tilde{\mathcal{C}}_q(\Delta; K)$ that $\varphi_\#$ is well-defined. Also, define $\varphi_\alpha : \tilde{H}_i(\Delta^\alpha; K) \to \tilde{H}_i(\Delta; K)$ by

$$\varphi_\alpha : z + B_i(\Delta^\alpha) \to \varphi_\#(z) + B_i(\Delta).$$

It is trivial that φ_α is onto.

(ii) The inclusion $\mathrm{link}_{\Delta^\alpha}(F) \supseteq (\mathrm{link}_\Delta(G))^\alpha$ is trivial. So we show the reverse inclusion. Let $\sigma \in \mathrm{link}_{\Delta^\alpha}(G^\alpha)$. Then $\sigma \cap G^\alpha = \emptyset$ and $\sigma \cup G^\alpha \in \Delta^\alpha$. We want to show $\pi(\sigma) \in \mathrm{link}_\Delta(G)$. Because in this case, $\pi(\sigma)^\alpha \in (\mathrm{link}_\Delta(G))^\alpha$ and since $\sigma \subseteq \pi(\sigma)^\alpha$, we conclude that $\sigma \in (\mathrm{link}_\Delta(G))^\alpha$.

Clearly, $\pi(\sigma) \cup G \in \Delta$. To show that $\pi(\sigma) \cap G = \emptyset$, suppose, on the contrary, that $x_i \in \pi(\sigma) \cap G$. Then $x_{ij} \in \sigma$ for some j. Especially, $x_{ij} \in G^\alpha$. Therefore $\sigma \cap G^\alpha \neq \emptyset$, a contradiction.

(iii) Let $\tau \in \mathrm{link}_{\Delta^\alpha} F$. Let $\tau \cap \pi(F)^\alpha = \emptyset$. It follows from $\tau \cup F \in \Delta^\alpha$ that $\pi(\tau)^\alpha \cup \pi(F)^\alpha \in \Delta^\alpha$. Now by $\tau \subset \pi(\tau)^\alpha$ it follows that $\tau \cup \pi(F)^\alpha \in \Delta^\alpha$. Hence $\tau \in \mathrm{link}_{\Delta^\alpha}(\pi(F)^\alpha)$. So we suppose that $\tau \cap \pi(F)^\alpha \neq \emptyset$. We write $\tau = (\tau \cap \pi(F)^\alpha) \cup (\tau \backslash \pi(F)^\alpha)$. It is clear that $\tau \cap \pi(F)^\alpha \subset \pi(F)^\alpha \backslash F$ and $\tau \backslash \pi(F)^\alpha \in \mathrm{link}_{\Delta^\alpha} \pi(F)^\alpha$. The reverse inclusion is trivial. $\square$

Remark 4.2. Let $\Delta = \langle x_1 x_2, x_2 x_3 \rangle$ be a complex on $[3]$ and $\alpha = (2, 1, 1) \in \mathbb{N}^3$. Then $\Delta^\alpha = \langle x_{11} x_{12} x_{21}, x_{21} x_{31} \rangle$ is a complex on $\{x_{11}, x_{12}, x_{21}, x_{31}\}$. Notice that Δ is pure but Δ^α is not. Therefore, the expansion of a pure simplicial complex is not necessarily pure.

Theorem 4.3. *Let Δ be a simplicial complex on $[n]$ of dimension $d-1$ and let $t \geq 0$ be the least integer that Δ is CM_t. Suppose that $\alpha = (k_1, \dots, k_n) \in \mathbb{N}^n$ such that $k_i > 1$ for some i and Δ^α is pure. Then Δ^α is $\mathrm{CM}_{t+e-k+1}$ but it is not CM_{t+e-k}, where $e = \dim(\Delta^\alpha) + 1$ and $k = \min\{k_i : k_i > 1\}$.*

Proof. We use induction on $e \geq 2$. If $e = 2$, then $\dim(\Delta^\alpha) = 1$ and Δ should be only in the form $\Delta = \langle x_1, \dots, x_n \rangle$. In particular, Δ^α is of the form

$$\Delta^\alpha = \langle \{x_{i_1 1}, x_{i_1 2}\}, \{x_{i_2 1}, x_{i_2 2}\}, \dots, \{x_{i_r 1}, x_{i_r 2}\} \rangle.$$

It is clear that Δ^α is CM_1 but it is not Cohen-Macaulay.

Assume that $e > 2$. Let $\{x_{ij}\} \in \Delta^\alpha$. We want to show that $\mathrm{link}_{\Delta^\alpha}(x_{ij})$ is CM_{e-k}. Consider the following cases:

Case 1: $k_i > 1$. Then

$$\mathrm{link}_{\Delta^\alpha}(x_{ij}) = \langle \{x_i\}^\alpha \backslash x_{ij} \rangle * (\mathrm{link}_\Delta(x_i))^\alpha.$$

$(\mathrm{link}_\Delta(x_i))^\alpha$ is of dimension $e - k_i - 1$ and, by induction hypothesis, it is $\mathrm{CM}_{t+e-k_i-k+1}$. On the other hand, $\langle \{x_i\}^\alpha \backslash x_{ij} \rangle$ is Cohen-Macaulay of dimension $k_i - 2$. Therefore, it follows from Theorem 1.1(i) of [4] that $\mathrm{link}_{\Delta^\alpha}(x_{ij})$ is CM_{t+e-k}.

Case 2: $k_i = 1$. Then

$$\mathrm{link}_{\Delta^\alpha}(x_{ij}) = (\mathrm{link}_\Delta(x_i))^\alpha$$

which is of dimension $e - 2$ and, by induction, it is CM_{t+e-k}.

Now suppose that $e > 2$ and $k_s = k$ for some $s \in [n]$. Let F be a facet of Δ such that x_s belongs to F.

If $\dim(\Delta) = 0$, then $k_l = k$ for all $l \in [n]$. In particular, $e = k$. It is clear that Δ^α is not CM_{t+e-k} (or Cohen-Macaulay). So suppose that $\dim(\Delta) > 0$. Choose $x_i \in F \backslash x_s$. Then

$$\mathrm{link}_{\Delta^\alpha}(x_{ij}) = \langle \{x_i\}^\alpha \backslash x_{ij} \rangle * (\mathrm{link}_\Delta(x_i))^\alpha.$$

By induction hypothesis, $(\mathrm{link}_\Delta(x_i))^\alpha$ is not CM_{t+e-k_i-k}. It follows from Theorem 3.1(ii) of [4] that $\mathrm{link}_{\Delta^\alpha}(x_{ij})$ is not $\mathrm{CM}_{t+e-k-1}$. Therefore Δ^α is not CM_{t+e-k}. □

Corollary 4.4. *Let Δ be a non-empty Cohen-Macaulay simplicial complex on $[n]$. Then for any $\alpha \in \mathbb{N}^n$, with $\alpha \neq \mathbf{1}$, Δ^α can never be Cohen-Macaulay.*

5. The contraction functor

Let $\Delta = \langle F_1, \dots, F_r \rangle$ be a simplicial complex on $[n]$. Consider the equivalence relation '$\sim$' on the vertices of Δ given by

$$x_i \sim x_j \Leftrightarrow \langle x_i \rangle * \mathrm{link}_\Delta(x_i) = \langle x_j \rangle * \mathrm{link}_\Delta(x_j).$$

In fact $\langle x_i \rangle * \mathrm{link}_\Delta(x_i)$ is the cone over $\mathrm{link}_\Delta(x_i)$, and the elements of $\langle x_i \rangle * \mathrm{link}_\Delta(x_i)$ are those faces of Δ, which contain x_i. Hence $\langle x_i \rangle * \mathrm{link}_\Delta(x_i) = \langle x_j \rangle * \mathrm{link}_\Delta(x_j)$, means the cone with vertex x_i is equal to the cone with vertex x_j. In other words, $x_i \sim x_j$ is equivalent to saying that for a facet $F \in \Delta$, F contains x_i if and only if it contains x_j.

Let $[\bar{m}] = \{\bar{y}_1, \dots, \bar{y}_m\}$ be the set of equivalence classes under $\sim$. Let $\bar{y}_i = \{x_{i1}, \dots, x_{ia_i}\}$. Set $\alpha = (a_1, \dots, a_m)$. For $F_t \in \Delta$, define $G_t = \{\bar{y}_i : \bar{y}_i \subset F_t\}$

and let Γ be a simplicial complex on the vertex set $[m]$ with facets $G_1, \dots, G_r$. We call Γ the *contraction of* Δ *by* α and α is called *the vector obtained from contraction*.

For example, consider the simplicial complex $\Delta = \langle x_1x_2x_3, x_2x_3x_4, x_1x_4x_5, x_2x_3x_5\rangle$ on the vertex set $[5] = \{x_1, \dots, x_5\}$. Then $\bar{y}_1 = \{x_1\}$, $\bar{y}_2 = \{x_2, x_3\}$, $\bar{y}_3 = \{x_4\}$, $\bar{y}_4 = \{x_5\}$ and $\alpha = (1, 2, 1, 1)$. Therefore, the contraction of Δ by α is $\Gamma = \langle \bar{y}_1\bar{y}_2, \bar{y}_2\bar{y}_3, \bar{y}_1\bar{y}_3\bar{y}_4, \bar{y}_2\bar{y}_4\rangle$ a complex on the vertex set $[\bar{4}] = \{\bar{y}_1, \dots, \bar{y}_4\}$.

Remark 5.1. Note that if Δ is a pure simplicial complex then the contraction of Δ is not necessarily pure (see the above example). In the special case where the vector $\alpha = (k_1, \dots, k_n) \in \mathbb{N}^n$ and $k_i = k_j$ for all i, j, it is easy to check that in this case Δ is pure if and only if Δ^α is pure. Another case is introduced in the following proposition.

Proposition 5.2. *Let Δ be a simplicial complex on $[n]$ and assume that $\alpha = (k_1, \dots, k_n) \in \mathbb{N}^n$ satisfies the following condition:*

(†) *for all facets $F, G \in \Delta$, if $x_i \in F\backslash G$ and $x_j \in G\backslash F$ then $k_i = k_j$.*

Then Δ is pure if and only if Δ^α is pure.

Proof. Let Δ be a pure simplicial complex and let $F, G \in \Delta$ be two facets of Δ. Then

$$|F^\alpha| - |G^\alpha| = \sum_{x_i\in F} k_i - \sum_{x_i\in G} k_i = \sum_{x_i\in F\backslash G} k_i - \sum_{x_i\in G\backslash F} k_i.$$

Now the condition (†) implies that $|F^\alpha| = |G^\alpha|$. This means that all facets of Δ^α have the same cardinality.

Let Δ^α be pure. Suppose that F, G are two facets in Δ. If $|F| > |G|$ then $|F\backslash G| > |G\backslash F|$. Therefore $\sum_{x_i\in F\backslash G} k_i > \sum_{x_i\in G\backslash F} k_i$. This implies that $|F^\alpha| = \sum_{x_i\in F} k_i > \sum_{x_i\in G} k_i = |G^\alpha|$, a contradiction. $\square$

There is a close relationship between a simplicial complex and its contraction. In fact, the expansion of the contraction of a simplicial complex is the same complex. The precise statement is the following.

Lemma 5.3. *Let Γ be the contraction of Δ by α. Then $\Gamma^\alpha \cong \Delta$.*

Proof. Suppose that Δ and Γ are on the vertex sets $[n] = \{x_1, \dots, x_n\}$ and $[\bar{m}] = \{\bar{y}_1, \dots, \bar{y}_m\}$, respectively. Let $\alpha = (a_1, \dots, a_m)$. For $\bar{y}_i \in \Gamma$, suppose that $\{\bar{y}_i\}^\alpha = \{\bar{y}_{i1}, \dots, \bar{y}_{ia_i}\}$. So Γ^α is a simplicial complex on the vertex set $[\bar{m}]^\alpha = \{\bar{y}_{ij} : i = 1, \dots, m,\ j = 1, \dots, a_i\}$. Now define $\varphi : [\bar{m}]^\alpha \to [n]$ by $\varphi(\bar{y}_{ij}) = x_{ij}$. Extending φ, we obtain the isomorphism $\varphi : \Gamma^\alpha \to \Delta$. $\square$

Proposition 5.4. *Let Δ be a simplicial complex and assume that Δ^α is Cohen-Macaulay for some $\alpha \in \mathbb{N}^n$. Then Δ is Cohen-Macaulay.*

Proof. By Lemma 4.1(i), for all $i \leq \dim(\mathrm{link}_\Delta F)$ and all $F \in \Delta$ there exists an epimorphism $\theta : \mathrm{link}_{\Delta^\alpha} F^\alpha \to \mathrm{link}_\Delta F$ such that

$$\tilde{H}_i(\mathrm{link}_{\Delta^\alpha} F^\alpha; K)/\ker(\theta) \cong \tilde{H}_i(\mathrm{link}_\Delta F; K).$$

Now suppose that $i < \dim(\mathrm{link}_\Delta F)$. Then $i < \dim(\mathrm{link}_{\Delta^\alpha} G^\alpha)$ and by Cohen-Macaulayness of Δ^α, $\tilde{H}_i(\mathrm{link}_{\Delta^\alpha} F^\alpha; K) = 0$. Therefor $\tilde{H}_i(\mathrm{link}_\Delta F; K) = 0$. This means that Δ is Cohen-Macaulay. □

It follows from Proposition 5.4 that:

Corollary 5.5. *The contraction of a Cohen-Macaulay simplicial complex Δ is Cohen-Macaulay.*

This can be generalized in the following theorem.

Theorem 5.6. *Let Γ be the contraction of a CM_t simplicial complex Δ, for some $t \geq 0$, by $\alpha = (k_1, \ldots, k_n)$. If $k_i \geq t$ for all i and Γ is pure, then Γ is Buchsbaum.*

Proof. If $t = 0$, then we saw in Corollary 5.5 that Γ is Cohen-Macaulay and so it is CM_t. Hence assume that $t > 0$. Let $\Delta = \langle F_1, \ldots, F_r \rangle$. We have to show that $\tilde{H}_i(\mathrm{link}_\Gamma G; K) = 0$, for all faces $G \in \Gamma$ with $|G| \geq 1$ and all $i < \dim(\mathrm{link}_\Gamma G)$.

Let $G \in \Gamma$ with $|G| \geq 1$. Then $|G^\alpha| \geq t$. It follows from Lemma 2.1 and CM_t-ness of Δ that

$$\tilde{H}_i(\mathrm{link}_\Gamma G; K) \cong \tilde{H}_i(\mathrm{link}_\Delta G^\alpha; K) = 0$$

for $i < \dim(\mathrm{link}_\Delta G^\alpha)$ and, particularly, for $i < \dim(\mathrm{link}_\Gamma G)$. Therefore Γ is Buchsbaum. □

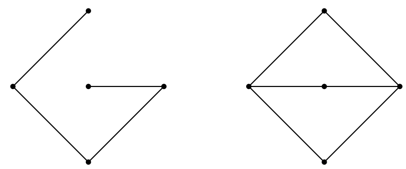
Corollary 5.7. *Let Γ be the contraction of a Buchsbaum simplicial complex Δ. If Γ is pure, then Γ is also Buchsbaum.*

Let $\mathcal{G}$ be a simple graph on the vertex set $[n]$ and let $\Delta_\mathcal{G}$ be its independence complex on $[n]$, i.e., a simplicial complex whose faces are the independent vertex sets of G. Let Γ be the contraction of $\Delta_\mathcal{G}$. In the following we show that Γ is the independence complex of a simple graph $\mathcal{H}$. We call $\mathcal{H}$ the *contraction* of $\mathcal{G}$.

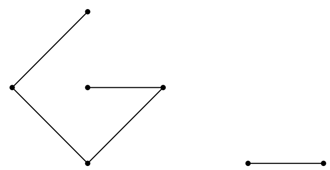
Lemma 5.8. *Let $\mathcal{G}$ be a simple graph. The contraction of $\Delta_\mathcal{G}$ is the independence complex of a simple graph $\mathcal{H}$.*

Proof. It suffices to show that I_Γ is a squarefree monomial ideal generated in degree 2. Let Γ be the contraction of $\Delta_\mathcal{G}$ and let $\alpha = (k_1, \ldots, k_n)$ be the vector obtained from the contraction. Let $[n] = \{x_1, \ldots, x_n\}$ be the vertex set of Γ. Suppose that $u = x_{i_1} \ldots x_{i_t} \in G(I_\Gamma)$. Then $u^\alpha \subset G(I_\Gamma)^\alpha = G(I_{\Delta_\mathcal{G}}) = G(I(\mathcal{G}))$. Since $u^\alpha = \{x_{i_1 j_1} \ldots x_{i_t j_t} : 1 \leq j_l \leq k_{i_l}, 1 \leq l \leq t\}$ we have $t = 2$ and the proof is completed. □

Example 5.9. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be, respectively, from left to right the following graphs:



The contraction of $\mathcal{G}_1$ and $\mathcal{G}_2$ are



The contraction of $\mathcal{G}_1$ is equal to itself but $\mathcal{G}_2$ is contracted to an edge and the vector obtained from contraction is $\alpha = (2, 3)$.

We recall that a simple graph is CM_t for some $t \geq 0$, if the associated independence complex is CM_t.

Remark 5.10. The simple graph $\mathcal{G}'$ obtained from $\mathcal{G}$ in Lemma 4.3 and Theorem 4.4 of [4] is the expansion of $\mathcal{G}$. Actually, suppose that $\mathcal{G}$ is a bipartite graph on the vertex set $V(\mathcal{G}) = V \cup W$ where $V = \{x_1, \dots, x_d\}$ and $W = \{x_{d+1}, \dots, x_{2d}\}$. Then for $\alpha = (n_1, \dots, n_d, n_1, \dots, n_d)$ we have $\mathcal{G}' = \mathcal{G}^\alpha$. It follows from Theorem 4.3 that if $\mathcal{G}$ is CM_t for some $t \geq 0$ then $\mathcal{G}'$ is $\mathrm{CM}_{t+n-n_{i_0}+1}$ where $n = \sum_{i=1}^d n_i$ and $n_{i_0} = \min\{n_i > 1 : i = 1, \dots, d\}$. This implies that the first part of Theorem 4.4 of [4] is an immediate consequence of Theorem 4.3 for $t = 0$.

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