Title:

Cohen-Macaulay-ness in codimension for simplicial complexes and expansion functor

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COHEN-MACAULAY-NESS IN CODIMENSION FOR SIMPLICIAL COMPLEXES AND EXPANSION FUNCTOR

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(Communicated by Siamak Yassemi)

Abstract. In this paper we show that expansion of a Buchsbaum simplicial complex is CM_t, for an optimal integer t ≥ 1. Also, by imposing extra assumptions on a CM_t simplicial complex, we prove that it can be obtained from a Buchsbaum complex.

Keywords: CM_t simplicial complex, expansion functor, simple graph

MSC(2010): Primary: 13H10; Secondary: 05C75.

1. Introduction

Set \([n] := \{x_1, \ldots, x_n\}\). Let \(K\) be a field and \(S = K[x_1, \ldots, x_n]\), a polynomial ring over \(K\). Let \(\Delta\) be a simplicial complex over \([n]\). For an integer \(t \geq 0\), Haghighi, Yassemi and Zaare-Nahandi introduced the concept of CM_t-ness which is the pure version of simplicial complexes Cohen-Macaulay in codimension \(t\) studied in [7]. A reason for the importance of CM_t simplicial complexes is that they generalize two notions for simplicial complexes: being Cohen-Macaulay and Buchsbaum. In particular, by the results from [9, 11], CM_0 is the same as Cohen-Macaulayness and CM_1 is identical with Buchsbaum property.

In [3], the authors described some combinatorial properties of CM_t simplicial complexes and gave some characterizations of them and generalized some results of [6, 8]. Then, in [4], they generalized a characterization of Cohen-Macaulay bipartite graphs from [5] and [2] on unmixed Buchsbaum graphs.

Bayati and Herzog defined the expansion functor in the category of finitely generated multigraded \(S\)-modules and studied some homological behaviors of this functor (see [1]). The expansion functor helps us to present other multigraded \(S\)-modules from a given finitely generated multigraded \(S\)-module which may have some of algebraic properties of the primary module. This allows to introduce new structures of a given multigraded \(S\)-module with the same
properties and especially to extend some homological or algebraic results for larger classes (see for example [1, Theorem 4.2]. There are some combinatorial versions of expansion functor which we will recall in this paper.

The purpose of this paper is the study of behaviors of expansion functor on CM\textsubscript{t} complexes. We first recall some notations and definitions of CM\textsubscript{t} simplicial complexes in Section 1. In the next section we describe the expansion functor in three contexts, the expansion of a simplicial complex, the expansion of a simple graph and the expansion of a monomial ideal. We show that there is a close relationship between these three contexts. In Section 3 we prove that the expansion of a CM\textsubscript{t} complex \( \Delta \) with respect to \( \alpha \) is CM\textsubscript{t+e-k+1} but it is not CM\textsubscript{t+e-k} where \( e = \dim(\Delta^\alpha) + 1 \) and \( k \) is the minimum of the components of \( \alpha \) (see Theorem 4.3). In Section 4, we introduce a new functor, called contraction, which acts in contrast to expansion functor. As a main result of this section we show that if the contraction of a CM\textsubscript{t} complex is pure and all components of the vector obtained from contraction are greater than or equal to \( t \) then it is Buchsbaum (see Theorem 5.6). The section is finished with a view towards the contraction of simple graphs.

2. Preliminaries

Let \( t \) be a non-negative integer. We recall from [3] that a simplicial complex \( \Delta \) is called CM\textsubscript{t} or Cohen-Macaulay in codimension \( t \) if it is pure and for every face \( F \in \Delta \) with \( \#(F) \geq t \), \( \text{link}_\Delta(F) \) is Cohen-Macaulay. Every CM\textsubscript{t} complex is also CM\textsubscript{r} for all \( r \geq t \). For \( t < 0 \), CM\textsubscript{t} means CM\textsubscript{0}. The properties CM\textsubscript{0} and CM\textsubscript{1} are the same as Cohen-Macaulay-ness and Buchsbaum-ness, respectively.

The link of a face \( F \) in a simplicial complex \( \Delta \) is denoted by \( \text{link}_\Delta(F) \) and is

\[
\text{link}_\Delta(F) = \{ G \in \Delta : G \cap F = \emptyset, G \cup F \in \Delta \}.
\]

The following lemma is useful for checking the CM\textsubscript{t} property of simplicial complexes:

**Lemma 2.1.** ([3, Lemma 2.3]) Let \( t \geq 1 \) and let \( \Delta \) be a nonempty complex. Then \( \Delta \) is CM\textsubscript{t} if and only if \( \Delta \) is pure and \( \text{link}_\Delta(v) \) is CM\textsubscript{t-1} for every vertex \( v \in \Delta \).

Let \( \mathcal{G} = (V(\mathcal{G}), E(\mathcal{G})) \) be a simple graph with vertex set \( V \) and edge set \( E \). The independence complex of \( \mathcal{G} \) is the complex \( \Delta_{\mathcal{G}} \) with vertex set \( V \) and with faces consisting of independent sets of vertices of \( \mathcal{G} \). Thus \( F \) is a face of \( \Delta_{\mathcal{G}} \) if and only if there is no edge of \( \mathcal{G} \) joining any two vertices of \( F \).

The edge ideal of a simple graph \( \mathcal{G} \), denoted by \( I(\mathcal{G}) \), is an ideal of \( S \) generated by all squarefree monomials \( x_i x_j \) with \( x_i x_j \in E(\mathcal{G}) \).

A simple graph \( \mathcal{G} \) is called CM\textsubscript{t} if \( \Delta_{\mathcal{G}} \) is CM\textsubscript{t} and it is called unmixed if \( \Delta_{\mathcal{G}} \) is pure.
For a monomial ideal $I \subset S$, we denote by $G(I)$ the unique minimal set of monomial generators of $I$.

3. The expansion functor in combinatorial and algebraic concepts

In this section we define the expansion of a simplicial complex and recall the expansion of a simple graph from [10] and the expansion of a monomial ideal from [1]. We show that these concepts are intimately related to each other.

(1) Let $\alpha = (k_1, \ldots, k_n) \in \mathbb{N}^n$. For $F = \{x_{i_1}, \ldots, x_{i_r}\} \subseteq \{x_1, \ldots, x_n\}$ define
$$F^\alpha = \{x_{i_1}, \ldots, x_{i_1k_1}, \ldots, x_{i_r}, x_{i_r}, \ldots, x_{i_rk_r}\}$$
as a subset of $[n]^\alpha := \{x_{11}, \ldots, x_{1k_1}, \ldots, x_{n1}, \ldots, x_{nk_n}\}$. $F^\alpha$ is called the expansion of $F$ with respect to $\alpha$.

For a simplicial complex $\Delta = \langle F_1, \ldots, F_r \rangle$ on $[n]$, we define the expansion of $\Delta$ with respect to $\alpha$ as the simplicial complex
$$\Delta^\alpha = \langle F_1^\alpha, \ldots, F_r^\alpha \rangle.$$

(2) The duplication of a vertex $x_i$ of a simple graph $G$ was first introduced by Schrijver [10] and it means extending its vertex set $V(G)$ by a new vertex $x_i'$ and replacing $E(G)$ by $E(G) \cup \{e \setminus \{x_i\} \cup \{x_i'\} : x_i \in e \in E(G)\}$.

For the $n$-tuple $\alpha = (k_1, \ldots, k_n) \in \mathbb{N}^n$, with positive integer entries, the expansion of the simple graph $G$ is denoted by $G^\alpha$ and it is obtained from $G$ by successively duplicating $k_i - 1$ times every vertex $x_i$.

(3) In [1] Bayati and Herzog defined the expansion functor in the category of finitely generated multigraded $S$-modules and studied some homological behaviors of this functor. We recall the expansion functor defined by them only in the category of monomial ideals and refer the reader to [1] for more general case in the category of finitely generated multigraded $S$-modules.

Let $S^\alpha$ be a polynomial ring over $K$ in the variables
$$x_{11}, \ldots, x_{1k_1}, \ldots, x_{n1}, \ldots, x_{nk_n}.$$
Whenever $I \subset S$ is a monomial ideal minimally generated by $u_1, \ldots, u_r$, the expansion of $I$ with respect to $\alpha$ is defined by
$$I^\alpha = \sum_{i=1}^r P_j^{\nu_j(u_i)} \cdots P_n^{\nu_n(u_i)} \subset S^\alpha$$
where $P_j = (x_{j1}, \ldots, x_{jk_j})$ is a prime ideal of $S^\alpha$ and $\nu_j(u_i)$ is the exponent of $x_j$ in $u_i$.

It was shown in [1] that the expansion functor is exact and so $(S/I)^\alpha = S^\alpha/I^\alpha$. In the following lemmas we describe the relations between the above three concepts of expansion functor.
Lemma 3.1. For a simplicial complex $\Delta$ we have $I^\alpha_\Delta = I_{\Delta^\alpha}$. In particular, $K[\Delta]^\alpha = K[\Delta^\alpha]$.

Proof. Let $\Delta = \langle F_1, \ldots, F_r \rangle$. Since $I_\Delta = \bigcap_{i=1}^r P_{F_i}$, it follows from Lemma 1.1 in [1] that $I^\alpha_\Delta = \bigcap_{i=1}^r P^\alpha_{F_i}$. The result is obtained by the fact that $P^\alpha_{F_i} = P_{(F_i)^\alpha}$.

Let $u = x_{i_1} \ldots x_{i_t} \in S$ be a monomial and $\alpha = (k_1, \ldots, k_n) \in \mathbb{N}^n$. We set $u^\alpha = G((u)^\alpha)$ and for a set $A$ of monomials in $S$, $A^\alpha$ is defined by

$$A^\alpha = \bigcup_{u \in A} u^\alpha.$$

One can easily obtain the following lemma.

Lemma 3.2. Let $I \subset S$ be a monomial ideal and $\alpha \in \mathbb{N}^n$. Then $G(I^\alpha) = G(I)^\alpha$.

Lemma 3.3. For a simple graph $\mathcal{G}$ on the vertex set $[n]$ and $\alpha \in \mathbb{N}^n$ we have $I(\mathcal{G}^\alpha) = I(\mathcal{G})^\alpha$.

Proof. Let $\alpha = (k_1, \ldots, k_n)$ and $P_j = (x_{j_1}, \ldots, x_{j_k})$. Then it follows from Lemma 11(ii,iii) of [1] that

$$I(\mathcal{G}^\alpha) = (x_i x_j : x_i, x_j \in E(\mathcal{G}), 1 \leq r \leq k_i, 1 \leq s \leq k_j) = \sum_{x_i, x_j \in E(\mathcal{G})} P_i P_j$$

$$= \sum_{x_i, x_j \in E(\mathcal{G})} (x_i)^\alpha (x_j)^\alpha = (\sum_{x_i, x_j \in E(\mathcal{G})} (x_i)(x_j))^\alpha = I(\mathcal{G})^\alpha.$$

4. The expansion of a CM$_t$ complex

The following proposition gives us some information about the expansion of a simplicial complex which are useful in the proof of the next results.

Proposition 4.1. Let $\Delta$ be a simplicial complex and let $\alpha \in \mathbb{N}^n$.

(i) For all $i \leq \dim(\Delta)$, there exists an epimorphism $\theta : \tilde{H}_i(\Delta^\alpha; K) \to \tilde{H}_i(\Delta; K)$.

In particular in this case

$$\tilde{H}_i(\Delta^\alpha; K)/\ker(\theta) \cong \tilde{H}_i(\Delta; K);$$

(ii) For $F \in \Delta^\alpha$ such that $F = G^\alpha$ for some $G \in \Delta$, we have

$$\text{link}_{\Delta^\alpha}(F) = (\text{link}_\Delta(G))^\alpha;$$
(iii) For $F \in \Delta^\alpha$ such that $F \neq G^\alpha$ for every $G \in \Delta$, we have
\[
\text{link}_{\Delta^\alpha} F = \langle U^\alpha \setminus F \rangle \ast \text{link}_{\Delta^\alpha} U^\alpha
\]
for some $U \in \Delta$ with $F \subseteq U^\alpha$. Here $\ast$ means the join of two simplicial complexes.

In the third case, $\text{link}_{\Delta^\alpha} F$ is a cone and so acyclic, i.e.,
\[
\widehat{H}_i(\text{link}_{\Delta^\alpha} F; K) = 0 \text{ for all } i > 0.
\]

Proof. (i) Consider the map $\pi : [n]^\alpha \to [n]$ by $\pi(x_{ij}) = x_i$ for all $i, j$. Let the simplicial map $\varphi : \Delta^\alpha \to \Delta$ be defined by $\varphi(\{x_{i_1j_1}, \ldots, x_{i_qj_q}\}) = \{\pi(x_{i_1j_1}), \ldots, \pi(x_{i_qj_q})\}$. Actually, $\varphi$ is an extension of $\pi$ to $\Delta^\alpha$ by linearity.

Define $\varphi_\# : \hat{C}_q(\Delta^\alpha; K) \to \hat{C}_q(\Delta; K)$, for each $q$, by
\[
\varphi_\#([x_{i_0j_0}, \ldots, x_{i_qj_q}]) = \begin{cases} 0 & \text{if for some indices } i_r = i_i \\ \varphi([x_{i_0j_0}], \ldots, \varphi([x_{i_qj_q}]) & \text{otherwise.} \end{cases}
\]

It is clear from the definitions of $\hat{C}_q(\Delta^\alpha; K)$ and $\hat{C}_q(\Delta; K)$ that $\varphi_\#$ is well-defined. Also, define $\varphi_\alpha : \hat{H}_i(\Delta^\alpha; K) \to \hat{H}_i(\Delta; K)$ by
\[
\varphi_\alpha : z + B_i(\Delta^\alpha) \to \varphi_\#(z) + B_i(\Delta).
\]

It is trivial that $\varphi_\alpha$ is onto.

(ii) The inclusion $\text{link}_{\Delta^\alpha}(F) \supseteq (\text{link}_{\Delta}(G))^\alpha$ is trivial. So we show the reverse inclusion. Let $g \in \text{link}_{\Delta^\alpha}(G) = \emptyset$ and $\sigma \cup G^\alpha \in \Delta^\alpha$. We want to show $\pi(\sigma) \in \text{link}_{\Delta}(G)$. Because in this case, $\pi(\sigma)^\alpha \in (\text{link}_{\Delta}(G))^\alpha$ and since $\sigma \subseteq \pi(\sigma)^\alpha$, we conclude that $\sigma \in (\text{link}_{\Delta}(G))^\alpha$.

Clearly, $\pi(\sigma) \cup G^\alpha \in \Delta$. To show that $\pi(\sigma) \cap G = \emptyset$, suppose, on the contrary, that $x_j \in \pi(\sigma) \cap G$. Then $x_{ij} \in \sigma$ for some $j$. Especially, $x_{ij} \in G^\alpha$. Therefore $\sigma \cap G^\alpha \neq \emptyset$, a contradiction.

(iii) Let $\tau \in \text{link}_{\Delta^\alpha}(F)$. Let $\tau \cap \pi(F)^\alpha = \emptyset$. It follows from $\tau \cup F \in \Delta^\alpha$ that $\pi(\tau)^\alpha \cup \pi(F)^\alpha \in \Delta^\alpha$. Now by $\tau \subseteq \pi(\tau)^\alpha$ it follows that $\tau \cup \pi(F)^\alpha \in \Delta^\alpha$. Hence $\tau \in \text{link}_{\Delta^\alpha}(\pi(F)^\alpha)$. So we suppose that $\tau \cap \pi(F)^\alpha = \emptyset$. We write $\tau = (\tau \cap \pi(F)^\alpha) \cup (\tau \cap \pi(F)^\alpha)$. It is clear that $\tau \cap \pi(F)^\alpha \subseteq \pi(F)^\alpha \setminus F$ and $\tau \cap \pi(F)^\alpha \in \text{link}_{\Delta^\alpha}(\pi(F)^\alpha)$. The reverse inclusion is trivial.

\[\square\]

Remark 4.2. Let $\Delta = \langle x_1, x_2, x_3 \rangle$ be a complex on $[3]$ and $\alpha = (2, 1, 1) \in \mathbb{N}^3$. Then $\Delta^\alpha = \langle x_1, x_1x_2, x_1x_2x_3 \rangle$ is a complex on $\{x_1, x_2, x_3\}$. Notice that $\Delta$ is pure but $\Delta^\alpha$ is not. Therefore, the expansion of a pure simplicial complex is not necessarily pure.

Theorem 4.3. Let $\Delta$ be a simplicial complex on $[n]$ of dimension $d - 1$ and let $t \geq 0$ be the least integer that $\Delta$ is $CM_t$. Suppose that $\alpha = (k_1, \ldots, k_n) \in \mathbb{N}^n$ such that $k_i > 1$ for some $i$ and $\Delta^\alpha$ is pure. Then $\Delta^\alpha$ is $CM_{t + e - k + 1}$ but it is not $CM_{t + e - k}$, where $e = \dim(\Delta^\alpha) + 1$ and $k = \min\{k_i : k_i > 1\}$. \[\square\]
Proof. We use induction on $e \geq 2$. If $e = 2$, then $\dim(\Delta^\alpha) = 1$ and $\Delta$ should be only in the form $\Delta = \langle x_1, \ldots, x_n \rangle$. In particular, $\Delta^\alpha$ is of the form

$$\Delta^\alpha = \langle \{x_{i_1,1}, x_{i_1,2}\}, \{x_{i_2,1}, x_{i_2,2}\}, \ldots, \{x_{i_r,1}, x_{i_r,2}\} \rangle.$$ 

It is clear that $\Delta^\alpha$ is $CM_1$ but it is not Cohen-Macaulay.

Assume that $e > 2$. Let $\{x_{ij}\} \in \Delta^\alpha$. We want to show that link$_{\Delta^\alpha}(x_{ij})$ is $CM_{e-k}$. Consider the following cases:

Case 1: $k_1 > 1$. Then

$$\text{link}_{\Delta^\alpha}(x_{ij}) = \langle \{x_i\}^\alpha \setminus x_{ij} \rangle * \text{link}_{\Delta}(x_i)^\alpha.$$ 

(link$_{\Delta}(x_i))^\alpha$ is of dimension $e - k_1 - 1$ and, by induction hypothesis, it is $CM_{i+e-k_1-k+1}$. On the other hand, $\langle \{x_i\}^\alpha \setminus x_{ij}\rangle$ is Cohen-Macaulay of dimension $k_1 - 2$. Therefore, it follows from Theorem 1.1(i) of [4] that link$_{\Delta^\alpha}(x_{ij})$ is $CM_{i+e-k}$.

Case 2: $k_1 = 1$. Then

$$\text{link}_{\Delta^\alpha}(x_{ij}) = \langle \{x_i\}^\alpha \setminus x_{ij} \rangle * \text{link}_{\Delta}(x_i)^\alpha.$$ 

which is of dimension $e - 2$ and, by induction, it is $CM_{i+e-k}$.

Now suppose that $e > 2$ and $k_s = k$ for some $s \in [n]$. Let $F$ be a facet of $\Delta$ such that $x_s$ belongs to $F$.

If $\dim(\Delta) = 0$, then $k_l = k$ for all $l \in [n]$. In particular, $e = k$. It is clear that $\Delta^\alpha$ is not $CM_{i+e-k}$ (or Cohen-Macaulay). So suppose that $\dim(\Delta) > 0$. Choose $x_i \in F \setminus x_s$. Then

$$\text{link}_{\Delta^\alpha}(x_{ij}) = \langle \{x_i\}^\alpha \setminus x_{ij} \rangle * \text{link}_{\Delta}(x_i)^\alpha.$$ 

By induction hypothesis, (link$_{\Delta}(x_i))^\alpha$ is not $CM_{i+e-k_1-k}$. It follows from Theorem 3.1(ii) of [4] that link$_{\Delta^\alpha}(x_{ij})$ is not $CM_{i+e-k_1-k}$. Therefore $\Delta^\alpha$ is not $CM_{i+e-k}$.

\[ \square \]

Corollary 4.4. Let $\Delta$ be a non-empty Cohen-Macaulay simplicial complex on $[n]$. Then for any $\alpha \in \mathbb{N}^n$, with $\alpha \neq 1$, $\Delta^\alpha$ can never be Cohen-Macaulay.

5. The contraction functor

Let $\Delta = \langle F_1, \ldots, F_r \rangle$ be a simplicial complex on $[n]$. Consider the equivalence relation `$\sim$' on the vertices of $\Delta$ given by

$$x_i \sim x_j \Leftrightarrow \langle x_i \rangle * \text{link}_{\Delta}(x_i) = \langle x_j \rangle * \text{link}_{\Delta}(x_j).$$ 

In fact $\langle x_i \rangle * \text{link}_{\Delta}(x_i)$ is the cone over link$_{\Delta}(x_i)$, and the elements of $\langle x_i \rangle * \text{link}_{\Delta}(x_i)$ are those faces of $\Delta$, which contain $x_i$. Hence $\langle x_i \rangle * \text{link}_{\Delta}(x_i) = \langle x_j \rangle * \text{link}_{\Delta}(x_j)$, means the cone with vertex $x_i$ is equal to the cone with vertex $x_j$. In other words, $x_i \sim x_j$ is equivalent to saying that for a facet $F \subseteq \Delta$, $F$ contains $x_i$ if and only if it contains $x_j$.

Let $[\tilde{m}] = \{\tilde{y}_1, \ldots, \tilde{y}_m\}$ be the set of equivalence classes under `$\sim$'. Let $\tilde{y}_l = \{x_{i_1}, \ldots, x_{i_a}\}$. Set $\alpha = (a_1, \ldots, a_m)$. For $F_l \in \Delta$, define $G_l = \{\tilde{y}_l : \tilde{y}_l \subseteq F_l\}$
and let $\Gamma$ be a simplicial complex on the vertex set $[m]$ with facets $G_1, \ldots, G_r$. We call $\Gamma$ the contraction of $\Delta$ by $\alpha$ and $\alpha$ is called the vector obtained from contraction.

For example, consider the simplicial complex $\Delta = \{x_1x_2x_3, x_2x_3x_4, x_1x_4x_5, x_2x_3x_5\}$ on the vertex set $[5] = \{x_1, \ldots, x_5\}$. Then $\bar{y}_1 = \{x_1\}$, $\bar{y}_2 = \{x_2, x_3\}$, $\bar{y}_3 = \{x_4\}$, $\bar{y}_4 = \{x_3\}$ and $\alpha = (1, 2, 1, 1)$. Therefore, the contraction of $\Delta$ by $\alpha$ is $\Gamma = \langle \bar{y}_1\bar{y}_2, \bar{y}_2\bar{y}_3, \bar{y}_1\bar{y}_3\bar{y}_4, \bar{y}_2\bar{y}_4\rangle$ a complex on the vertex set $[4] = \{\bar{y}_1, \ldots, \bar{y}_4\}$.

**Remark 5.1.** Note that if $\Delta$ is a pure simplicial complex then the contraction of $\Delta$ is not necessarily pure (see the above example). In the special case where the vector $\alpha = (k_1, \ldots, k_n) \in \mathbb{N}^n$ and $k_i = k_j$ for all $i, j$, it is easy to check that in this case $\Delta$ is pure if and only if $\Delta^\alpha$ is pure. Another case is introduced in the following proposition.

**Proposition 5.2.** Let $\Delta$ be a simplicial complex on $[n]$ and assume that $\alpha = (k_1, \ldots, k_n) \in \mathbb{N}^n$ satisfies the following condition:

1. for all facets $F, G \in \Delta$, if $x_i \in F \setminus G$ and $x_j \in G \setminus F$ then $k_i = k_j$.

Then $\Delta$ is pure if and only if $\Delta^\alpha$ is pure.

**Proof.** Let $\Delta$ be a pure simplicial complex and let $F, G \in \Delta$ be two facets of $\Delta$. Then

$|F^\alpha| - |G^\alpha| = \sum_{x_i \in F} k_i - \sum_{x_i \in G} k_i = \sum_{x_i \in F \setminus G} k_i - \sum_{x_i \in G \setminus F} k_i.$

Now the condition (1) implies that $|F^\alpha| = |G^\alpha|$. This means that all facets of $\Delta^\alpha$ have the same cardinality.

Let $\Delta^\alpha$ be pure. Suppose that $F, G$ are two facets in $\Delta$. If $|F| > |G|$ then $|F \setminus G| > |G \setminus F|$. Therefore $\sum_{x_i \in F \setminus G} k_i > \sum_{x_i \in G \setminus F} k_i$. This implies that $|F^\alpha| = \sum_{x_i \in F} k_i > \sum_{x_i \in G} k_i = |G^\alpha|$, a contradiction. $\square$

There is a close relationship between a simplicial complex and its contraction. In fact, the expansion of the contraction of a simplicial complex is the same complex. The precise statement is the following.

**Lemma 5.3.** Let $\Delta$ be the contraction of $\Delta$ by $\alpha$. Then $\Gamma^\alpha \cong \Delta$.

**Proof.** Suppose that $\Delta$ and $\Gamma$ are on the vertex sets $[n] = \{x_1, \ldots, x_n\}$ and $[\bar{m}] = \{\bar{y}_1, \ldots, \bar{y}_m\}$, respectively. Let $\alpha = (a_1, \ldots, a_m)$. For $\bar{y}_i \in \Gamma$, suppose that $\bar{y}_i^\alpha = \{\bar{y}_{i1}, \ldots, \bar{y}_{ia_i}\}$. So $\Gamma^\alpha$ is a simplicial complex on the vertex set $[\bar{m}]^\alpha = \{\bar{y}_{ij} : i = 1, \ldots, m, j = 1, \ldots, a_i\}$. Now define $\varphi : [\bar{m}]^\alpha \to [n]$ by $\varphi(\bar{y}_{ij}) = x_{ij}$. Extending $\varphi$, we obtain the isomorphism $\varphi : \Gamma^\alpha \to \Delta$. $\square$

**Proposition 5.4.** Let $\Delta$ be a simplicial complex and assume that $\Delta^\alpha$ is Cohen-Macaulay for some $\alpha \in \mathbb{N}^n$. Then $\Delta$ is Cohen-Macaulay.
Proof. By Lemma 4.1(i), for all \( i \leq \dim(\text{link}_\Delta F) \) and all \( F \in \Delta \) there exists an epimorphism \( \theta : \text{link}_\Delta F^\alpha \to \text{link}_\Delta F \) such that
\[
\tilde{H}_i(\text{link}_\Delta F^\alpha; K)/\ker(\theta) \cong \tilde{H}_i(\text{link}_\Delta F; K).
\]
Now suppose that \( i < \dim(\text{link}_\Delta F) \). Then \( i < \dim(\text{link}_G G) \) and by Cohen-Macaulayness of \( \Delta^\alpha \), \( \tilde{H}_i(\text{link}_\Delta F^\alpha; K) = 0 \). Therefore \( \tilde{H}_i(\text{link}_\Delta F; K) = 0 \). This means that \( \Delta \) is Cohen-Macaulay. \( \square \)

It follows from Proposition 5.4 that:

**Corollary 5.5.** The contraction of a Cohen-Macaulay simplicial complex \( \Delta \) is Cohen-Macaulay.

This can be generalized in the following theorem.

**Theorem 5.6.** Let \( \Gamma \) be the contraction of a CM\(_t\) simplicial complex \( \Delta \), for some \( t \geq 0 \), by \( \alpha = (k_1, \ldots, k_n) \). If \( k_i \geq t \) for all \( i \) and \( \Gamma \) is pure, then \( \Gamma \) is Buchsbaum.

Proof. If \( t = 0 \), then we saw in Corollary 5.5 that \( \Gamma \) is Cohen-Macaulay and so it is CM\(_t\). Hence assume that \( t > 0 \). Let \( \Delta = (F_1, \ldots, F_r) \). We have to show that \( \tilde{H}_i(\text{link}_\Gamma G; K) = 0 \), for all faces \( G \in \Gamma \) with \( |G| \geq 1 \) and all \( i < \dim(\text{link}_\Gamma G) \).

Let \( G \in \Gamma \) with \( |G| \geq 1 \). Then \( |G^\alpha| \geq t \). It follows from Lemma 2.1 and CM\(_t\)-ness of \( \Delta \) that
\[
\tilde{H}_i(\text{link}_\Gamma G; K) \cong \tilde{H}_i(\text{link}_\Delta G^\alpha; K) = 0
\]
for \( i < \dim(\text{link}_\Delta G^\alpha) \) and, particularly, for \( i < \dim(\text{link}_\Gamma G) \). Therefore \( \Gamma \) is Buchsbaum. \( \square \)

**Corollary 5.7.** Let \( \Gamma \) be the contraction of a Buchsbaum simplicial complex \( \Delta \). If \( \Gamma \) is pure, then \( \Gamma \) is also Buchsbaum.

Let \( \mathcal{G} \) be a simple graph on the vertex set \([n]\) and let \( \Delta_\mathcal{G} \) be its independence complex on \([n]\), i.e., a simplicial complex whose faces are the independent vertex sets of \( \mathcal{G} \). Let \( \Gamma \) be the contraction of \( \Delta_\mathcal{G} \). In the following we show that \( \Gamma \) is the independence complex of a simple graph \( \mathcal{H} \). We call \( \mathcal{H} \) the contraction of \( \mathcal{G} \).

**Lemma 5.8.** Let \( \mathcal{G} \) be a simple graph. The contraction of \( \Delta_\mathcal{G} \) is the independence complex of a simple graph \( \mathcal{H} \).

Proof. It suffices to show that \( I_\Gamma \) is a squarefree monomial ideal generated in degree 2. Let \( \Gamma \) be the contraction of \( \Delta_\mathcal{G} \) and let \( \alpha = (k_1, \ldots, k_n) \) be the vector obtained from the contraction. Let \( [n] = \{x_1, \ldots, x_n\} \) be the vertex set of \( \Gamma \). Suppose that \( u = x_{i_1} \ldots x_{i_t} \in G(I_\Gamma) \). Then \( u^\alpha \subseteq G(I_\Gamma)^\alpha = G(I_{\Delta_\mathcal{G}}) = G(I(\mathcal{G})) \).

Since \( u^\alpha = \{x_{i_1j_1} \ldots x_{i_lj_l} : 1 \leq j_1 \leq k_{i_1}, 1 \leq l \leq t \} \) we have \( t = 2 \) and the proof is completed. \( \square \)
Example 5.9. Let $G_1$ and $G_2$ be, respectively, from left to right the following graphs:

The contraction of $G_1$ and $G_2$ are

The contraction of $G_1$ is equal to itself but $G_2$ is contracted to an edge and the vector obtained from contraction is $\alpha = (2, 3)$.

We recall that a simple graph is CM$_t$ for some $t \geq 0$, if the associated independence complex is CM$_t$.

Remark 5.10. The simple graph $G'$ obtained from $G$ in Lemma 4.3 and Theorem 4.4 of [4] is the expansion of $G$. Actually, suppose that $G$ is a bipartite graph on the vertex set $V(G) = V \cup W$ where $V = \{x_1, \ldots, x_d\}$ and $W = \{x_{d+1}, \ldots, x_{2d}\}$. Then for $\alpha = (n_1, \ldots, n_d, n_1, \ldots, n_d)$ we have $G' = G^\alpha$. It follows from Theorem 4.3 that if $G$ is CM$_t$ for some $t \geq 0$ then $G'$ is CM$_{t+n-n_{i_0}+1}$ where $n = \sum_{i=1}^{d} n_i$ and $n_{i_0} = \min\{n_i > 1 : i = 1, \ldots, d\}$. This implies that the first part of Theorem 4.4 of [4] is an immediate consequence of Theorem 4.3 for $t = 0$.

Acknowledgment

The author would like to thank Hassan Haghighi from K. N. Toosi University of Technology and Rahim Zaare-Nahandi from University of Tehran for careful reading an earlier version of this article and for their helpful comments. The research was in part supported by a grant from IPM (No. 93130029).

References


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