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RADICAL OF \(-\)-IDEALS IN \(PMV\)-ALGEBRAS

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Abstract. In this paper, we introduce the notion of the radical of a \(PMV\)-algebra \(A\) and we characterize radical \(A\) via elements of \(A\). Also, we introduce the notion of the radical of a \(-\)-ideal in \(PMV\)-algebras. Several characterizations of this radical is given. We define the notion of a semimaximal \(-\)-ideal in a \(PMV\)-algebra. Finally we show that \(A/I\) has no nilpotent elements if and only if \(I\) is a semi-maximal \(-\)-ideal of \(A\).

Keywords: \(PMV\)-algebra, \(-\)-ideal, \(-\)-prime ideal, radical.


1. Introduction

C. Chang introduced the notion of \(MV\)-algebras to provide a proof for the completeness of the Lukasiewicz axioms for infinite valued propositional logic [2]. In fact \(MV\)-algebras are now algebraic counterparts of Lukasiewicz many valued logics.

A. Dvurecenskij and A. Di Nola in [4] introduced the notion of product \(MV\)-algebras, i.e., \(MV\)-algebras with product which is defined on the whole \(MV\)-algebra and is associative and left/right distributive with respect to a partial addition. They concluded that the category of product \(MV\)-algebras is categorically equivalent to the category of associative unital \(l\)-rings. Some examples are presented and compared with \(MV\)-algebras. In addition, they introduced and studied \(MVf\)-algebras [4].

In [9], we introduced the notion of the radical of an ideal in a \(MV\)-algebra and gave several characterizations of this radical. We defined the notion of a semi-maximal ideal in an \(MV\)-algebra and proved some theorems which give relations between this semi-maximal ideal and other types of ideals in \(MV\)-algebras [9].

In this paper, we introduce the notion of the radical of a \(PMV\)-algebra \(A\) and give several characterizations of radical \(A\). We introduce the notion of the
radical of \( -\)ideal of \( PMV\)-algebras. We have also presented several different characterizations and many important properties of the radical of a \( -\)ideal in a \( PMV\)-algebra. This leads us to introduce the notion of semi-maximal \( -\)ideal. Finally, we show that \( I \) is a semi-maximal \( -\)ideal of \( A \) if and only if \( A/I \) has no nilpotent elements of \( A \).

2. Preliminaries

In this section, we recall some basic notions in \( MV\)-algebras and summarize some of their basic properties. For more details about these concepts, we refer the reader to [2–4].

**Definition 2.1.** [2] An \( MV\)-algebra is a structure \((A, \oplus, *, 0)\), where \( \oplus \) is a binary operation, \( * \) is a unary operation, and \( 0 \) is a constant satisfying the following conditions, for any \( a, b \in A \):

\begin{align*}
(MV1) & \ (A, \oplus, 0) \text{ is an abelian monoid,} \\
(MV2) & \ (a^*) = a, \\
(MV3) & \ 0^* \oplus a = 0^*, \\
(MV4) & \ (a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a.
\end{align*}

We say that the element \( x \in A \) has order \( n \), and we write \( \text{ord}(x) = n \), if \( n \) is the smallest natural number such that \( nx = 1 \), where \( 1 = 0^* \) and \( nx := x \oplus x \oplus \ldots \oplus x \). In this case we say that the element \( x \) has a finite order, and write \( \text{ord}(x) < \infty \). An \( MV\)-algebra \( A \) is locally finite if every non-zero element of \( A \) is of finite order. Also we have \( a^n = a^{n-1} \oplus a \) and \( na = (n-1)a \oplus a \), where \( a \oplus b = (a^* \oplus b^*)^* \) [3].

If we define the auxiliary operations \( \odot, \lor \) and \( \land \) on \( A \) as:

\[ a \odot b = (a^* \oplus b^*)^*, \quad a \lor b = a \oplus (b \odot a^*) = b \oplus (b^* \odot a), \]

\[ a \land b = a \odot (b \oplus a^*) = b \odot (b^* \oplus a), \]

then \((A, \odot, 1)\) is an abelian monoid and the structure \( L(A) := (A, \lor, \land, 0, 1) \) is a bounded distributive lattice [3].

An element \( a \in A \) is called complemented if there is an element \( b \in A \) such that \( a \lor b = 1 \) and \( a \land b = 0 \). We denote the set of complemented elements of \( A \) by \( B(A) \).

**Lemma 2.2.** [3] In each \( MV\)-algebra \( A \), the following relations hold for all \( x, y, z \in A \):

1. \( x \leq y \) if and only if \( y^* \leq x^* \),
2. If \( x \leq y \), then \( x \oplus z \leq y \oplus z \) and \( x \odot z \leq y \odot z \),
3. \( x \leq y \) if and only if \( x^* \oplus y = 1 \) if and only if \( x \odot y^* = 0 \),
4. \( x, y \leq x \oplus y \) and \( x \odot y \leq x, y \),
5. \( x \oplus x^* = 1 \) and \( x \odot x^* = 0 \).
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(6) If \(x \in B(A)\), then \(x \odot x = x\) and \(x \oplus x = x\).
(7) \(x \odot (y \land z) = (x \odot y) \land (x \odot z)\).

An ideal in an \(MV\)-algebra is defined as:

**Definition 2.3.** [2] An ideal of an \(MV\)-algebra \(A\) is a nonempty subset \(I\) of \(A\), satisfying the following conditions:

(I1) If \(x \in I\), \(y \in A\) and \(y \leq x\), then \(y \in I\),

(I2) If \(x, y \in I\), then \(x \oplus y \in I\).

We denote the set of all ideals of an \(MV\)-algebra \(A\) by \(Id(A)\).

**Definition 2.4.** [3] Let \(I\) be an ideal of an \(MV\)-algebra \(A\). Then \(I\) is proper if \(I \neq A\). A proper ideal \(P\) is prime if for \(x, y \in A\), \(x \land y \in P\) implies \(x \in P\) or \(y \in P\). Equivalently, \(P\) is prime if and only if for all \(x, y \in A\), \(x \odot y^* \in P\) or \(y \odot x^* \in P\).

**Theorem 2.5.** [3, 14] Let \(I\) be a proper ideal of \(A\). Then the following statements hold:

1. Any prime ideal of \(A\) is contained in a unique maximal ideal of \(A\),
2. If \(a \in A - I\), then there is a prime ideal \(P\) of \(A\) such that \(I \subseteq P\) and \(a \not\in P\). In particular for every element \(a \in A\), \(a \neq 0\), there exists a prime ideal \(P\) such that \(a \not\in P\).

**Definition 2.6.** [9] Let \(I\) be a proper ideal of \(A\). The intersection of all maximal ideals of \(A\) which contain \(I\) is called the radical of \(I\) and it is denoted by \(\text{Rad}(I)\).

**Theorem 2.7.** [9] Let \(I\) be a proper ideal of \(A\). Then

\[
\text{Rad}(I) = \{a \in A : na \odot a \in I, \text{ for all } n \in \mathbb{N}\}.
\]

We will denote by \(\mathcal{MV}\) the category whose objects are \(MV\)-algebras and whose morphisms are \(MV\)-algebra homomorphisms. A crucial result in the theory of \(MV\)-algebras is the categorical equivalence between the category of \(MV\)-algebras and the category of Abelian \(l\)-groups with strong unit [13]. We recall that an \(lu\)-group is an algebra \((G, +, -0, \lor, \land, u)\), where the following properties hold:

(a) \((G, +, -, 0)\) is a group,
(b) \((G, \lor, \land)\) is a lattice,
(c) For any \(x, y, a, b \in G\), \(x \leq y\) implies \(a + x + b \leq a + y + b\),
(d) \(u > 0\) is strong unit for \(G\) (that is, for all \(x \in G\) there is some natural number \(n \geq 1\) such that \(-nu \leq x \leq nu\) [1]).

We refer to [1] for a detailed study of \(l\)-groups theory. Given an Abelian \(l\)-group \((G, +, 0, \leq)\) and a positive element \(u > 0\) in \(G\), the interval \([0, u]\) can be endowed with an \(MV\)-algebra structure as follows:

\[
x \oplus y := (x + y) \land u \quad \text{and} \quad x^* := u - x,
\]
for any \( x, y \in [0, u] \). Moreover, the lattice operations on \([0, u]\) are the restriction of the lattice operations on \( G \). The \( MV \)-algebra \(([0, u], \oplus, *, 0, u)\) will be denoted by \([0, u]_G\). If \( G \) is an \( l \)-group then a strong unit is a positive element \( u > 0 \) from \( G \) with the property that for any \( g \in G \) there is integer number \( n \geq 0 \) such that \( g \leq nu \). In the sequel, the Abelian \( l \)-groups with strong unit will be simply called \( lu \)-groups. We shall denote by \( UG \) the category of \( lu \)-groups. The elements of this category are pairs \((G, u)\) where \( G \) is an Abelian \( l \)-group and \( u \) is a strong unit of \( G \). In the sequel, the Abelian \( l \)-groups with strong unit will be simply called \( lu \)-groups. We shall denote by \( UG \) the category of \( lu \)-groups. The elements of this category are pairs \((G, u)\) where \( G \) is an Abelian \( l \)-group and \( u \) is a strong unit of \( G \). The morphisms will be \( l \)-group homomorphisms which preserve the strong unit. The functor that establishes the categorical equivalence between \( MV \) and \( UG \) is

\[
\Gamma : UG \rightarrow MV.
\]

such that \( \Gamma(G, u) := [0, u]_G \) for any \( lu \)-group \((G, u)\), \( \Gamma(h) := h \mid_{[0,u]} \) for any \( lu \)-groups homomorphism \( h \).

The categorical equivalence between \( MV \)-algebras and \( lu \)-groups leads also to the problem of defining a product operation on \( MV \)-algebras, in order to obtain structures corresponding to \( l \)-rings. We recall that an \( l \)-ring \([5]\) is a structure \((R, +, \cdot, 0, \leq)\), where \((R, +, 0, \leq)\) is an \( l \)-group such that, for any \( x, y \in R \)

\[
x \geq 0 \text{ and } y \geq 0 \implies x \cdot y \geq 0.
\]

**Definition 2.8.** \([4]\) A product \( MV \)-algebra (or \( PMV \)-algebra, for short) is a structure \((A, \oplus, *, 0)\), where \((A, \oplus, *, 0)\) is an \( MV \)-algebra and \( * \) is a binary associative operation on \( A \) such that the following property is satisfied:

if \( x + y \) is defined, then \( x \cdot z + y \cdot z \) and \( z \cdot x + z \cdot y \) are defined and

\[
(x + y) \cdot z = x \cdot z + y \cdot z, \quad z \cdot (x + y) = z \cdot x + z \cdot y,
\]

where + is a partial addition on \( A \), as follows:

for any \( x, y \in A \), \( x + y \) is defined if and only if \( x \leq y^* \)

and in this case, \( x + y := x \oplus y \).

If \( A \) is a \( PMV \)-algebra, then a unity for the product is an element \( e \in A \) such that \( e \cdot x = x \cdot e = x \) for any \( x \in A \). A \( PMV \)-algebra that has unity for the product is called unital.

A \( - \)ideal of a \( PMV \)-algebra \( A \) is an ideal \( I \) of \( MV \)-algebra \( A \) such that \( a \in I \) and \( b \in A \) entail \( a \cdot b \in I \) and \( b \cdot a \in I \). We denote by \( Id_\cdot(A) \) the set of \( - \)ideals of a \( PMV \)-algebra \( A \).

We will refer to \([4,12]\) for the basic properties of \( PMV \)-algebras. Obviously, a \( PMV \)-algebra homomorphism will be an \( MV \)-algebra homomorphism which also commutes with the product operation. We shall denote by \( PPMV \) the category of product \( MV \)-algebras with the corresponding homomorphisms.

In the sequel, an \( lu \)-ring will be a pair \((R, u)\) where \((R, +, \cdot, \leq)\) is an \( l \)-ring and \( u \) is a strong unit of \( R \) such that \( u \cdot u \leq u \). We imply that the interval
[0, u] of an lu-ring \((R, u)\) is closed under the product of \(R\). Thus, if we consider the restriction of \(\cdot\) to \([0, u] \times [0, u]\), then the interval \([0, u]\) has a canonical PMV-algebra structure:
\[
x \oplus y := (x + y) \wedge u, \quad x^* := u - x, \quad x \cdot y := x \cdot y,
\]
for any \(0 \leq x, y \leq u\). We shall denote this structure by \([0, u]_R\).

If \(\mathcal{UR}\) is the category of lu-rings, whose objects are pairs \((R, u)\) as above and whose morphisms are l-rings homomorphisms which preserve the strong unit, then we get a functor
\[
\Gamma : \mathcal{UR} \to \mathcal{PMV},
\]
\[
\Gamma(R, u) := [0, u]_R, \text{ for any lu-ring } (R, u),
\]
\[
\Gamma(h) := h |_{[0,u]} \text{ for any lu-rings homomorphism } h.
\]
In [4] it is proved that \(\Gamma\) establishes a categorical equivalence between \(\mathcal{UR}\) and \(\mathcal{PMV}\).

**Definition 2.9.** [8] Let \(P\) be a \(\cdot\)-ideal of \(A\). \(P\) is called a \(\cdot\)-prime if (i) \(P \neq A\), (ii) for every \(a, b \in P\), if \(a \cdot b \in P\), then \(a \in P\) or \(b \in P\).

**Definition 2.10.** [3] An element \(a\) in MV-algebra \(A\) is said to be infinitesimal if and only if \(a \neq 0\) and \(na \leq a^*\) for each integer \(n \geq 0\). The set of all infinitesimals in \(A\) will be denoted by \(\text{Inf}(A)\).

**Lemma 2.11.** [4] If \(A\) is a PMV-algebra, then for any \(a, b \in A\),
(i) \(a \cdot 0 = 0 = 0 \cdot a\),
(ii) if \(a \leq b\), then for any \(c \in A\), \(a \cdot c \leq b \cdot c\) and \(c \cdot a \leq c \cdot b\).

We recall that in an MV-algebra \(A\), the Chang distance the function is defined by
\[
d : A \times A \to A, \quad d(a, b) := (a \otimes b^*) \oplus (b \otimes a^*) [2].
\]
In the following lemma, we state and prove some properties of PMV-algebras.

**Lemma 2.12.** [10] If \(A\) is a PMV-algebra, then the following properties hold for any \(x, y, \alpha \in A\),
(a) \((nx) \cdot y = x \cdot (ny)\), for any \(n \in \mathbb{N}\),
(b) \(x \cdot y^* \leq (x \cdot y)^*\),
(c) \((x \cdot y)^* = x^* \cdot y + (1 \cdot y)^*\),
(d) \((\alpha \cdot x) \odot (\alpha \cdot y)^* \leq \alpha \cdot (x \odot y)^*\),
(e) \(\alpha \cdot (x \oplus y) \leq \alpha \cdot x \oplus \alpha \cdot y\),
(f) \(d(\alpha \cdot x, \alpha \cdot y) \leq \alpha \cdot d(x, y)\).

Moreover, if \(A\) is a unital PMV-algebra, then
\((x \cdot y)^* = x^* \cdot y + y^*\).

**Lemma 2.13.** [4] If \(A\) is a unital PMV-algebra, then:
(a) The unity for the product is \(e = 1\),
(b) \(x \cdot y \leq x \land y\) for any \(x, y \in A\).
Theorem 2.14. [4] A finite MV-algebra $A$ admits a product $\cdot$ such that $a \cdot 1 = a = 1 \cdot a$ for any $a \in A$ if and only if $A$ is a Boolean algebra, i.e., $a \oplus a = a$ for any $a \in A$. If it is the case, then $a \cdot b = a \wedge b \in A$.

Definition 2.15. [10] A nonempty subset of a PMV-algebra $S \subseteq A$ is called \textit{\~{n}}-closed system in $A$ if $1 \in S$ and $x, y \in S$ implies $x \cdot y \in S$.

We denote by $S(A)$ the set of all \textit{\~{n}}-closed systems of $A$.

Remark 2.16. [10] Let $A$ be a PMV-algebra. Then $I(a) = \{ x \in A : x \leq y \oplus ma \oplus n(\alpha \cdot a), \text{ for some } y \in I, \text{ integers } n, m \geq 0, \alpha \in A \}$.

Proposition 2.17. [10] Let $A$ be a PMV-algebra.

(i) If $N \subseteq A$ is a nonempty set, then we have $(N) = \{ x \in A : x \leq x_1 \oplus \cdots \oplus x_n \oplus \alpha_1 \cdot y_1 \oplus \cdots \oplus \alpha_m \cdot y_m \text{ for some } x_1, \ldots, x_n, y_1, \ldots, y_m \in N, \alpha_1, \ldots, \alpha_m \in A \}$, where by $(N)$, we mean the ideal generated by $N$.

In particular, for $a \in A$,

$$(a) = \{ x \in A : x \leq na \oplus m(\alpha \cdot a) \text{ for some integer } n, m \geq 0, \alpha \in A \},$$

(ii) If $I_1, I_2 \in Idp(A)$, then $I_1 \vee I_2 = (I_1 \cup I_2) = \{ a \in A : a \leq x_1 \oplus x_2 \text{ for some } x_1 \in I_1 \text{ and } x_2 \in I_2 \}$.

3. Radical of \textit{\~{n}}-ideals in PMV-algebras

From now on $(A, \oplus, *, 0)$ (or simply $A$) is a PMV-algebra.

Definition 3.1. The intersection of all maximal \textit{\~{n}}-ideals of $A$ is called the radical of $A$ and it is denoted by $Rad(A)$.

Lemma 3.2. If $I$ is a proper \textit{\~{n}}-ideal of $A$, then the following are equivalent:

(i) $I$ is a maximal \textit{\~{n}}-ideal of $A$,

(ii) for any $a \in A$, $a \not\in I$ if and only if $(na \oplus m(\alpha \cdot a))^* \in I$, for some integers $n, m > 0$ and $a \in A$.

Proof. (i) \Rightarrow (ii) Suppose that $I$ is a maximal \textit{\~{n}}-ideal of $A$. Since $a \not\in I$, $I \vee (a) = A$. So by Proposition 2.17, there exist $x \in I$ and $n, m > 0$ and $a \in A$ such that $[na \oplus (m(\alpha \cdot a))] \oplus x = 1$. We deduce that $(na \oplus m(\alpha \cdot a))^* \leq x \in I$. This results $(na \oplus m(\alpha \cdot a))^* \in I$, for some $n, m \in \mathbb{N}$ and $a \in A$.

Conversely, if $a \in I$, then $na \in I$. Also, by \textit{\~{n}}-ideal property and Lemma 2.12(a), we have $\alpha(ma) \in I$ and $m(\alpha \cdot a) = (ma) \cdot a = \alpha \cdot (ma) \in I$. So $(na \oplus m(\alpha-a)) \in I$. Since $I$ is a proper \textit{\~{n}}-ideal, we conclude that $(na \oplus m(\alpha-a))^* \notin I$.

(ii) \Rightarrow (i) Suppose there exists a \textit{\~{n}}-ideal $J$ such that $I \not\subseteq J$. So there exists an $a \in J - I$. Hence $a \not\in I$ and by hypothesis, we conclude that $(na \oplus m(\alpha-a))^* \in I$, for some $n, m \in \mathbb{N}$ and $a \in A$. Hence $(na \oplus m(\alpha-a))^* \in J$. Since $a \in J$, we obtain $(na \oplus m(\alpha-a)) \in J$. Thus $1 = (na \oplus m(\alpha-a))^* \oplus (na \oplus m(\alpha-a)) \in J$,
Now, if \( x \in A \), then \( x \in J \) because \( x \leq 1 \in J \). It follows that \( A \subseteq J \). Thus \( A = J \).

By the following theorem, we characterize \( \text{Rad}(A) \) via elements of \( A \).

**Theorem 3.3.** Let \( A \) be a PMV-algebra. Then

\[
\text{Rad}(A) = \{ x \in A : nx \oplus m(\alpha \cdot x) \leq x^* , \text{ for any } n, m \in \mathbb{N} \text{ and } \alpha \in A \}
\]

\( \cup \text{Inf}(A) \cup \{ 0 \} \).

**Proof.** Suppose that \( kx \oplus t(\alpha \cdot x) \leq x^* \), for any \( k, t \in \mathbb{N} \) and \( \alpha \in A \) and \( 0 < x \notin \text{Inf}(A) \). Let \( x \notin \text{Rad}(A) \). Then there exists a maximal \( -\)ideal \( I \) of \( A \) such that \( x \notin I \). We see that \( [nx \oplus m(\alpha \cdot x)] \circ x = 0 \in I \). Since \( x \notin I \), it follows from Lemma 3.2, that \( (nx \oplus m(\alpha \cdot x))^* \in I \), for some \( n, m \in \mathbb{N} \) and \( \alpha \in A \). Hence \( (nx \oplus m(\alpha \cdot x))^* \oplus ((nx \oplus m(\alpha \cdot a))^* \circ x) \in I \). So \( x \leq (nx \oplus m(\alpha \cdot x))^* \circ x \in I \).

Then \( x \in I \), which is a contradiction. Thus \( x \in \text{Rad}(A) \).

Conversely, let \( x \in \text{Rad}(A) \) and suppose that there exist \( k, t \in \mathbb{N} \) and \( \beta \in A \) such that \( kx \oplus t(\beta \cdot x) \notin x^* \) and there exists \( m \in \mathbb{N} \) such that \( mx \leq x^* \) and \( x > 0 \). Hence \( x \in I \), for any maximal \( -\)ideal \( I \) and \( 0 \neq [kx \oplus t(\beta \cdot x)] \circ x \).

Let \( x \geq 0 \) and \( 0 \neq mx \circ x \), for some \( m \in \mathbb{N} \), this results \( nx \circ x \leq x \in I \), for all \( n \in \mathbb{N} \). It follows from Theorem 2.7 that \( x \in \text{Rad}(I) \) in \( \text{MV-algebra} \). Also by Theorem 2.5 (2), since \( mx \circ x \neq 0 \), there exists a prime ideal \( P \) of \( A \) such that \( mx \circ (x^*)^* = mx \circ x \notin P \). Since \( P \) is a prime ideal of \( \text{MV-algebra} \), \( x^* \circ (mx)^* \in P \). Hence by Theorem 2.5 (1), there exists a unique maximal ideal \( J \) of \( A \) such that \( P \subseteq J \). Therefore \( (mx)^* \circ x^* \in J \). If \( x \in J \), then \( mx \in J \), also we have \( (mx)^* \leq x \circ (mx)^* = x \circ (x^* \circ (mx)^*) \in J \). Thus \( mx \circ (mx)^* = 1 \in J \), which is a contradiction. Therefore \( x \notin J \). We conclude that \( I \subseteq P \subseteq J \) and \( x \notin J \). Hence \( x \notin \text{Rad}(I) \), which is a contradiction.

Therefore \( nx \oplus m(\alpha \cdot x) \leq x^* \), for all \( n, m \in \mathbb{N} \) and \( \alpha \in A \) or \( 0 < x \notin \text{Inf}(A) \).

**Lemma 3.4.** If \( S \) is \( -\)closed system in \( A \) and \( I \) is a \( -\)ideal of \( A \) such that \( S \cap I = \emptyset \), then there exists a \( -\)prime \( P \) of \( A \) such that \( I \subseteq P \) and \( P \cap S = \emptyset \).

**Proof.** Let \( T = \{ J \in \text{Id}(A) : I \subseteq J, J \cap S = \emptyset \} \). A routine application of Zorn’s lemma shows that \( T \) has a maximal element \( P \). Suppose by contrary that \( P \) is not a \( -\)prime of \( A \). That is, there exist \( a, b \in A \) such that \( a \cdot b \in P \) but \( a \notin P \) and \( b \notin P \).

By the maximality of \( P \), we deduce that \( P(a), P(b) \notin T \), hence \( P(a) \cap S \neq \emptyset \) and \( P(b) \cap S \neq \emptyset \), that is, there exist \( p_1 \in P(a) \cap S \) and \( p_2 \in P(b) \cap S \). By Remark 2.16, \( p_1 \leq y \oplus ma \oplus n(\alpha \cdot a) \) and \( p_2 \leq x \oplus kb \oplus t(\beta \cdot b) \), where \( x, y \in P \) and \( m, n, k, t \in \mathbb{N} \).

Then by Lemma 2.12 (e), we have \( p_1 \cdot p_2 \leq x \cdot y \oplus x \cdot ma \oplus x \cdot n(\alpha \cdot a) \oplus kb \cdot y \oplus kb \cdot ma \oplus kb \cdot n(\alpha \cdot a) \oplus (t(\beta \cdot b) \cdot y \oplus t(\beta \cdot b) \cdot ma \oplus t(\beta \cdot b) \cdot n(\alpha \cdot a)) \).

Since \( x, y \in P \) and \( a \cdot b \in P \), we imply that \( p_1 \cdot p_2 \in P \) but \( p_1 \cdot p_2 \in S \), hence \( P \cap S \neq \emptyset \), which is a contradiction. Hence \( P \) is a \( -\)prime of \( A \).
Definition 3.5. Let $I$ be a proper $\sim$-ideal of $A$. The intersection of all $\sim$-prime ideals of $A$ which contain $I$ is called the radical of $I$ and it is denoted by $\text{Rad}(I)$. If there are not $\sim$-prime ideals of $A$ containing $I$, then $\text{Rad}(I) = A$.

Example 3.6. Let $\Omega = \{1, 2\}$ and $A = \mathcal{P}(\Omega)$, which is a PMV-algebra with $\oplus = \cup$ and $\odot = \cdot = \cap$. Obviously, $P_1 = \{\emptyset, \{1\}\}$ and $P_2 = \{\emptyset, \{2\}\}$ are $\sim$-prime ideals of $A$. Hence $\text{Rad}(P_1) = P_1$ and $\text{Rad}(\emptyset, \{2\}) = P_2$ and $\text{Rad}(\emptyset) = \{\emptyset, \{1\}\} \cap \{\emptyset, \{2\}\} = \emptyset$.

Example 3.7. Let $M_2(\mathbb{R})$ be the ring of square matrices of order 2 with real elements and 0 be the matrix with all of its entries 0. If we define the order relation on components $A = (a_{ij})_{i,j=1,2} \geq 0$ if $a_{ij} \geq 0$ for all $i,j = 1,2$ such that $v = \left( \begin{array}{cc} 1/2 & 1/2 \\ 1/2 & 1/2 \end{array} \right)$, then $A = \Gamma(M_2(\mathbb{R}), v) = [0, v]$ is a PMV-algebra. Obviously, $\text{Id}(A) = \{(0), A\}$. In [8], it is showed that $P = \{0\}$ is not a $\sim$-prime ideal of $A$. Hence $\text{Rad}(0) = \{0\}$.

Example 3.8. Let $X$ be a compact topological space and $C(X)$ be the Riesz space of the real continuous functions defined on $X$, then the constant function $1(x) = 1$, for any $x \in X$ is a strong unit in $C(X)$. Then $A = \Gamma(C(X), 1)$ with the usual product of functions is a PMV-algebra. Consider $P = \{0\}$ (0 is the zero function). It is clear that $P$ is a $\sim$-prime ideal of $A$. Hence $\text{Rad}(0) = \{0\}$.

Example 3.9. Let $G = \oplus\{Z_i\}_{i \in \mathbb{N}}$ be the lexicographic product of denumerable infinite copies of the abelian $l$-group $Z$ of the relative integers and $e^i \in G$ such that $c^i_k = 0$ if $k \neq i$ and $c^i_i = 1$ if $k = i$, then $G$ with the usual product is an lu-ring. It follows from [4] that $A = \Gamma(G, u) = [0, u]$ is a PMV-algebra, where $\Gamma$ is a functor from the category of abelian lu-ring to the category PMV-algebras and $u = (1, 0, 0, 0, \ldots)$ is the strong unit of $A$, where $\leq$ is the lexicographic order on $G$.

If we set $P_i = (0, e^i)$, then $P_i \subseteq P_j$, for $i > j$. We have $(0, e^1) \cdot (0, e^2) = 0 \in P_1$, while $(0, e^1) \notin P_1$, $(0, e^2) \notin P_1$, $i \neq 1,2$, hence $P_1$ is not a $\sim$-prime ideal of $A$. Thus $\text{Rad}(P_1) = A$.

By the following lemma, we characterize $\text{Rad}(I)$ via elements of $A$, where $I$ is an arbitrary $\sim$-ideal of $A$.

Lemma 3.10. Let $I$ be $\sim$-ideal of $A$. Then

$$\text{Rad}(I) = \{a \in A : a^n = a \cdot a \cdots a \in I, \text{ for some } n \in \mathbb{N}\}.$$ 

Proof. Set $T = \{a \in A : a^n \in I, \text{ for some } n > 0\}$. Let $r \in T$. Then there exists an integer number $n > 0$ such that $r^n \in I$.

Now for any $\sim$-prime ideal $P$ containing $I$, we have $r^n \in P$. Since $P$ is a $\sim$-prime ideal of $A$, $r \in P$. Hence $T \subseteq \text{Rad}(I)$.

Conversely, let $r \in \text{Rad}(I)$. We show that $r \in T$. By contrary, suppose that $r \notin T$, so $r^n \notin I$, for all $n > 0$. Consider $S = \{r^n \oplus x : n \in \mathbb{N} \cup \{0\}, x \in I\}$. 

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Firstly, \( S \) is \( \sim \)-closed system in \( A \). By Theorem 2.12(e), for \( x, y \in I \) and \( n, m \in \mathbb{N} \), we have
\[
(r^n \oplus x) \cdot (r^m \oplus y) \leq r^{n+m} \oplus r^n \cdot y \oplus r^m \cdot x \cdot y,
\]
for some \( z \in I \). Hence \( S \) is a \( \sim \)-closed system.

Now, we claim that \( S \cap I = \emptyset \). If \( a \in S \cap I \), then there exist \( n \in \mathbb{N} \cup \{0\} \) and \( x \in I \) such that \( a = r^n \oplus x \). Hence \( r^n \leq a \in I \), we conclude that \( r^n \in I \), which is a contradiction. Thus \( S \cap I = \emptyset \). It follows from Theorem 3.4 that there exists a \( \sim \)-prime ideal \( P \) of \( A \) such that \( I \subseteq P \) and \( P \cap S = \emptyset \). Hence \( r \in P \) and \( r = r \oplus 0 \in S \). Therefore \( r \in P \cap S \), which is a contradiction. This results \( r \in T \). Thus \( \text{Rad}(I) \subseteq T \) and the proof is complete. \( \square \)

We recall that \( x \in I \rightarrow J \) if and only if \( \{x\} \cap I \subseteq J \), for ideals \( I \) and \( J \) of \( A \), where \( I \rightarrow J = \{x \in A | I \cap (x) \subseteq J\} \) [14].

**Theorem 3.11.** Let \( I \) and \( J \) be proper \( \sim \)-ideals of \( A \) and \( a, b \in A \). Then the following conditions hold:

1. If \( x \in B(A) \), for any \( x \in A \), then \( a \oplus b \in I \),
2. If \( I \subseteq J \), then \( \text{Rad}(I) \subseteq \text{Rad}(J) \),
3. If \( A \) is a unital PMV-algebra, then \( \text{Rad}(I) = A \) iff \( I = A \),
4. \( \text{Rad}(\text{Rad}(I)) = \text{Rad}(I) \),
5. \( \text{Rad}(I) \cup \text{Rad}(J) \subseteq \text{Rad}(I \cup J) \),
6. \( \text{Rad}(I) \rightarrow \text{Rad}(J) \subseteq I \rightarrow \text{Rad}(J) \),
7. \( \text{Rad}(I \rightarrow J) \subseteq \text{Rad}(I \rightarrow \text{Rad}(J)) \),
8. If for every \( a \in I \) there exists \( k \in \mathbb{N} \) such that \( ka \in J \), then \( \text{Rad}(I) \subseteq \text{Rad}(J) \).

**Proof.**

1. Let \( a, b \in \text{Rad}(I) \). Then \( a \oplus b \in \text{Rad}(I) \) and \( (a \oplus b)^n \in I \), for some \( n \in \mathbb{N} \). It follows from Lemma 2.14 that \( (a \oplus b)^n = (a \oplus b) \). We deduce that \( a \oplus b \in I \).

2. It is clear.

3. Let \( \text{Rad}(I) = A \). Then \( 1 \in \text{Rad}(I) \), so \( 1^1 \in I \), for some \( n \in \mathbb{N} \). Therefore \( I = A \). The converse is clear.

4. By (2), we have \( \text{Rad}(I) \subseteq \text{Rad}(\text{Rad}(I)) \). It is enough to show that \( \text{Rad}(\text{Rad}(I)) \subseteq \text{Rad}(I) \). Let \( x \in \text{Rad}(\text{Rad}(I)) \). Then there exists \( n \in \mathbb{N} \) such that \( x^n \in \text{Rad}(I) \). We imply that \( (x^n)^m \in I \), for some \( m \in \mathbb{N} \). Hence \( x^{nm} \in I \). Therefore \( x \in \text{Rad}(I) \), that is \( \text{Rad}(\text{Rad}(I)) \subseteq \text{Rad}(I) \). Thus \( \text{Rad}(\text{Rad}(I)) = \text{Rad}(I) \).

5. The proof is clear by (2).

6. Let \( x \in \text{Rad}(I) \rightarrow \text{Rad}(J) \). Then \( \{x\} \cap \text{Rad}(I) \subseteq \text{Rad}(J) \). Hence \( I \cap \{x\} \subseteq \text{Rad}(J) \), that is \( x \in I \rightarrow \text{Rad}(J) \).
(7) Let \( x \in \text{Rad}(I \to J) \). Then \( x^n \in I \to J \), for some \( n \in \mathbb{N} \). Hence \( I \cap (x^n) \subseteq J \subseteq \text{Rad}(J) \), for some \( n \in \mathbb{N} \). Hence \( x^n \in I \to \text{Rad}(J) \), for some \( n \in \mathbb{N} \), so \( x \in \text{Rad}(I \to \text{Rad}(J)) \).

(8) Let \( a \in I \). Assume that there is \( k \in \mathbb{N} \) such that \( ka \in J \). We have \( a \leq ka \), thus \( a \in J \). Hence \( I \subseteq J \) and by (2), we have \( \text{Rad}(I) \subseteq \text{Rad}(J) \). □

In the following example, we show that the inclusions in parts (2) and (5) of Theorem 3.11 could be proper.

**Example 3.12.** Consider \( PMV \)-algebra \( A = P(\Omega) \) as in Example 3.6, we have \( \text{Rad}(P_1) \cup \text{Rad}(P_2) = P_1 \cup P_2 = \{\emptyset, \{1\}, \{2\}\} \), but \( \{1, 2\} \in (P_1 \cup P_2) \subseteq \text{Rad}(P_1 \cup P_2) \), since \( \{1, 2\} = \{1\} \oplus \{2\} \), then \( \{1, 2\} \in \text{Rad}(P_1 \cup P_2) \) but \( \{1, 2\} \not\in \text{Rad}(P_1) \cup \text{Rad}(P_2) \). Hence \( \text{Rad}(P_1) \cup \text{Rad}(P_2) \neq \text{Rad}(P_1 \cup P_2) \), therefore the equality of Theorem 3.11 (5), is not true in general.

Also, in Example 3.9, we have \( \text{Rad}(P_1) = \text{Rad}(\{(0, 0)\}) = A \), while \( A \not\subseteq \{(0, 0)\} \), hence the converse of Theorem 3.11, (2) is not true in general.

**Theorem 3.13.** Let \( \{I_i\}_{i \in I} \) be a family of proper \( \text{-}\)-ideals of \( A \). Then
\[
\text{Rad}(\bigcap_{i \in I} I_i) = \bigcap_{i \in I} \text{Rad}(I_i).
\]

*Proof.* We have \( \bigcap_{i \in I} I_i \subseteq I_i \subseteq \text{Rad}(I_i) \), for all \( i \in I \). Then by Theorem 3.11 (2), we get that \( \text{Rad}(\bigcap_{i \in I} I_i) \subseteq \text{Rad}(I_i) \) for all \( i \in I \). Therefore \( \text{Rad}(\bigcap_{i \in I} I_i) \subseteq \bigcap_{i \in I} \text{Rad}(I_i) \).

Conversely, let \( x \in \bigcap_{i \in I} \text{Rad}(I_i) \). Then \( x \in \text{Rad}(I_i) \), for all \( i \in I \) and so \( x^n \in I_i \), for all \( i \in I \) and for some \( n \in \mathbb{N} \). Hence \( x^n \in \bigcap_{i \in I} I_i \), for some \( n \in \mathbb{N} \), that is \( x \in \text{Rad}(\bigcap_{i \in I} I_i) \). Therefore \( \text{Rad}(\bigcap_{i \in I} I_i) = \bigcap_{i \in I} \text{Rad}(I_i) \). □

**Proposition 3.14.** Let \( f : A \to B \) be a \( PMV \)-homomorphism. Then \( \text{Rad}(\ker(f)) = f^{-1}(\text{Rad}(\{(0)\})) \).

*Proof.* By Theorem 3.11, we have
\[
\begin{align*}
    a \in \text{Rad}(\ker(f)) & \iff a^n \in \ker(f), \text{for some } n \in \mathbb{N}, \\
    & \iff f(a^n) = 0, \text{for some } n \in \mathbb{N}, \\
    & \iff f(a)^n = 0, \text{for some } n \in \mathbb{N}, \\
    & \iff f(a) \in \text{Rad}(\{(0)\}), \\
    & \iff a \in f^{-1}(\text{Rad}(\{(0)\})).
\end{align*}
\]

□

**Theorem 3.15.** Let \( I \) be a proper \( \text{-}\)-ideal of \( A \). Then \( \text{Rad}(I) \cap B(A) \subseteq I \).

*Proof.* Let \( x \in \text{Rad}(I) \cap B(A) \). Then \( x \in \text{Rad}(I) \) and \( x \in B(A) \). So \( x^n \in I \), for some \( n \in \mathbb{N} \) and by Lemma 2.14, \( x^n = x \cdot x \cdots x = x \land x \cdots \land x = x \in I \). Hence \( x \in I \). Therefore \( \text{Rad}(I) \cap B(A) \subseteq I \). □
Corollary 3.16. $\text{Rad} \{0\} \cap B(A) = \{0\}$.

By the following example we show that the inclusion in Theorem 3.15 could be proper.

Example 3.17. In Example 3.9, it is clear that $B(A) = \{(0,0), u\}$, hence $\text{Rad}(P_1) \cap B(A) = P_1 \cap \{(0,0), u\} = \{(0,0)\} \neq P_1$.

Theorem 3.18. Let $I$ be a proper \(\ldots\)-ideal of $A$. Then the following statements hold:

1. $\text{Rad}(0/I) = \text{Rad}(I)/I$,
2. If $\text{Rad}(I) \subseteq B(A)$, then $\text{Rad}(I) = I$ and $B(A/\text{Rad}(I)) = B(A)/\text{Rad}(I)$,
3. If $a$ is of finite order, then $a/\text{Rad}(I)$ is of finite order, for any $a \in A$.

Proof. (1) In the following by $I \subseteq N$, we means $N$ is a \(\ldots\)-ideal of $A$ containing $I$, then we have

$$\text{Rad}(0/I) = \bigcap_{N \in \text{Spec}(A), I \subseteq N} (N/I) = (\bigcap_{N \in \text{Spec}(A), I \subseteq N} N)/I = \text{Rad}(I)/I.$$  

(2) Let $\text{Rad}(I) \subseteq B(A)$. By Theorem 3.15, we have $\text{Rad}(I) = I$ and

$$B(A)/\text{Rad}(I) = \{e/\text{Rad}(I) : e \in B(A)\},$$

$$= \{e/\text{Rad}(I) : e \vee e^* = 1\},$$

$$= \{e/\text{Rad}(I) : e/\text{Rad}(I) \vee (e/\text{Rad}(I))^* = 1/\text{Rad}(I)\},$$

$$= B(A/\text{Rad}(I)).$$

(3) Suppose that $0 \neq a \in A$ is of finite order. Then there exists $n \in \mathbb{N}$ such that $na = 1$, for all $0 \neq a \in A$. Hence

$$1/\text{Rad}(I) = na/\text{Rad}(I) = n(a/\text{Rad}(I)).$$

Note. It follows from Theorem 3.18(3) that, if $A$ is locally finite $MV$-algebra, then $A/\text{Rad}(I)$ is locally finite.

In the following example, we show that converse of Theorem 3.18(3), is not true in general.

Example 3.19. In Example 3.6, $P_1 = \{0,\{1\}\}$ is a \(\ldots\)-ideal of $A$. We get $\text{Rad}(P_1) = P_1$ and $\{2\}/\text{Rad}(P_1) = \{1,2\}/\text{Rad}(P_1)$. Since $d(\{2\}, \{1,2\}) = \{2\} \ominus \{1,2\} \ast \ominus \{1,2\} \ominus \{2\} = 0 \ominus \{1\} = \{1\} \in P_1$. Hence $\{2\}/\text{Rad}(P_1) \in A/\text{Rad}(P_1)$ is of finite order, while $\{2\}$ is not of finite order.

Definition 3.20. The set of nilpotent elements of a $PMV$-algebra $A$ is

$$\text{Nil}(A) = \{x \in A : x^n = x \cdots x = 0, \text{ for some } n \geq 1\}.$$ 

Corollary 3.21. Let $I$ be a \(\ldots\)-ideal of a $PMV$-algebra $A$. Then $\text{Nil}(A) \subseteq \text{Rad}(I)$. 

Remark 3.22. If $I$ is a $\sim$-ideal of $A$, then from Lemma 3.10 $a \in \text{Rad}(I)$ if and only if $a/I \in \text{Nil}(A/I)$.

4. Semi-maximal $\sim$-ideals in $PMV$-algebras

Definition 4.1. Let $I$ be a proper ideal of $A$. If $\text{Rad}(I) = I$, then $I$ is called a semi-maximal $\sim$-ideal of $A$.

By Lemma 3.10, a $\sim$-ideal $I$ of $A$ is a semi-maximal if and only if

$$I = \{ a \in A : a^n \in I \text{ for some } n \in \mathbb{N} \}.$$

Example 4.2. In Example 3.6, we have $\text{Rad}(P_1) = P_1$, hence $P_1$ is a semi-maximal $\sim$-ideal.

Example 4.3. In Example 3.9, $\{(0,0)\}$ is not a semi-maximal $\sim$-ideal of $A$.

Proposition 4.4. Let $A, B$ be $PMV$-algebras and $f : A \to B$ be a $PMV$-homomorphism. Then the following statements hold:

(a) If $I$ is a semi-maximal $\sim$-ideal of $B$, then $f^{-1}(I)$ is a semi-maximal $\sim$-ideal of $A$.

(b) If $f$ is onto and $I$ is a semi-maximal $\sim$-ideal of $A$ with $\text{Ker}(f) \subseteq I$, then $f(I)$ is a semi-maximal $\sim$-ideal of $B$.

Proof. (a) It is enough to show that $f^{-1}(\text{Rad}(I)) = \text{Rad}(f^{-1}(I))$, since then $f^{-1}(I) = f^{-1}(\text{Rad}(I)) = \text{Rad}(f^{-1}(I))$. Now, for $x \in A$, we have

$$x \in f^{-1}(\text{Rad}(I)) \iff f(x^n) \in I, \text{ for some } n \in \mathbb{N},$$

$$\iff f(x^n) \in I, \text{ for some } n \in \mathbb{N},$$

$$\iff x^n \in f^{-1}(I), \text{ for some } n \in \mathbb{N},$$

$$\iff x \in \text{Rad}(f^{-1}(I)).$$

(b) Let $I$ be a semi-maximal $\sim$-ideal of $A$. We can easily check that $f(I)$ is a $\sim$-ideal of $B$. It is sufficient to show that $\text{Rad}(f(I)) = f(\text{Rad}(I)) = f(I)$.

Let $x \in f(\text{Rad}(I))$. Then there exists $t \in \text{Rad}(I)$ such that $x = f(t)$. Hence $t^n \in I$, for some $n \in \mathbb{N}$. This results $x^n = f(t)^n \in f(I)$, for some $n \in \mathbb{N}$, we obtain $x \in \text{Rad}(f(I))$. Then $f(\text{Rad}(I)) \subseteq \text{Rad}(f(I))$.

Conversely, let $x \in \text{Rad}(f(I))$. Then $x^n \in f(I)$, for some $n \in \mathbb{N}$. Since $f$ is onto, there exists $t \in I$ such that $x = f(t)$. Thus for some $n \in \mathbb{N}$, we have

$$f(t)^n \in f(I) \Rightarrow f(t)^n = f(c), \text{ for some } c \in I,$$

$$\Rightarrow t^n \circ c^* \in \text{Ker} f \subseteq I,$$

$$\Rightarrow [t^n \circ c^*] \oplus c \in I,$$

$$\Rightarrow t^n \leq c \lor t^n \in I,$$
Let suppose that.

If $3.11(1)$ By Theorem $3.10$ and only if $A=I$

$3.11(2)$ We have $a^2$ for every $I$

Then $\text{Rad}$ $a$ $2$

Therefore $n$ $I$

Proof. (1) By Theorem $4.7$, $\text{Rad}(I)$ is a semi-maximal $-ideal$ of $A$. Let $J$ be a semi-maximal $-ideal$ such that $I \subseteq J$. Then $\text{Rad}(I) \subseteq \text{Rad}(J) = J$.

(2) We have $\text{Rad}(I)/I \subseteq \text{Rad}(\text{Rad}(I))/I$. We show that $\text{Rad}(\text{Rad}(I))/I \subseteq \text{Rad}(I)/I$. Take $a/I \in \text{Rad}(\text{Rad}(I))/I$, then $(a/I)^n \in \text{Rad}(I)/I$, for some $n \in \mathbb{N}$. Hence $(a^n)/I = b/I$, for some $b \in \text{Rad}(I)$ and $n \in \mathbb{N}$, so $d(a^n, b) \in I \subseteq \text{Rad}(I)$. Therefore $((a^n) \circ b^n) \oplus [b \circ (a^n)'] \in \text{Rad}(I)$ and $b \in \text{Rad}(I)$. It follows that $[(a^n) \circ b^n] \oplus b \in \text{Rad}(I)$ and $(a^n) \forall b \in \text{Rad}(I)$. Hence $a^n \in \text{Rad}(I)$, for some $n \in \mathbb{N}$, that is $a \in \text{Rad}(\text{Rad}(I))$. Thus $a/I \in \text{Rad}(\text{Rad}(I))/I = \text{Rad}(I)/I$. □

Corollary 4.6. Let $\{I_i\}_{i \in I}$ be finite family of semi-maximal $-ideals$ of $A$. Then $\bigcap_{i \in I} I_i$ is a semi-maximal $-ideal$ of $A$.

Proof. Let $\{I_i\}$ be finite family of semi-maximal ideals of $A$. Hence $\text{Rad}(I_i) = I_i$, for every $i \in I$, so by Theorem 3.11, we have

\[
\text{Rad} \left( \bigcap_{i \in I} I_i \right) = \bigcap_{i \in I} \text{Rad}(I_i) = \bigcap_{i \in I} I_i.
\]

□

By the following theorem we prove that $I$ is a semi-maximal $-ideal$ of $A$ if and only if $A/I$ has no nilpotent elements of $A$.

Theorem 4.7. If $A$ is a $PMV$-algebra and $I$ is a $-ideal$ of $A$, then $A/I$ has no nilpotent elements if and only if $I$ is a semi-maximal $-ideal$ of $A$.

Proof. Suppose that $A/I$ has no nilpotent elements and $a \in \text{Rad}(I)$. Then from Lemma 3.10, we deduce that $a^n \in I$, for some integer $n > 0$. So $(a/I)^n = (a^n)/I = 0/I$. Since $A/I$ has no nilpotent elements, $a/I = 0/I$. This implies $a \in I$. Therefore $\text{Rad}(I) \subseteq I$ and $I$ is a semi-maximal $-ideal$ of $A$.

Conversely, let $I$ be a semi-maximal ideal of $A$ and $0 \neq a/I$ be a nilpotent element of $A/I$. Then $(a/I)^n = a^n/I = 0/I$, for some integer $n > 0$. Hence $a^n \in I$, for some integer $n > 0$ and so $a \in \text{Rad}(I)$. Since $I$ is a semi-maximal $-ideal$ of $A$, $a \in \text{Rad}(I) = I$, $a \in I$. So $a/I = 0/I$, which is a contradiction. Therefore $A/I$ has no nilpotent elements. □
5. Conclusion

$MV$-algebras were originally introduced by C. Chang in [2] in order to give an algebraic counterpart of the Łukasiewicz many valued logic.

A. Dvurečenskij and A. Di Nola in [4] introduced the notion of $PMV$-algebras, that is $MV$-algebras whose product operation $(\cdot)$ is defined on the whole $MV$-algebra.

In this paper, we introduced the notion of the radical of a $PMV$-algebra and characterized radical $A$ via elements of $A$. We also presented several different characterizations and many important properties of the radical of a $\sim$-ideal in a $PMV$-algebra. We introduced the notion of a semi-maximal $\sim$-ideal. We proved that if $I$ is a $\sim$-ideal of a $PMV$-algebra $A$, $I$ is a semi-maximal $\sim$-ideal of $A$ if and only if $A/\text{Rad}(I)$ has no nilpotent elements.

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