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RADICAL OF $\cdot$-IDEALS IN PMV-ALGEBRAS

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Abstract. In this paper, we introduce the notion of the radical of a PMV-algebra A and we charactrize radical A via elements of A. Also, we introduce the notion of the radical of a $\cdot$-ideal in PMV-algebras. Several characterizations of this radical is given. We define the notion of a semimaximal $\cdot$-ideal in a PMV-algebra. Finally we show that A/I has no nilpotent elements if and only if I is a semi-maximal $\cdot$-ideal of A.
Keywords: PMV-algebra, $\cdot$-ideal, $\cdot$-prime ideal, radical.
MSC(2010): Primary: 06D35; Secondary: 06B10.

1. Introduction

C. Chang introduced the notion of MV-algebras to provide a proof for the completeness of the Łukasiewicz axioms for infinite valued propositional logic [2]. In fact MV-algebras are now algebraic counterparts of Łukasiewicz many valued logics.

A. Dvurecenskij and A. Di Nola in [4] introduced the notion of product MV-algebras, i.e., MV-algebras with product which is defined on the whole MV-algebra and is associative and left/right distributive with respect to a partial addition. They concluded that the category of product MV-algebras is categorically equivalent to the category of assocative unital l-rings. Some examples are presented and compared with MV-algebras. In addition, they introduced and studied MVf-algebras [4].

In [9], we introduced the notion of the radical of an ideal in a MV-algebra and gave several characterizations of this radical. We defined the notion of a semi-maximal ideal in an MV-algebra and proved some theorems which give relations between this semi-maximal ideal and other types of ideals in MV-algebras [9].

In this paper, we introduce the notion of the radical of a PMV-algebra A and give several characterizations of radical A. We introduce the notion of the

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radical of $\cdot$-ideal of PMV-algebras. We have also presented several different characterizations and many important properties of the radical of a $\cdot$-ideal in a PMV-algebra. This leads us to introduce the notion of semi-maximal $\cdot$-ideal. Finally, we show that I is a semi-maximal $\cdot$-ideal of A if and only if A/I has no nilpotent elements of A.

2. **Preliminaries**

In this section, we recall some basic notions in MV-algebras and summarize some of their basic properties. For more details about these concepts, we refer the reader to [2–4].

Definition 2.1. [2] An MV-algebra is a structure $(A, \oplus, {}^*, 0)$, where $\oplus$ is a binary operation, * is a unary operation, and 0 is a constant satisfying the following conditions, for any $a, b \in A$:
$(MV1)$ (A, $\oplus$,0) is an abelian monoid,
$(MV2)$ $(a^*)^* = a$,
$(MV3)$ $0^* \oplus a = 0^*$,
$(MV4)$ $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

We say that the element $x \in A$ has order n, and we write $ord(x) = n$, if n is the smallest natural number such that $nx = 1$, where $1 = 0^*$ and $nx := \underbrace{x \oplus x \oplus \cdots \oplus x}_{n \quad \text{time}}$. In this case we say that the element x has a finite order, and write $ord(x) < \infty$. An MV-algebra A is locally finite if every non-zero element of A is of finite order. Also we have $a^n = a^{n-1} \odot a$ and $na = (n-1)a \oplus a$, where $a \odot b = (a^* \oplus b^*)^*$ [3].

If we define the auxiliary operations $\odot, \vee$ and $\wedge$ on A as:

$$a \odot b = (a^* \oplus b^*)^*, \quad a \vee b = a \oplus (b \odot a^*) = b \oplus (b^* \odot a),$$

$$a \wedge b = a \odot (b \oplus a^*) = b \odot (b^* \oplus a),$$

then $(A, \odot, 1)$ is an abelian monoid and the structure $L(A) := (A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice [3].

An element $a \in A$ is called complemented if there is an element $b \in A$ such that $a \vee b = 1$ and $a \wedge b = 0$. We denote the set of complemented elements of A by $B(A)$.

Lemma 2.2. *[3] In each MV-algebra A, the following relations hold for all $x, y, z \in A$:*
(1) $x \leq y$ if and only if $y^* \leq x^*$,
(2) If $x \leq y$, then $x \oplus z \leq y \oplus z$ and $x \odot z \leq y \odot z$,
(3) $x \leq y$ if and only if $x^* \oplus y = 1$ if and only if $x \odot y^* = 0$,
(4) $x, y \leq x \oplus y$ and $x \odot y \leq x, y$,
(5) $x \oplus x^* = 1$ and $x \odot x^* = 0$,

(6) *If* $x \in B(A)$, *then* $x \odot x = x$ *and* $x \oplus x = x$,
(7) $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$.

An ideal in an MV-algebra is defined as:

Definition 2.3. [2] An ideal of an MV-algebra A is a nonempty subset I of A, satisfying the following conditions:
$(I1)$ If $x \in I$, $y \in A$ and $y \leq x$, then $y \in I$,
$(I2)$ If $x, y \in I$, then $x \oplus y \in I$.
We denote the set of all ideals of an MV-algebra A by $Id(A)$.

Definition 2.4. [3] Let I be an ideal of an MV-algebra A. Then I is proper if $I \neq A$. A proper ideal P is prime if for $x, y \in A$, $x \wedge y \in P$ implies $x \in P$ or $y \in P$. Equivalently, P is prime if and only if for all $x, y \in A$, $x \odot y^* \in P$ or $y \odot x^* \in P$ [3].

Theorem 2.5. *[3, 14] Let I be a proper ideal of A. Then the following statements hold:*
(1) *Any prime ideal of A is contained in a unique maximal ideal of A,*
(2) *If $a \in A - I$, then there is a prime ideal P of A such that $I \subseteq P$ and $a \notin P$. In particular for every element $a \in A$, $a \neq 0$, there exists a prime ideal P such that $a \notin P$.*

Definition 2.6. [9] Let I be a proper ideal of A. The intersection of all maximal ideals of A which contain I is called the radical of I and it is denoted by $Rad(I)$.

Theorem 2.7. *[9] Let I be a proper ideal of A. Then*

$$Rad(I) = \{a \in A : na \odot a \in I, \quad \textit{for all} \quad n \in \mathbb{N}\}.$$

We will denote by $\mathcal{MV}$ the category whose objects are MV-algebras and whose morphisms are MV-algebra homomorphisms. A crucial result in the theory of MV-algebras is the categorical equivalence between the category of MV-algebras and the category of Abelian l-groups with strong unit [13]. We recall that an lu-group is an algebra (G, +, -, 0, $\vee$, $\wedge$, u), where the following properties hold:
(a) (G, +, -, 0) is a group,
(b) (G, $\vee$, $\wedge$) is a lattice,
(c) For any $x, y, a, b \in G$, $x \leq y$ implies $a + x + b \leq a + y + b$,
(d) $u > 0$ is strong unit for G (that is, for all $x \in G$ there is some natural number $n \geq 1$ such that $-nu \leq x \leq nu$) [1].

We refer to [1] for a detailed study of l-groups theory. Given an Abelian l-group (G,+,0,$\leqslant$) and a positive element $u > 0$ in G, the interval $[0, u]$ can be endowed with an MV-algebra structure as follows:

$$x \oplus y := (x + y) \wedge u \quad \textit{and} \quad x^* := u - x,$$

for any $x, y \in [0, u]$. Moreover, the lattice operations on $[0, u]$ are the restriction of the lattice operations on G. The MV-algebra $([0, u], \oplus, *, 0, u)$ will be denoted by $[0, u]_G$. If G is an l-group then a strong unit is a positive element $u > 0$ from G with the property that for any $g \in G$ there is integer number $n \geq 0$ such that $g \leq nu$. In the sequel, the Abelian l-groups with strong unit will be simply called lu-groups. We shall denote by $\mathcal{UG}$ the category of lu-groups. The elements of this category are pairs (G, u) where G is an Abelian l-group and u is a strong unit of G. The morphisms will be l-group homomorphisms which preserve the strong unit. The functor that establishes the categorical equivalence between $\mathcal{MV}$ and $\mathcal{UG}$ is

$$\Gamma : \mathcal{UG} \longrightarrow \mathcal{MV}.$$

such that $\Gamma(G, u) := [0, u]_G$ for any lu-group (G, u), $\quad \Gamma(h) := h\mid_{[0,u]}$ for any lu-groups homomorphism h.

The categorical equivalence between MV-algebras and lu-groups leads also to the problem of defining a product operation on MV-algebras, in order to obtain structures corresponding to l-rings. We recall that an l-ring [5] is a structure $(R, +, \cdot, 0, \leq)$, where $(R, +, 0, \leq)$ is an l-group such that, for any $x, y \in R$

$$x \geq 0 \text{ and } y \geq 0 \text{ implies } x \cdot y \geq 0.$$

Definition 2.8. [4] A product MV-algebra (or PMV-algebra, for short) is a structure $(A, \oplus, {}^*, \cdot, 0)$, where $(A, \oplus, {}^*, 0)$ is an MV-algebra and $\cdot$ is a binary associative operation on A such that the following property is satisfied:
if $x + y$ is defined, then $x \cdot z + y \cdot z$ and $z \cdot x + z \cdot y$ are defined and

$$(x + y) \cdot z = x \cdot z + y \cdot z, \quad z \cdot (x + y) = z \cdot x + z \cdot y,$$

where $+$ is a partial addition on A, as follows:

$$\text{for any } x, y \in A, \quad x + y \text{ is defined if and only if } x \leq y^*$$

and in this case, $x + y := x \oplus y$.

If A is a PMV-algebra, then a unity for the product is an element $e \in A$ such that $e \cdot x = x \cdot e = x$ for any $x \in A$. A PMV-algebra that has unity for the product is called unital.

A $\cdot$-ideal of a PMV-algebra A is an ideal I of MV-algebra A such that $a \in I$ and $b \in A$ entail $a \cdot b \in I$ and $b \cdot a \in I$. We denote by $Id_p(A)$ the set of $\cdot$-ideals of a PMV-algebra A.

We will refer to [4,12] for the basic properties of PMV-algebras. Obviously, a PMV-algebra homomorphism will be an MV-algebra homomorphism which also commutes with the product operation. We shall denote by $\mathcal{PMV}$ the category of product MV-algebras with the corresponding homomorphisms.

In the sequel, an lu-ring will be a pair (R, u) where $(R, \oplus, \cdot, \leq)$ is an l-ring and u is a strong unit of R such that $u \cdot u \leq u$. We imply that the interval

$[0, u]$ of an lu-ring (R, u) is closed under the product of R. Thus, if we consider the restriction of $\cdot$ to $[0, u] \times [0, u]$, then the interval $[0, u]$ has a canonical PMV-algebra structure:

$$x \oplus y := (x + y) \wedge u, \quad x^* := u - x, \quad x \cdot y := x \cdot y,$$

for any $0 \leq x, y \leq u$. We shall denote this structure by $[0, u]_R$.

If $\mathcal{UR}$ is the category of lu-rings, whose objects are pairs (R, u) as above and whose morphisms are l-rings homomorphisms which preserve the strong unit, then we get a functor

$$\Gamma : \mathcal{UR} \to \mathcal{PMV},$$
$$\Gamma(R, u) := [0, u]_R, \text{ for any lu-ring } (R, u),$$
$$\Gamma(h) := h \mid_{[0,u]} \text{ for any lu-rings homomorphism h.}$$

In [4] it is proved that Γ establishes a categorical equivalence between $\mathcal{UR}$ and $\mathcal{PMV}$.

Definition 2.9. [8] Let P be a $\cdot$-ideal of A. P is called a $\cdot$-prime if (i) $P \neq A$, (ii) for every $a, b \in A$, if $a \cdot b \in P$, then $a \in P$ or $b \in P$.

Definition 2.10. [3] An element a in MV-algebra A is said to be infinitesimal if and only if $a \neq 0$ and $na \leq a^*$ for each integer $n \geq 0$. The set of all infinitesimals in A will be denoted by $Inf(A)$.

Lemma 2.11. *[4] If A is PMV-algebra, then for any $a, b \in A$,*
(i) $a \cdot 0 = 0 = 0 \cdot a$,
(ii) if $a \leq b$, then for any $c \in A$, $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$.

We recall that in an MV-algebra A, the Chang distance the function is defined by $d : A \times A \longrightarrow A, \qquad d(a, b) := (a \odot b^*) \oplus (b \odot a^*)$ [2].
In the following lemma, we state and prove some properties of PMV-algebras.

Lemma 2.12. *[10] If A is a PMV-algebra, then the following properties hold for any $x, y, \alpha \in A$,*
(a) $(nx) \cdot y = x \cdot (ny)$, for any $n \in \mathbb{N}$,
(b) $x \cdot y^* \leq (x \cdot y)^*$,
(c) $(x \cdot y)^* = x^* \cdot y + (1 \cdot y)^*$,
(d) $(\alpha \cdot x) \odot (\alpha \cdot y)^* \leq \alpha \cdot (x \odot y^*)$,
(e) $\alpha \cdot (x \oplus y) \leq \alpha \cdot x \oplus \alpha \cdot y$,
(f) $d(\alpha \cdot x, \alpha \cdot y) \leq \alpha \cdot d(x, y)$,
Moreover, if A is a unital PMV-algebra, then
$(x \cdot y)^* = x^* \cdot y + y^*$.

Lemma 2.13. *[4] If A is a unital PMV-algebra, then:*
(a) The unity for the product is $e = 1$,
(b) $x \cdot y \leq x \wedge y$ for any $x, y \in A$.

Theorem 2.14. *[4] A finite MV-algebra A admits a product $\cdot$ such that $a \cdot 1 = a = 1 \cdot a$ for any $a \in A$ if and only if A is a Boolean algebra, i.e., $a \oplus a = a$ for any $a \in A$. If it is the case, then $a \cdot b = a \wedge b \in A$.*

Definition 2.15. [10] A nonempty subset of a PMV-algebra $S \subseteq A$ is called $\cdot$-closed system in A if $1 \in S$ and $x, y \in S$ implies $x \cdot y \in S$.

We denote by $S(A)$ the set of all $\cdot$-closed systems of A.

Remark 2.16. [10] Let A be a PMV-algebra. Then
$I(a) = \{x \in A : x \leq y \oplus ma \oplus n(\alpha \cdot a),\ \text{for some } y \in I,\ \text{integers } n, m \geq 0, \alpha \in A\}$.

Proposition 2.17. *[10] Let A be a PMV-algebra.*
(i) If $N \subseteq A$ is a nonempty set, then we have $(N] = \{x \in A : x \leq x_1 \oplus \cdots \oplus x_n \oplus \alpha_1 \cdot y_1 \oplus \cdots \oplus \alpha_m \cdot y_m \quad \text{for some} \quad x_1, \cdots, x_n, y_1, \cdots y_m \in N, \alpha_1, \cdots \alpha_m \in A\}$, where by $(N]$, we mean the ideal generated by N.

In particular, for $a \in A$,

$$(a] = \{x \in A : x \leq na \oplus m(\alpha \cdot a) \quad \text{for some integer } n, m \geq 0, \alpha \in A\},$$

(ii) If $I_1, I_2 \in Id_p(A)$, then
$I_1 \vee I_2 = (I_1 \cup I_2] = \{a \in A : a \leq x_1 \oplus x_2 \quad \text{for some} \quad x_1 \in I_1 \quad \text{and} \quad x_2 \in I_2\}$.

3. Radical of $\cdot$-ideals in PMV-algebras

From now on $(A, \oplus, *, 0)$ (or simply A) is a PMV-algebra.

Definition 3.1. The intersection of all maximal $\cdot$- ideals of A is called the radical of A and it is denoted by $Rad(A)$.

Lemma 3.2. *If I is a proper $\cdot$-ideal of A, then the following are equivalent:*
(i) I is a maximal $\cdot$-ideal of A,
(ii) for any $a \in A$, $a \notin I$ if and only if $(na \oplus m(\alpha \cdot a))^* \in I$, for some integers $n, m > 0$ and $\alpha \in A$.

Proof. $(i) \Rightarrow (ii)$ Suppose that I is a maximal $\cdot$-ideal of A. Since $a \notin I$, $I \vee (a] = A$. So by Proposition 2.17, there exist $x \in I$ and $n, m > 0$ and $\alpha \in A$ such that $[na \oplus (m(\alpha \cdot a))] \oplus x = 1$. We deduce that $(na \oplus m(\alpha \cdot a))^* \leq x \in I$. This results $(na \oplus m(\alpha \cdot a))^* \in I$, for some $n, m \in \mathbb{N}$ and $\alpha \in A$.

Conversely, if $a \in I$, then $na \in I$. Also, by $\cdot$-ideal property and Lemma 2.12(a), we have $\alpha(ma) \in I$ and $m(\alpha \cdot a) = (m\alpha) \cdot a = \alpha \cdot (ma) \in I$. So $(na \oplus m(\alpha \cdot a)) \in I$. Since I is a proper $\cdot$-ideal, we conclude that $(na \oplus m(\alpha \cdot a))^* \notin I$.

$(ii) \Rightarrow (i)$ Suppose there exists a $\cdot$-ideal J such that $I \subsetneq J$. So there exists an $a \in J - I$. Hence $a \notin I$ and by hypothesis, we conclude that $(na \oplus m(\alpha \cdot a))^* \in I$, for some $n, m \in \mathbb{N}$ and $\alpha \in A$. Hence $(na \oplus m(\alpha \cdot a))^* \in J$. Since $a \in J$, we obtain $(na \oplus m(\alpha \cdot a)) \in J$. Thus $1 = (na \oplus m(\alpha \cdot a))^* \oplus (na \oplus m(\alpha \cdot a)) \in J$, Now, if $x \in A$, then $x \in J$ because $x \leq 1 \in J$. It follows that $A \subseteq J$. Thus $A = J$. $\square$

By the following theorem, we characterize $Rad(A)$ via elements of A.

Theorem 3.3. *Let A be a PMV-algebra A. Then*
$Rad(A) = \{x \in A : nx \oplus m(\alpha \cdot x) \leq x^*, \text{ for any } n, m \in \mathbb{N} \text{ and } \alpha \in A\}$
$\cup Inf(A) \cup \{0\}$.

Proof. Suppose that $kx \oplus t(\alpha \cdot x) \leq x^*$, for any $k, t \in \mathbb{N}$ and $\alpha \in A$ and $0 < x \notin Inf(A)$. Let $x \notin Rad(A)$. Then there exists a maximal $\cdot$-ideal I of A such that $x \notin I$. We see that $[nx \oplus m(\alpha \cdot x)] \odot x = 0 \in I$. Since $x \notin I$, it follows from Lemma 3.2, that $(nx \oplus m(\alpha \cdot x))^* \in I$, for some $n, m \in \mathbb{N}$ and $\alpha \in A$. Hence $(nx \oplus m(\alpha \cdot x))^* \oplus [((nx \oplus m(\alpha \cdot a))^*)^* \odot x] \in I$. So $x \leq (nx \oplus m(\alpha \cdot x))^* \vee x \in I$. Then $x \in I$, which is a contradiction. Thus $x \in Rad(A)$.

Conversely, let $x \in Rad(A)$ and suppose that there exist $k, t \in \mathbb{N}$ and $\beta \in A$ such that $kx \oplus t(\beta \cdot x) \nleq x^*$ and there exists $m \in \mathbb{N}$ such that $mx \nleq x^*$ and $x > 0$. Hence $x \in I$, for any maximal $\cdot$-ideal I and $0 \neq [kx \oplus t(\beta \cdot x)] \odot x$, $x > 0$ and $0 \neq mx \odot x$, for some $m \in \mathbb{N}$, this results $nx \odot x \leq x \in I$, for all $n \in \mathbb{N}$. It follows from Theorem 2.7 that $x \in Rad(I)$ in MV-algebra A. Also by Theorem 2.5 (2), since $mx \odot x \neq 0$, there exists a prime ideal P of A such that $mx \odot (x^*)^* = mx \odot x \notin P$. Since P is a prime ideal of MV-algebra A, $x^* \odot (mx)^* \in P$. Hence by Theorem 2.5 (1), there exists a unique maximal ideal J of A such that $P \subseteq J$. Therefore $(mx)^* \odot x^* \in J$. If $x \in J$, then $mx \in J$, also we have $(mx)^* \leq x \vee (mx)^* = x \oplus (x^* \odot (mx)^*) \in J$. Thus $mx \oplus (mx)^* = 1 \in J$, which is a contradiction. Therefore $x \notin J$. We conclude that $I \subseteq P \subseteq J$ and $x \notin J$. Hence $x \notin Rad(I)$, which is a contradiction. Therefore $nx \oplus m(\alpha \cdot x) \leq x^*$, for all $n, m \in \mathbb{N}$ and $\alpha \in A$ or $0 < x \in Inf(A)$. $\square$

Lemma 3.4. *If S is $\cdot$-closed system in A and I is a $\cdot$-ideal of A such that $S \cap I = \emptyset$, then there exists a $\cdot$-prime P of A such that $I \subseteq P$ and $P \cap S = \emptyset$.*

Proof. Let $T = \{J \in Id(A) : I \subseteq J, J \cap S = \emptyset\}$. A routine application of Zorn's lemma shows that T has a maximal element P. Suppose by contrary that P is not a $\cdot$-prime of A. That is, there exist $a, b \in A$ such that $a \cdot b \in P$ but $a \notin P$ and $b \notin P$.

By the maximality of P, we deduce that $P(a), P(b) \notin T$, hence $P(a) \cap S \neq \emptyset$ and $P(b) \cap S \neq \emptyset$, that is, there exist $p_1 \in P(a) \cap S$ and $p_2 \in P(b) \cap S$. By Remark 2.16, $p_1 \leq y \oplus ma \oplus n(\alpha \cdot a)$ and $p_2 \leq x \oplus kb \oplus t(\beta \cdot b)$, where $x, y \in P$ and $m, n, k, t \in \mathbb{N}$.

Then by Lemma 2.12 (e), we have $p_1 \cdot p_2 \leq x \cdot y \oplus x \cdot ma \oplus x \cdot n(\alpha \cdot a) \oplus kb \cdot y \oplus kb \cdot ma \oplus kb \cdot n(\alpha \cdot a) \oplus t(\beta \cdot b) \cdot y \oplus t(\beta \cdot b) \cdot ma \oplus t(\beta \cdot b) \cdot n(\alpha \cdot a)$.

Since $x, y \in P$ and $a \cdot b \in P$, we imply that $p_1 \cdot p_2 \in P$ but $p_1 \cdot p_2 \in S$, hence $P \cap S \neq \emptyset$, which is a contradiction. Hence P is a $\cdot$-prime of A. $\square$

Definition 3.5. Let I be a proper $\cdot$-ideal of A. The intersection of all $\cdot$-prime ideals of A which contain I is called the radical of I and it is denoted by $Rad(I)$. If there are not $\cdot$-prime ideals of A containing I, then $Rad(I) = A$.

Example 3.6. Let $\Omega = \{1, 2\}$ and $\mathcal{A} = \mathcal{P}(\Omega)$, which is a PMV-algebra with $\oplus = \cup$ and $\odot = \cdot = \cap$. Obviously, $P_1 = \{\emptyset, \{1\}\}$ and $P_2 = \{\emptyset, \{2\}\}$ are $\cdot$-prime ideals of A. Hence $Rad(P_1) = P_1$ and $Rad\{\emptyset, \{2\}\} = P_2$ and $Rad\{\emptyset\} = \{\emptyset, \{1\}\} \cap \{\emptyset, \{2\}\} = \{\emptyset\}$.

Example 3.7. Let $M_2(\mathbb{R})$ be the ring of square matrices of order 2 with real elements and 0 be the matrix with all of its entries 0. If we define the order relation on components $A = (a_{ij})_{i,j=1,2} \geq 0$ iff $a_{ij} \geq 0$ for all $i, j = 1, 2$ such that $v = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$, then $A = \Gamma(M_2(\mathbb{R}), v) = [0, v]$ is a PMV-algebra. Obviously, $Id(A) = \{\{0\}, A\}$. In [8], it is showed that $P = \{0\}$ is not a $\cdot$-prime ideal of A. Hence $Rad\{0\} = A$.

Example 3.8. Let X be a compact topological space and $C(X)$ be the Riesz space of the real continuous functions defined on X, then the constant function $1(x) = 1$, for any $x \in X$ is a strong unit in $C(X)$. Then $A = \Gamma(C(X), 1)$ with the usual product of functions is a PMV-algebra. Consider $P = \{\mathbf{0}\}$ ($\mathbf{0}$ is the zero function). It is clear that P is a $\cdot$-prime ideal of A. Hence $Rad(\{0\}) = \{0\}$.

Example 3.9. Let $G = \oplus\{Z_i\}_{i\in\mathbb{N}}$ be the lexicographic product of denumerable infinite copies of the abelian l-group $\mathbb{Z}$ of the relative integers and $e^i \in G$ such that $e^i_k = 0$ if $k \neq i$ and $e^i_k = 1$ if $k = i$, then G with the usual product is an lu-ring. It follows from [4] that $A = \Gamma(G, u) = [0, u]$ is a PMV-algebra, where Γ is a functor from the category of abelian lu-ring to the category PMV-algebras and $u = (1, 0, 0, 0, \ldots)$ is the strong unit of A, where $\leq$ is the lexicographic order on G.

If we set $P_i =< (0, e^i) >$, then $P_i \subseteq P_j$, for $i > j$. We have $(0, e^1) \cdot (0, e^2) = \mathbf{0} \in P_i$, while $(0, e^1) \notin P_i$, $(0, e^2) \notin P_i$, $i \neq 1, 2$, hence P_i is not a $\cdot$-prime ideal of A. Thus $Rad(P_i) = A$.

By the following lemma, we characterize $Rad(I)$ via elements of A, where I is an arbitrary $\cdot$-ideal of A.

Lemma 3.10. *Let I be $\cdot$-ideal of A. Then*

$$Rad(I) = \{a \in A : a^n = a \cdot a \cdot \cdots \cdot a \in I, \text{ for some } n \in \mathbb{N}\}.$$

Proof. Set $T = \{a \in A : a^n \in I, \text{ for some } n > 0\}$. Let $r \in T$. Then there exists an integer number $n > 0$ such that $r^n \in I$.

Now for any $\cdot$-prime ideal P containing I, we have $r^n \in P$. Since P is a $\cdot$-prime ideal of A, $r \in P$. Hence $T \subseteq Rad(I)$.

Conversely, let $r \in Rad(I)$. We show that $r \in T$. By contrary, suppose that $r \notin T$, so $r^n \notin I$, for all $n > 0$. Consider $S = \{r^n \oplus x : n \in \mathbb{N} \cup \{0\}, x \in I\}$.

Firstly, S is $\cdot$-closed system in A. By Theorem 2.12(e),for $x, y \in I$ and $n, m \in \mathbb{N}$, we have

$$\begin{aligned}(r^n \oplus x) \cdot (r^m \oplus y) &\leq r^{n+m} \oplus \underbrace{r^n \cdot y \oplus x \cdot r^m \oplus x \cdot y}_{z}, \\ &= r^{n+m} \oplus z,\end{aligned}$$

for some $z \in I$. Hence S is a $\cdot$-closed system.

Now, we claim that $S \cap I = \emptyset$. If $a \in S \cap I$, then there exist $n \in \mathbb{N} \cup \{0\}$ and $x \in I$ such that $a = r^n \oplus x$. Hence $r^n \leq a \in I$, we conclude that $r^n \in I$, which is a contradiction. Thus $S \cap I = \emptyset$. It follows from Theorem 3.4 that there exists a $\cdot$-prime ideal P of A such that $I \subseteq P$ and $P \cap S = \emptyset$. Hence $r \in P$ and $r = r \oplus 0 \in S$. Therefore $r \in P \cap S$, which is a contradiction. This results $r \in T$. Thus $Rad(I) \subseteq T$ and the proof is complete. □

We recall that $x \in I \to J$ if and only if $(x] \cap I \subseteq J$, for ideals I and J of A, where $I \to J = \{x \in A | I \cap (x] \subseteq J\}$ [14].

Theorem 3.11. *Let I and J be proper $\cdot$-ideals of A and $a, b \in A$. Then the following condition hold:*
(1) *If $x \in B(A)$, for any $x \in A$, then $a \oplus b \in I$,*
(2) *If $I \subseteq J$, then $Rad(I) \subseteq Rad(J)$,*
(3) *If A is a unital PMV-algebra, then $Rad(I) = A$ iff $I = A$,*
(4) *$Rad(Rad(I)) = Rad(I)$,*
(5) *$Rad(I) \cup Rad(J) \subseteq Rad(I \cup J]$,*
(6) *$Rad(I) \to Rad(J) \subseteq I \to Rad(J)$,*
(7) *$Rad(I \to J) \subseteq Rad(I \to Rad(J))$,*
(8) *If for every $a \in I$ there exists $k \in \mathbb{N}$ such that $ka \in J$, then $Rad(I) \subseteq Rad(J)$.*

Proof. (1) Let $a, b \in Rad(I)$. Then $a \oplus b \in Rad(I)$ and $(a \oplus b)^n \in I$, for some $n \in \mathbb{N}$. It follows from Lemma 2.14 that $(a \oplus b)^n = (a \oplus b)$. We deduce that $a \oplus b \in I$.

(2) It is clear.

(3) Let $Rad(I) = A$. Then $1 \in Rad(I)$, so $1 = 1^n \in I$, for some $n \in \mathbb{N}$. Therefore $I = A$. The converse is clear.

(4) By (2), we have $Rad(I) \subseteq Rad(Rad(I))$. It is enough to show that $Rad(Rad(I)) \subseteq Rad(I)$. Let $x \in Rad(Rad(I))$. Then there exists $n \in \mathbb{N}$ such that $x^n \in Rad(I)$. We imply that $(x^n)^m \in I$, for some $m \in \mathbb{N}$. Hence $x^{nm} \in I$. Therefore $x \in Rad(I)$, that is $Rad(Rad(I)) \subseteq Rad(I)$. Thus $Rad(Rad(I)) = Rad(I)$.

(5) The proof is clear by (2).

(6) Let $x \in Rad(I) \to Rad(J)$. Then $(x] \cap Rad(I) \subseteq Rad(J)$. Hence $I \cap (x] \subseteq Rad(J)$, that is $x \in I \to Rad(J)$.

(7) Let $x \in Rad(I \to J)$. Then $x^n \in I \to J$, for some $n \in \mathbb{N}$. Hence $I \cap (x^n] \subseteq J \subseteq Rad(J)$, for some $n \in \mathbb{N}$. Hence $x^n \in I \to Rad(J)$, for some $n \in \mathbb{N}$, so $x \in Rad(I \to Rad(J))$.

(8) Let $a \in I$. Assume that there is $k \in \mathbb{N}$ such that $ka \in J$. We have $a \leq ka$, thus $a \in J$. Hence $I \subseteq J$ and by (2), we have $Rad(I) \subseteq Rad(J)$. □

In the following example, we show that the inclusions in parts (2) and (5) of Theorem 3.11 could be proper.

Example 3.12. Consider PMV-algebra $A = P(\Omega)$ as in Example 3.6, we have $Rad(P_1) \cup Rad(P_2) = P_1 \cup P_2 = \{\emptyset, \{1\}, \{2\}\}$, but $\{1,2\} \in (P_1 \cup P_2] \subseteq Rad(P_2 \cup P_2]$, since $\{1,2\} = \{1\} \oplus \{2\}$, then $\{1,2\} \in Rad(P_1 \cup P_2]$ but $\{1,2\} \notin Rad(P_1) \cup Rad(P_2)$. Hence $Rad(P_1) \cup Rad(P_2) \neq Rad(P_1 \cup P_2]$, therefore the equality of Theorem 3.11 (5), is not true in general.

Also, in Example 3.9, we have $Rad(P_i) = Rad(\{(0, \mathbf{0})\}) = A$, while $A \nsubseteq \{(0, \mathbf{0})\}$, hence the converse of Theorem 3.11, (2) is not true in general.

Theorem 3.13. *Let $\{I_i\}_{i \in I}$ be a family of proper $\cdot$-ideals of A. Then*

$$Rad(\cap_{i \in I} I_i) = \cap_{i \in I} Rad(I_i).$$

Proof. We have $\cap_{i \in I} I_i \subseteq I_i \subseteq Rad(I_i)$, for all $i \in I$. Then by Theorem 3.11 (2), we get that $Rad(\cap_{i \in I} I_i) \subseteq Rad(I_i)$ for all $i \in I$. Therefore $Rad(\cap_{i \in I} I_i) \subseteq \cap_{i \in I} Rad(I_i)$.

Conversely, let $x \in \cap_{i \in I} Rad(I_i)$. Then $x \in Rad(I_i)$, for all $i \in I$ and so $x^n \in I_i$, for all $i \in I$ and for some $n \in \mathbb{N}$. Hence $x^n \in \cap_{i \in I} I_i$, for some $n \in \mathbb{N}$, that is $x \in Rad(\cap_{i \in I} I_i)$. Therefore $Rad(\cap_{i \in I} I_i) = \cap_{i \in I} Rad(I_i)$.

□

Proposition 3.14. *Let $f : A \to B$ be a PMV-homomorphism. Then $Rad(Ker(f)) = f^{-1}(Rad(\{0\}))$.*

Proof. By Theorem 3.11, we have

$$\begin{aligned} a \in Rad(ker(f)) &\Leftrightarrow a^n \in ker(f), \text{for some} \quad n \in \mathbb{N}, \\ &\Leftrightarrow f(a^n) = 0, \text{for some} \quad n \in \mathbb{N}, \\ &\Leftrightarrow f(a)^n = 0, \text{for some} \quad n \in \mathbb{N}, \\ &\Leftrightarrow f(a) \in Rad(\{0\}), \\ &\Leftrightarrow a \in f^{-1}(Rad(\{0\})). \end{aligned}$$

□

Theorem 3.15. *Let I be a proper $\cdot$-ideal of A. Then $Rad(I) \cap B(A) \subseteq I$.*

Proof. Let $x \in Rad(I) \cap B(A)$. Then $x \in Rad(I)$ and $x \in B(A)$. So $x^n \in I$, for some $n \in \mathbb{N}$ and by Lemma 2.14, $x^n = x \cdot x \cdot \cdots \cdot x = x \wedge x \cdots \wedge x = x \in I$. Hence $x \in I$. Therefore $Rad(I) \cap B(A) \subseteq I$. □

Corollary 3.16. $Rad(\{0\}) \cap B(A) = \{0\}$.

By the following example we show that the inclusion in Theorem 3.15 could be proper.

Example 3.17. In Example 3.9, it is clear that $B(A) = \{(0, \mathbf{0}), u\}$, hence $Rad(P_i) \cap B(A) = P_i \cap \{(0, \mathbf{0}), u\} = \{(0, \mathbf{0})\} \neq P_i$.

Theorem 3.18. *Let I be a proper $\cdot$-ideal of A. Then the following statements hold:*
(1) $Rad(\{0\}/I) = Rad(I)/I$,
(2) If $Rad(I) \subseteq B(A)$, then $Rad(I) = I$ and $B(A/Rad(I)) = B(A)/Rad(I)$,
(3) If a is of finite order, then $a/Rad(I)$ is of finite order, for any $a \in A$.

Proof. (1) In the following by $I \subseteq N$, we means N is a $\cdot$-ideal of A containing I, then we have

$$Rad(\{0\}/I) = \bigcap_{\substack{N \in Spec(A) \\ I \subseteq N}} (N/I) = (\bigcap_{\substack{N \in Spec(A) \\ I \subseteq N}} N)/I = Rad(I)/I.$$

(2) Let $Rad(I) \subseteq B(A)$. By Theorem 3.15, we have $Rad(I) = I$ and

$$\begin{aligned} B(A)/Rad(I) &= \{e/Rad(I) : e \in B(A)\}, \\ &= \{e/Rad(I) : e \vee e^* = 1\}, \\ &= \{e/Rad(I) : e/Rad(I) \vee (e/Rad(I))^* = 1/Rad(I)\}, \\ &= B(A/Rad(I)). \end{aligned}$$

(3) Suppose that $0 \neq a \in A$ is of finite order. Then there exists $n \in \mathbb{N}$ such that $na = 1$, for all $0 \neq a \in A$. Hence

$$1/Rad(I) = na/Rad(I) = n(a/Rad(I)).$$

□

Note. It follows from Theorem 3.18(3) that, if A is locally finite MV-algebra, then $A/Rad(I)$ is locally finite.

In the following example, we show that converse of Theorem 3.18(3), is not true in general.

Example 3.19. In Example 3.6, $P_1 = \{\emptyset, \{1\}\}$ is a $\cdot$-ideal of A. We get $Rad(P_1) = P_1$ and $\{2\}/Rad(P_1) = \{1,2\}/Rad(P_1)$. Since $d(\{2\}, \{1,2\}) = \{2\} \odot \{1,2\}^* \oplus \{1,2\} \odot \{2\}^* = \emptyset \oplus \{1\} = \{1\} \in P_1$. Hence $\{2\}/Rad(P_1) \in A/Rad(P_1)$ is of finite order, while $\{2\}$ is not of finite order.

Definition 3.20. The set of nilpotent elements of a PMV-algebra A is

$$Nil(A) = \{x \in A : x^n = x \cdot \ldots \cdot x = 0, \text{ for some } n \geq 1\}.$$

Corollary 3.21. *Let I be a $\cdot$-ideal of a PMV-algebra A. Then $Nil(A) \subseteq Rad(I)$.*

Remark 3.22. If I is a $\cdot$-ideal of A, then from Lemma 3.10 $a \in Rad(I)$ if and only if $a/I \in Nil(A/I)$.

4. Semi-maximal $\cdot$-ideals in PMV-algebras

Definition 4.1. Let I be a proper ideal of A. If $Rad(I) = I$, then I is called a semi-maximal $\cdot$-ideal of A.

By Lemma 3.10, a $\cdot$-ideal I of A is a semi-maximal if and only if

$$I = \{a \in A : a^n \in I \quad \text{for some} \quad n \in \mathbb{N}\}.$$

Example 4.2. In Example 3.6, we have $Rad(P_1) = P_1$, hence P_1 is a semi-maximal $\cdot$-ideal.

Example 4.3. In Example 3.9, $\{(0, \mathbf{0})\}$ is not a semi-maximal $\cdot$-ideal of A.

Proposition 4.4. *Let A, B be PMV-algebras and $f : A \to B$ be a PMV-homomorphism. Then the following statements hold:*
(a) If I is a semi-maximal $\cdot$-ideal of B, then $f^{-1}(I)$ is a semi-maximal $\cdot$-ideal of A,
(b) If f is onto and I is a semi-maximal $\cdot$-ideal of A with $Ker(f) \subseteq I$, then $f(I)$ is a semi-maximal $\cdot$-ideal of B.

Proof. (a) It is enough to show that $f^{-1}(Rad(I)) = Rad(f^{-1}(I))$, since then $f^{-1}(I) = f^{-1}(Rad(I)) = Rad(f^{-1}(I))$. Now, for $x \in A$, we have

$$\begin{aligned} x \in f^{-1}(Rad(I)) \quad &\Leftrightarrow \quad f(x)^n \in I, \text{ for some } \quad n \in \mathbb{N}, \\ &\Leftrightarrow \quad f(x^n) \in I, \text{ for some } \quad n \in \mathbb{N}, \\ &\Leftrightarrow \quad x^n \in f^{-1}(I), \text{ for some } \quad n \in \mathbb{N}, \\ &\Leftrightarrow \quad x \in Rad(f^{-1}(I)). \end{aligned}$$

(b) Let I be a semi-maximal $\cdot$-ideal of A. We can easily check that $f(I)$ is a $\cdot$-ideal of B. It is sufficient to show that $Rad(f(I)) = f(Rad(I)) = f(I)$.

Let $x \in f(Rad(I))$. Then there exists $t \in Rad(I)$ such that $x = f(t)$. Hence $t^n \in I$, for some $n \in \mathbb{N}$. This results $x^n = f(t)^n \in f(I)$, for some $n \in \mathbb{N}$, we obtain $x \in Rad(f(I))$. Then $f(Rad(I)) \subseteq Rad(f(I))$.

Conversely, let $x \in Rad(f(I))$. Then $x^n \in f(I)$, for some $n \in \mathbb{N}$. Since f is onto, there exists $t \in I$ such that $x = f(t)$. Thus for some $n \in \mathbb{N}$, we have

$$\begin{aligned} f(t)^n \in f(I) \quad &\Rightarrow \quad f(t)^n = f(c), \text{ for some } \quad c \in I, \\ &\Rightarrow \quad t^n \odot c^* \in Kerf \subseteq I, \\ &\Rightarrow \quad [t^n \odot c^*] \oplus c \in I, , \\ &\Rightarrow \quad t^n \leq c \vee t^n \in I, \end{aligned}$$

$$\begin{aligned} &\Rightarrow \quad t^n \in I, \\ &\Rightarrow \quad t \in Rad(I), \\ &\Rightarrow \quad x \in f(Rad(I)), \\ &\Rightarrow \quad Rad(f(I)) \subseteq f(Rad(I)). \end{aligned}$$

Therefore $f(I)$ is a semi-maximal $\cdot$-ideal of B. □

Theorem 4.5. *Let I be a $\cdot$-ideal of A. Then the following statements hold:*
(1) *$Rad(I)$ is the smallest semi-maximal $\cdot$-ideal of A such that $I \subseteq Rad(I)$,*
(2) *$Rad(I)/I$ is a semi-maximal $\cdot$-ideal of A/I.*

Proof. (1) By Theorem 3.11(4), $Rad(I)$ is a semi-maximal $\cdot$-ideal of A. Let J be a semi-maximal $\cdot$-ideal such that $I \subseteq J$. Then $Rad(I) \subseteq Rad(J) = J$.
(2) We have $Rad(I)/I \subseteq Rad(Rad(I)/I)$. We show that $Rad(Rad(I)/I) \subseteq Rad(I)/I$. Take $a/I \in Rad(Rad(I)/I)$, then $(a/I)^n \in Rad(I)/I$, for some $n \in \mathbb{N}$. Hence $(a^n)/I = b/I$, for some $b \in Rad(I)$ and $n \in \mathbb{N}$, so $d(a^n, b) \in I \subseteq Rad(I)$. Therefore $((a^n) \odot b^*) \oplus [b \odot (a^n)^*] \in Rad(I)$ and $b \in Rad(I)$. It follows that $[(a^n) \odot b^*] \oplus b \in Rad(I)$ and $(a^n) \vee b \in Rad(I)$. Hence $a^n \in Rad(I)$, for some $n \in \mathbb{N}$, that is $a \in Rad(Rad(I))$. Thus $a/I \in Rad(Rad(I))/I = Rad(I)/I$. □

Corollary 4.6. *Let $\{I_i\}_{i \in I}$ be finite family of semi-maximal $\cdot$-ideals of A. Then $\bigcap\limits_{i \in I} I_i$ is a semi-maximal $\cdot$-ideal of A.*

Proof. Let $\{I_i\}$ be finite family of semi-maximal ideals of A. Hence $Rad(I_i) = I_i$, for every $i \in I$, so by Theorem 3.11, we have

$$Rad(\bigcap_{i \in I} I_i) = \bigcap_{i \in I} Rad(I_i) = \bigcap_{i \in I} I_i.$$

□

By the following theorem we prove that I is a semi-maximal $\cdot$-ideal of A if and only if A/I has no nilpotent elements of A.

Theorem 4.7. *If A is a PMV-algebra and I is a $\cdot$-ideal of A, then A/I has no nilpotent elements if and only if I is a semi-maximal $\cdot$-ideal of A.*

Proof. Suppose that A/I has no nilpotent elements and $a \in Rad(I)$. Then from Lemma 3.10, we deduce that $a^n \in I$, for some integer $n > 0$. So $(a/I)^n = (a^n)/I = 0/I$. Since A/I has no nilpotent elements, $a/I = 0/I$. This implies $a \in I$. Therefore $Rad(I) \subseteq I$ and I is a semi-maximal $\cdot$-ideal of A.

Conversely, let I be a semi-maximal ideal of A and $0 \neq a/I$ be a nilpotent element of A/I. Then $(a/I)^n = a^n/I = 0/I$, for some integer $n > 0$. Hence $a^n \in I$, for some integer $n > 0$ and so $a \in Rad(I)$. Since I is a semi-maximal $\cdot$-ideal of A, $a \in Rad(I) = I$, $a \in I$. So $a/I = 0/I$, which is a contradiction. Therefore A/I has no nilpotent elements. □

5. Conclusion

MV-algebras were originally introduced by C. Chang in [2] in order to give an algebraic counterpart of the Łukasiewics many valued logic.

A. Dvurecenskij and A. Di Nola in [4] introduced the notion of PMV-algebras, that is MV-algebras whose product operation $(\cdot)$ is defined on the whole MV-algebra.

In this paper, we introduced the notion of the radical of a PMV-algebra and charactrized radical A via elements of A. We also presented several different characterizations and many important properties of the radical of a $\cdot$-ideal in a PMV-algebra. We introduced the notion of a semi-maximal $\cdot$-ideal. We proved that if I is a $\cdot$-ideal of a PMV-algebra A, I is a semi-maximal $\cdot$-ideal of A if and only if $A/Rad(I)$ has no nilpotent elements.

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